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EINSTEIN RELATION FOR RANDOM WALKS IN RANDOM ENVIRONMENT[1]

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In this article, we consider the speed of the random walks in a (uniformly elliptic and i.i.d.) random environment (RWRE) under perturbation. We obtain the derivative of the speed of the RWRE w.r.t. the perturbation, under the assumption that one of the following holds: (i) the environment is balanced and the perturbation satisfies a Kalikow-type ballisticity condition, (ii) the environment satisfies Sznitman's ballisticity condition. This is a generalized version of the Einstein relation for RWRE.

Our argument is based on a modification of Lebowitz–Rost's argument developed in [*Stochastic Process. Appl.* **54** (1994) 183–196] and a new regeneration structure for the perturbed balanced environment.

1. Introduction. In the 1905, Einstein ([9], pages 1–18) investigated the movement of suspended particles in a liquid under the influence of an external force. He established the following mobility–diffusivity relation:

$$\lim_{\lambda\to 0}\frac{v_\lambda}{\lambda}\sim D, \tag{ER}$$

where λ is the size of the perturbation, D is the diffusion constant of the equilibrium state and v_λ is the effective speed of the random motion in the perturbed media. General derivations of this principle assume reversibility.

Recently, there has been much interest in studying the Einstein relation for reversible motions in a perturbed random media, where the perturbation is proportional to the original environment; see [1, 10, 14, 17]. However, it is not clear whether (ER) still holds in nonreversible set-up, for example, random walks in random environments (RWRE), and several interesting questions are either open or not discussed: is v_λ monotone (in an appropriate sense) and differentiable with respect to λ? What if the perturbation of the environment is not propositional to the original one? If the original environment is ballistic, (ER) is not expected to hold, but what can we say about the derivative of the velocity?

Motivated by these questions, we study the speed of RWRE under general perturbations, where the original environment is either balanced or ballistic. In the

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balanced case, when the perturbation is proportional to the original environment, we obtain the Einstein relation. (This result forms part of the author's doctoral thesis [11].) Moreover, we provide a new interpretation of this relation. Namely, in our context, the Einstein relation is a consequence of the weak convergence of the invariant measures for the "environment viewed from the point of view of the particle" process, which holds even for more general perturbations that satisfies a Kalikow-type condition. In the ballistic case, we can quantify the rate of the weak convergence. As a corollary, we obtain the derivative of the speed w.r.t. the size of the perturbation (for both the balanced and the ballistic cases).

We define the model as follows.

An (uniformly elliptic) *environment* $\omega : \mathbb{Z}^d \times \{e \in \mathbb{Z}^d : |e| = 1\} \to [\kappa, 1)$ is a function that satisfies

$$\sum_{e : |e|=1} \omega(x, e) = 1 \qquad \forall x \in \mathbb{Z}^d,$$

where $\kappa > 0$ and $|\cdot|$ is the l^2-norm. The random walks in the environment ω starting from x is the Markov chain $(X_n)_{n\geq 0}$ with transition probability P_ω specified by

$$P_\omega^x(X_0 = x) = 1,$$
$$P_\omega^x(X_{n+1} = y + e | X_n = y) = \omega(y, e).$$

Following Sabot [18], we consider a perturbed environment

$$\omega^\lambda := \omega + \lambda\xi, \qquad \lambda \in [0, \kappa/2),$$

where $\xi : \mathbb{Z}^d \times \{e \in \mathbb{Z}^d : |e| = 1\} \to [-1, 1]$ satisfies

$$\sum_{e : |e|=1} \xi(x, e) = 0 \qquad \forall x.$$

We denote the local environment at x as $\omega_x := (\omega(x, e))_{e : |e|=1}$ and write

$$\zeta := (\omega, \xi).$$

We endow the set Ω of all ζ with a probability measure $\mathcal{P}$ such that $(\zeta_x)_{x\in\mathbb{Z}^d}$ are independent and identically distributed (i.i.d.).

The measure $P^x_{\omega^\lambda}$ for a fixed ω is called the *quenched law*. The average over all quenched environments, $\mathbb{P}^x_\lambda := \mathcal{P} \otimes P^x_{\omega^\lambda}$, is called the *annealed law*. Expectations with respect to $P^x_{\omega^\lambda}$ and $\mathbb{P}^x_\lambda$ are denoted by $E^x_{\omega^\lambda}$ and $\mathbb{E}^x_\lambda$, respectively. We omit the superscript when x is the origin $o := (0, \dots, 0)$, for example, we write P^o_ω as P_ω. We define the *local drift* of a function $f : \mathbb{Z}^d \times \{e \in \mathbb{Z}^d : |e| = 1\} \to \mathbb{R}$ by

$$d(f) := \sum_{e : |e|=1} f(o, e)e$$

and its spatial shift $\theta^x f$ as

$$\theta^x f(y, e) := f(x + y, e).$$

When the original environment ω is deterministic and homogeneous (i.e., $\omega = \theta^x\omega$, $\forall x$), Sabot ([18], Theorem 1) got the following perturbation expansion for $d \geq 2$:

> If one of $E_{\mathcal{P}}[d(\xi)] \neq o$ and $d(\omega) \neq o$ holds, then, for $\lambda > 0$ small enough, $\lim_{n\to\infty} X_n/n := v_\lambda$ exists $\mathbb{P}_\lambda$-almost surely, and
>
> $$v_\lambda = d(\omega) + \lambda E_{\mathcal{P}}[d(\xi)] + \lambda^2 d_2 + o(\lambda^{3-\varepsilon}) \qquad \forall \varepsilon > 0.$$
>
> The constant d_2 can be expressed in terms of the Green function.

(Sabot also obtained the expansion for $d = 1$, with d_2 replaced by $d_{2,\lambda}$. But in this case v_λ can be explicitly computed, and hence is not as interesting. See remarks in [18], page 2999.) Note that the condition for the above expansion is essentially that ω^λ is *ballistic* for all small $\lambda > 0$, that is, $\lim_{n\to\infty} X_n/n \neq 0$ is a deterministic constant, $\mathbb{P}_\lambda$-a.s.

The purpose of our article is to generate Sabot's first-order expansion to the case where the original environment is random. For RWRE in $\mathbb{Z}^d$, $d \geq 2$, two notable ballisticity conditions are Kalikow's condition and Sznitman's (T′) condition, which are introduced in [13] and [22], respectively. We recall that the (T′) condition is conjectured to be equivalent to the ballisticity of RWRE, and it implies Kalikow's condition. In this paper we are interested in two cases:

(i) The original environment has zero drift (or *balanced*), and (ω, xi) satisfies a Kalikow-type condition for small $\lambda > 0$: for some $\ell \in S^{d-1}$,

$$\text{(K)} \qquad \inf_{f\in\mathcal{F}} E_{\mathcal{P}}\left[\frac{d(\xi)\cdot\ell}{\sum_{e:|e|=1}\omega(o,e)f(e)}\right]\Big/ E_{\mathcal{P}}\left[\frac{1}{\sum_{e:|e|=1}\omega(o,e)f(e)}\right] > 0,$$

$\mathcal{F}$ denotes the collection of nonzero functions $f : \{e : |e| = 1\} \to [0, 1]$.

(ii) The original environment satisfies Sznitman's ballisticity condition (T′).

Condition (K) guarantees that ω^λ has a speed of size $\sim c\lambda$. Note that it is satisfied for some interesting cases, for example, it holds for a perturbation that is "either neutral or pointing to the right" (see Remark 9). For the definition of Sznitman's (T′) condition, we refer to equation (0.5) in [22].

1.1. *Results*. Before the statement of our results, let us recall that one of the main tools in the study of RWRE is the environment viewed from the point of view of the particle process $(\bar{\zeta}_n)_{n\in\mathbb{N}}$, which is defined as

$$\bar{\zeta}_n = (\bar{\omega}_n, \bar{\xi}_n) := \theta^{X_n}\zeta, \qquad n \in \mathbb{N}.$$

Lawler [16] proved that for balanced environment, there exists an ergodic invariant measure for $(\bar{\zeta}_n)$ which is absolutely continuous with respect to $\mathcal{P}$. For ballistic environment whose regeneration time has finite moment (e.g., an environment that satisfies Sznitman's condition), it is shown in [23], Theorem 3.1, that the law of $\bar{\zeta}_n$ converges weakly to an invariant measure. Recently, Berger, Cohen and Rosenthal [2] proved that for dimensions $d \geq 4$, this measure is ergodic and absolutely continuous with respect to the original law of the environment.

We denote by $\mathcal{Q}$ (for both the balanced and the ballistic cases) the invariant measure of $(\bar{\zeta}_n)$ viewed from the original RWRE, and by $\mathcal{Q}_\lambda$ the invariant measure of $(\bar{\zeta}_n)$ viewed from the perturbed RWRE.

Our main results are the following.

THEOREM 1. *Assume that the original environment is balanced* [*i.e.*, $d(\omega) = o$ *almost surely*] *and* $\mathcal{P}$ *satisfies* (K), *then*

$$\mathcal{Q}_\lambda \Rightarrow \mathcal{Q} \qquad \text{as } \lambda \to 0,$$

where $\Rightarrow$ *denotes weak convergence.*

THEOREM 2. *Assume the* $\mathcal{P}$*-law of* ω *satisfies Sznitman's condition* (T$'$). *Then, there exists a linear operator* Λ *such that*

$$\lim_{\lambda \to 0} \frac{\mathcal{Q}_\lambda f - \mathcal{Q} f}{\lambda} = \Lambda f$$

for all (*a.s.*) *bounded* $f : \Omega \to \mathbb{R}$ *which is*

(1) $$\sigma(\omega_x : |x| < N_f)\text{-measurable for some constant } N_f \geq 1.$$

Here, $\mathcal{Q} f$ *denotes the expectation of* f *under* $\mathcal{Q}$. *Moreover,* Λ *can be expressed in terms of the regeneration times*; *see* (51).

As a corollary of the above theorems, we obtain the following.

COROLLARY 3. *If either* (i) *or* (ii) *is satisfied, then there is* $\lambda_0 \in [0, \kappa/2)$ *such that for* $\lambda \in (0, \lambda_0)$, *the limit*

$$\lim_{n \to \infty} \frac{X_n}{n} =: v_\lambda$$

exists $\mathbb{P}_\lambda$*-almost surely and* [*for the convenience of the notation, we set* $\Lambda \equiv 0$ *when* $\mathcal{P}$ *satisfies* (i)]:

$$\lim_{\lambda \to 0} \frac{v_\lambda - v_0}{\lambda} = \mathcal{Q}\big(d(\xi)\big) + \Lambda\big(d(\omega)\big).$$

Recalling that for random walks in balanced random environment, Lawler [16] proved that the scaling limit of $X_{\lfloor \cdot n \rfloor}/\sqrt{n}$ converges to a Brownian motion with diffusion matrix

$$D := \big(E_{\mathcal{Q}}[2\omega(o, e_i)]\delta_{ij}\big)_{1 \leq i,j \leq d},$$

the Einstein relation of a balanced random environment is an immediate consequence of Corollary 3.

PROPOSITION 4 (Einstein relation). *Assume that* $\mathcal{P}$*-almost surely, the original environment is balanced, and*

(2) $$\xi(x, e) = \omega(x, e) e \cdot \ell \qquad \forall x, e.$$

Then $\mathbb{P}_\lambda$*-almost surely*, $v_\lambda := \lim_{n\to\infty} X_n/n = \lambda E_{Q_\lambda}[d(\omega)]$, *and*

$$\lim_{\lambda\to 0}\frac{v_\lambda}{\lambda} = D\ell. \tag{3}$$

The zero-drift case (Theorem 1) is more delicate and makes the main part of the paper. Its proof consists of proving the following two theorems.

THEOREM 5. *Assume that the original environment is balanced. Then, for* $\mathcal{P}$*-almost every* ζ *and any bounded measurable function* $f:\Omega\to\mathbb{R}$,

$$\lim_{\lambda\to 0}\frac{\lambda^2}{t}E_{\omega^\lambda}\left[\sum_{i=0}^{\lceil t/\lambda^2\rceil} f(\bar{\zeta}_i)\right] = \mathcal{Q}f \qquad \forall t>0.$$

THEOREM 6. *Assume that the original environment is balanced and* $\mathcal{P}$ *satisfies* (K). *Then for any* f *that satisfies* (1),

$$\left|\mathcal{Q}_\lambda f - \frac{\lambda^2}{t}\mathbb{E}_\lambda \sum_{i=0}^{\lceil t/\lambda^2\rceil} f(\bar{\zeta}_i)\right| \le \frac{C\|f\|_\infty}{\sqrt{t}}$$

for all $\lambda\in(0,1/N_f)$ *and* $t>0$.

Our proof of Theorem 5 is an adaption of the argument of Lebowitz and Rost [17] (see also [10], Proposition 3.1) to the discrete setting. Namely, using a change of measure argument, we observe that the $\mathbb{P}_\lambda$-law of the rescaled process $\lambda X_{\cdot/\lambda^2}$ converges to a Brownian motion with drift. For the proof of Theorem 6, we want to follow the strategy of Gantert, Mathieu and Piatnitski [10]. Arguments in [10], Proposition 5.1, show that if there is a sequence of random times $\tau_n \sim n/\lambda^2$ (called the *regeneration times*) that divides the random path into i.i.d. parts, then good moment estimates of the regeneration times yield the Einstein relation. [Note that the usual definition of regeneration times, i.e., the $T(n)$'s in Section 6, does not give the correct scale.] Their definition of the regeneration times, which is a variant of that in [19], crucially employs a heat kernel estimate [10], Lemma 5.2, for reversible diffusions. However, due to the lack of reversibility, we do not have a heat kernel estimate for RWRE. In this paper, we construct the regeneration times differently, so that they divide the random path into 1-*dependent* pieces. Moreover, our regeneration times have good moment bounds, which lead to a proof of Theorem 6. The key ingredients in our construction are Kuo and Trudinger's [15] Harnack inequality for discrete harmonic functions and the "ε-coins" trick introduced in [7].

The proof of the ballistic case (Theorem 2) uses a modification of Lebowitz and Rost's argument and the (usual) regeneration structure for a ballistic RWRE. The reason that the ballistic case is easier to analyze is that the original environment already has a regeneration structure, which provides us enough information on the rate of the convergence to the stationary measure. [Recall that Sznitman's (T′) condition implies that the inter-regeneration time has stretch-exponential moment.]

The structure of the paper is as follows. We will prove Theorem 5 in Section 2. In Section 3, using Kalikow's random walks, we obtain estimates that will be useful in deriving the moment bounds of the regenerations. In Section 4, we present our new construction of the regeneration times and show that they have good moment bounds. Sections 5 and 6 are devoted to the proofs of Theorems 1 and 2. With these two theorems, we obtain the derivative of the speed (w.r.t. the size of the perturbation) in Section 7.

Throughout this paper, we use c, C to denote finite positive constants that depend only on the environment measure $\mathcal{P}$ (and implicitly, on the dimension d and the ellipticity constant κ). They may differ from line to line. We also use c_i, C_i to distinguish different constants that are fixed throughout. Let $\{e_1, \dots, e_d\}$ be the natural basis of $\mathbb{Z}^d$.

2. Proof of Theorem 5. We first consider the Radon–Nikodym derivative of the measure P_{ω^λ} with respect to P_ω. For $s > 0$, put

$$\begin{aligned} G(s,\lambda) &= G(s,\lambda;\zeta, X.) = \log \prod_{j=0}^{\lceil s\rceil -1}\left[1+\lambda\frac{\bar{\xi}_j(o, X_{j+1}-X_j)}{\bar{\omega}_j(o, X_{j+1}-X_j)}\right] \\ &=: \log \prod_{j=0}^{\lceil s\rceil -1}[1+\lambda a(\bar{\zeta}_j, \Delta X_j)], \end{aligned}$$

where $\Delta X_i := X_{i+1} - X_i$ and

$$a(\zeta, e) := \frac{\xi(o,e)}{\omega(o,e)}.$$

Then, for any measurable function F on $C([0,s],\mathbb{R}^d)$,

$$E_{\omega^\lambda} F(X_r : 0 \le r \le s) = E_\omega\big[F(X_s : 0 \le r \le s)e^{G(s,\lambda)}\big].$$

In particular,

$$E_\omega e^{G(s,\lambda)} = E_{\omega^\lambda}[1] = 1 \tag{4}$$

for any $\lambda \in (0,1)$ and $s > 0$. Moreover, by Taylor's expansion,

$$\begin{aligned} G(s,\lambda) &= \sum_{j=0}^{\lceil s\rceil -1} \log\big(1+\lambda a(\bar{\zeta}_j, \Delta X_j)\big) \\ &= \sum_{j=0}^{\lceil s\rceil -1}\left[\lambda a(\bar{\zeta}_j,\Delta X_j) - \frac{\lambda^2 a(\bar{\zeta}_j,\Delta X_j)^2}{2}\right] + \lambda^3\lceil s\rceil H \qquad (5)\\ &= \lambda \sum_{j=0}^{\lceil s\rceil -1} a(\bar{\zeta}_j,\Delta X_j) - \frac{\lambda^2}{2}\sum_{j=0}^{\lceil s\rceil -1} a(\bar{\zeta}_j,\Delta X_j)^2 + \lambda^3\lceil s\rceil H, \end{aligned}$$

where the random variable $H = H(\lambda, \zeta, X.)$ satisfies $0 \le H \le (1+\kappa^{-1})/3$. Setting

$$h(\zeta) = \sum_{e:|e|=1} \xi(o,e)^2/\omega(o,e),$$

we have

$$\left(\sum_{j=0}^{n}\left[a(\bar{\zeta}_j, \Delta X_j)^2 - h(\bar{\zeta}_j)\right]\right)_{n\ge 0}$$

is a P_ω-martingale with bounded increments. Thus, P_ω-almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n}\left[a(\bar{\zeta}_j, \Delta X_j)^2 - h(\bar{\zeta}_j)\right] = 0.$$

Further, recall that $\mathcal{Q}$ is the ergodic invariant measure for $(\bar{\zeta}_n)_{n\ge 0}$ (under P_ω) and $\mathcal{Q} \approx \mathcal{P}$. Hence, by the ergodic theorem, $\mathcal{P} \otimes P_\omega$-almost surely,

$$\lim_{\lambda\to 0} \lambda^2 \sum_{j=0}^{\lceil t/\lambda^2\rceil - 1} a(\bar{\zeta}_j, \Delta X_j)^2 = \lim_{\lambda\to 0} \lambda^2 \sum_{j=0}^{\lceil t/\lambda^2\rceil - 1} h(\bar{\zeta}_{j-1}) = t E_{\mathcal{Q}} h \tag{6}$$

and

$$\lim_{\lambda\to 0} \frac{\lambda^2}{t} \sum_{i=0}^{\lceil t/\lambda^2\rceil} f(\bar{\zeta}_i) = E_{\mathcal{Q}} f. \tag{7}$$

Moreover, observing that $J_n := \sum_{j=0}^{n} a(\bar{\zeta}_j, \Delta X_j)$ is a P_ω-martingale, by (6) and [8], Theorem 7.7.2, we get an invariance principle:

> For $\mathcal{P}$-almost every ζ, the process $(\lambda J_{s/\lambda^2})_{s\ge 0}$ converges weakly (under P_ω) to a Brownian motion $(N_s)_{s\ge 0}$ with diffusion constant $E_{\mathcal{Q}} h$.

Hence, by (5), (6), (7) and the invariance principle, for $\mathcal{P}$-almost all ζ,

$$\frac{\lambda^2}{t} \sum_{i=0}^{\lceil t/\lambda^2\rceil} f(\bar{\zeta}_i) e^{G(t/\lambda^2, \lambda)} \tag{8}$$

converges weakly to

$$(E_{\mathcal{Q}} f) \exp(N_t - t E_{\mathcal{Q}} h/2).$$

Next, we will prove that for $\mathcal{P}$-almost every ζ, this convergence is also in $L^1(P_\omega)$. It suffices to show that the class $(e^{G(t/\lambda^2, \lambda)})_{\lambda\in(0,1)}$ is uniformly integrable under P_ω, $\mathcal{P}$-a.s. Indeed, for any $\gamma > 1$, it follows from (5) and the estimate on H that

$$\begin{aligned}\gamma G(t/\lambda^2,\lambda) &\le G(t/\lambda^2,\gamma\lambda)\\ &\qquad + \frac{(\gamma^2-\gamma)\lambda^2}{2}\sum_{j=0}^{\lceil t/\lambda^2\rceil-1} a(\bar{\zeta}_j,\Delta X_j) + C(1+\gamma^3)\lambda t\\ &< G(t/\lambda^2,\gamma\lambda) + C\gamma^3(t+1).\end{aligned}$$

Hence, for $\gamma > 1$ and all $\lambda \in (0,1)$,

$$E_\omega \exp(\gamma G(t/\lambda^2,\lambda)) \le e^{C\gamma^3(t+1)} E_\omega \exp(G(t/\lambda^2,\gamma\lambda)) \overset{\text{by (4)}}{=} e^{C\gamma^3(t+1)}, \tag{9}$$

which implies the uniform integrability of $(e^{G(t/\lambda^2,\lambda)})_{\lambda\in(0,1)}$. So the E_ω-expectation of (8) also converges to the expectation of its weak limit (for $\mathcal{P}$-almost every ζ) and

$$\lim_{\lambda\to 0} E_{\omega^\lambda}\left[\frac{\lambda^2}{t}\sum_{i=0}^{\lceil t/\lambda^2\rceil} f(\bar{\zeta}_i)\right] = (E_{\mathcal{Q}} f) E[\exp(N_t - t E_{\mathcal{Q}} h/2)].$$

The theorem follows by noting that $t E_{\mathcal{Q}} h = E N_t^2$ and that

$$E[\exp(N_t - E N_t^2/2)] = 1.$$

3. Kalikow's auxiliary random walks. In this section, we will recall Kalikow's auxiliary random walks and use it to obtain some estimates that will be useful later.

For any connected strict subset U of $\mathbb{Z}^d$, let

$$\begin{aligned}\partial U &= \{x \in \mathbb{Z}^d \setminus U : \exists y \in U, |y-x| = 1\},\\ T_U &= \inf\{n \ge 0 : X_n \in \partial U\}.\end{aligned}$$

Define on $U \cup \partial U$ a Markov chain with transition probability

$$\hat{P}_U(x,x+e) = \begin{cases} \dfrac{E_{\mathcal{P}} E_\omega[\sum_{n=0}^{T_U} 1_{X_n=x}\omega(x,e)]}{E_{\mathcal{P}} E_\omega[\sum_{n=0}^{T_U} 1_{X_n=x}]}, & x \in U, |e| = 1,\\ 1, & x \in \partial U, e = o,\end{cases} \tag{10}$$

and set

$$\hat{d}_U(x) := \sum_e e \hat{P}_U(x,x+e).$$

We say that the *Kalikow's condition relative to $\ell \in S^{d-1}$* holds if there exists $\delta > 0$ such that

$$\inf_{U,x\in U} \hat{d}_U(x)\cdot \ell \ge \delta. \tag{11}$$

The interest of this Markov chain lies in the fact that $\hat{P}_U$ and $\mathbb{P}$ have the same exit distribution from U ([13], Proposition 1):

$$\text{if } \hat{P}_U(T_U < \infty) = 1, \qquad \text{then } \hat{P}_U(X_{T_U} \in \cdot) = \mathbb{P}(X_{T_U} \in \cdot).$$

THEOREM 7 ([23], Theorem 2.3). *If* (11) *holds, then there exists a deterministic* $v \in \mathbb{R}^d$ *such that*

$$\lim_{n\to\infty} X_n/n = v, \qquad \mathbb{P}\text{-}a.s.$$

It is also shown in [13], (11), that (11) has the following sufficient condition:

$$\inf_{f\in\mathcal{F}} E_{\mathcal{P}}\biggl[\frac{d(\omega)\cdot\ell}{\sum_e \omega(o,e)f(e)}\biggr]\Big/E_{\mathcal{P}}\biggl[\frac{1}{\sum_e \omega(o,e)f(e)}\biggr] \ge \delta, \tag{12}$$

where $\mathcal{F}$ is the same as in (K).

PROPOSITION 8. *Assume* (i). *Then for some* $\lambda_0 > 0$ *and all* $\lambda \in [0, \lambda_0)$, *there is a deterministic constant* $v_\lambda \in \mathbb{R}^d$ *such that*

$$\lim_{t\to\infty} \frac{X_t}{t} = v_\lambda, \qquad \mathbb{P}_\lambda\text{-}almost\ surely.$$

PROOF. By (K), there exist $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ and

$$\inf_{f\in\mathcal{F}} E_{\mathcal{P}}\biggl[\frac{d(\xi)\cdot\ell}{\sum_{e:|e|=1}\omega^\lambda(o,e)f(e)}\biggr]\Big/E_{\mathcal{P}}\biggl[\frac{1}{\sum_{e:|e|=1}\omega^\lambda(o,e)f(e)}\biggr] > 0.$$

Noting that $d(\omega) = 0$, there is $\rho > 0$ such that the law of ω^λ satisfies (12), with δ replaced by $\lambda\rho$. This implies

$$\inf_{U, x\in U} \hat{d}_U(x)\cdot\ell \ge \lambda\rho. \tag{13}$$

The proposition follows. $\square$

REMARK 9. Although (K) looks complicated, it includes some simple cases:

(a) (K) holds when

$$E_{\mathcal{P}}\bigl[\bigl(d(\xi)\cdot\ell\bigr)_+\bigr] > \frac{1}{\varepsilon}E_{\mathcal{P}}\bigl[\bigl(d(\xi)\cdot\ell\bigr)_-\bigr].$$

For instance, (K) is satisfied when the perturbation is "either neutral or pointing to the right." See [23], Proposition 2.4.

(b) When ω and ξ are independent, (K) is equivalent to $E_{\mathcal{P}}[d(\xi)] \neq o$.

3.1. *Auxiliary estimates*. In this subsection, we consider perturbed RWRE that satisfies (i). Making use of Kalikow's random walks, we obtain some auxiliary estimates that will be useful in getting the regeneration moment bounds in Section 4.

From now on, we assume that (i) holds with

$$\ell = e_1.$$

(The same arguments work also for general $\ell \in S^{d-1}$, but with cumbersome notations.) Recall that (i) implies (13):

$$\inf_{U, x \in U} \hat{d}_U(x) \cdot e_1 \geq \lambda\rho.$$

Let

$$\lambda_1 := 0.5/\lceil (2\lambda)^{-1} \rceil$$

so that $0.5/\lambda_1$ is an integer and

$$\frac{1}{2\lambda} \leq \frac{1}{2\lambda_1} < \frac{1}{2\lambda} + 1.$$

For any $k \in \frac{1}{2}\mathbb{Z}$, $x \in \mathbb{Z}^d$, set

$$\mathcal{H}_k^x = \mathcal{H}_n^x(\lambda, \ell) := \{y \in \mathbb{Z}^d : (y - x) \cdot e_1 = k/\lambda_1\}, \tag{14}$$

$$T_k := \inf\{t \geq 0 : (X_t - X_0) \cdot e_1 = k/\lambda_1\}. \tag{15}$$

For $n \in \mathbb{N}$, we call $\mathcal{H}_n^x$ *the nth level* (with respect to x). Since the random walk is transient in the e_1 direction, T_k's are finite $\mathbb{P}_\lambda$-almost surely.

PROPOSITION 10. *Let (X'_n) be a simple random walk on $\mathbb{Z}$ with*

$$\frac{P(X'_{i+1} = x + 1 | X'_i = x)}{P(X'_{i+1} = x - 1 | X'_i = x)} = q \neq 1 \qquad \forall x \in \mathbb{Z}.$$

Then for any $i, j \in \mathbb{Z}^+$ and $-j \leq 0 \leq i$,

$$P(X' \text{ visits } -j \text{ before visiting } i | X'_0 = 0) = \frac{q^i - 1}{q^{i+j} - 1}.$$

In particular, when $q < 1$,

$$P(X' \text{ never visits } -j | X'_0 = 0) = 1 - q^{-j}.$$

The proof is omitted.

PROPOSITION 11. *Assume* (i). *There exists $\lambda_0 \in (0, 1)$ such that for $\lambda \in (0, \lambda_0)$ and any $n, m \in \mathbb{N}/2$:*

(a) $\frac{e^{5m/\kappa} - 1}{e^{5(m+n)/\kappa} - 1} \leq \mathbb{P}_\lambda(T_{-n} < T_m) \leq \frac{e^{m\rho/2} - 1}{e^{(n+m)\rho/2} - 1}$;

(b) $\frac{e^{5m/\kappa} - 1}{e^{5(m+n)/\kappa} - 1} \leq P_{\omega^\lambda}(T_{-n} < T_m) \leq \frac{1 - e^{-5m/\kappa}}{1 - e^{-5(m+n)/\kappa}}$, *$\mathcal{P}$-almost surely.*

PROOF. (a) Let $U=\{z\in\mathbb{Z}^d: n/\lambda_1\le z\cdot e_1\le m/\lambda_1\}$. With abuse of notation, we let $\hat{X}$ be the Markov chain defined at (10), with ω replaced by ω^λ (because we are interested in the perturbed environment). Since [by (13), (10) and $\lambda<\kappa/2$]

$$1+\rho\lambda\le\frac{\hat{P}_U(x,x+e_1)}{\hat{P}_U(x,x-e_1)}\le 1+\frac{4\lambda}{\kappa},$$

we can couple two Markov chains X', X'' on $\mathbb{Z}$ to $\hat{X}$ such that for all $i\in\mathbb{N}$, $x\in\mathbb{Z}$,

$$\frac{P(X'_{i+1}=x+1|X'_i=x)}{P(X'_{i+1}=x-1|X'_i=x)}=1+\rho\lambda,$$
$$\frac{P(X''_{i+1}=x+1|X''_i=x)}{P(X''_{i+1}=x-1|X''_i=x)}=1+\frac{4\lambda}{\kappa},$$

and

$$X'_i\le\hat{X}_i\cdot e_1\le X''_i\qquad\forall i\in\mathbb{N}.$$

Hence, by Proposition 10, we obtain

$$\frac{(1+4\lambda/\kappa)^{m/\lambda_1}-1}{(1+4\lambda/\kappa)^{(n+m)/\lambda_1}-1}\le\hat{P}_U(T_{-n}<T_m)\le\frac{(1+\rho\lambda)^{m/\lambda_1}-1}{(1+\rho\lambda)^{(n+m)/\lambda_1}-1}.$$

Taking λ small enough, inequality (a) is proved.

(b) Observe that for $\lambda\in(0,\kappa/2)$, $\mathcal{P}$-almost surely,

$$1-\frac{4\lambda}{\kappa}\le\frac{\omega^\lambda(x,e_1)}{\omega^\lambda(x,-e_1)}\le 1+\frac{4\lambda}{\kappa}.$$

Inequality (b) then follows from the same argument as in the proof of (a). □

THEOREM 12. *Assume that ω is balanced. Let*

$$\tilde{T}_n:=T_n\wedge T_{-n}.$$

There exists a constant $s>0$ such that for any uniformly elliptic balanced environment ω and all $\lambda\in(0,\kappa/2)$, $n\in\mathbb{N}$,

$$E_{\omega^\lambda}\big[e^{s\lambda^2\tilde{T}_n/n}\big]<C.$$

The proof, which uses coupling, is given in the Appendix.

PROPOSITION 13. *Assume that ω is balanced. There exists a constant C_0 such that for $\mathcal{P}$-almost all (ω,ξ) and $\lambda\in(0,\kappa/2)$,*

$$P_{\omega^\lambda}\big(|X_{T_{0.5}}|<C_0/\lambda\big)>1/C_0.$$

PROOF. Let $Y_n := X_n - \lambda \sum_{i=0}^{n-1} d(\theta^{X_i}\xi)$. Then (Y_n) is a P_{ω^λ}-martingale. Recall the definition of $\tilde{T}_n$ in Theorem 12. For any $K > 0$ and $\tilde{K} := K/(4d)$,

$$
\begin{aligned}
&P_{\omega^\lambda}(|X_{T_{0.5}}| < K/\lambda) \\
&\quad \geq 1 - P_{\omega^\lambda}\Big(\max_{t \leq \tilde{K}/\lambda^2} |X_t| \geq K/\lambda\Big) - P_{\omega^\lambda}(\tilde{T}_{0.5} > \tilde{K}/\lambda^2) - P_{\omega^\lambda}(T_{0.5} > T_{-0.5}).
\end{aligned}
$$

By Proposition 11(b) and Theorem 12, it suffices to show that

$$P_{\omega^\lambda}\Big(\max_{t \leq \tilde{K}/\lambda^2} |X_t| \geq K/\lambda\Big)$$

can be sufficiently small if K is large. Indeed,

$$
\begin{aligned}
P_{\omega^\lambda}\Big(\max_{t \leq \tilde{K}/\lambda^2} |X_t| \geq K/\lambda\Big) &\leq P_{\omega^\lambda}\Big(\max_{t \leq \tilde{K}/\lambda^2} |Y_t| \geq CK/\lambda\Big) \\
&\leq Ce^{-(c(K/\lambda)^2)/(K/\lambda^2)} = Ce^{-cK},
\end{aligned}
$$

where we used Azuma–Hoeffding inequality in the last inequality. □

LEMMA 14. *Assume* (i); *then* $\mathbb{P}_\lambda(|X_{T_n} - \frac{n}{\lambda_1}e_1| \geq \frac{n}{\lambda_1}) \leq Ce^{-cn}$.

PROOF. Observe that

$$\mathbb{P}_\lambda\left(\left|X_{T_n} - \frac{n}{\lambda_1}e_1\right| \geq \frac{n}{\lambda_1}\right) = \hat{P}_{U_n}\left(\left|\hat{X}_{T_n} - \frac{n}{\lambda_1}e_1\right| \geq \frac{n}{\lambda_1}\right),$$

where $U_n = \{x : x \cdot e_1 \leq n/\lambda_1\}$. For $j \geq 0$, let

$$\hat{Y}_j := \hat{X}_j - \sum_{i=0}^{j-1} \hat{d}_{U_n}(X_i).$$

Then, for $k_n := \frac{2n}{\rho\lambda^2}$,

$$
\begin{aligned}
&\hat{P}_{U_n}\left(\left|\hat{X}_{T_n} - \frac{n}{\lambda_1}e_1\right| \geq \frac{5n}{\rho\lambda_1}\right) \\
&\quad = \hat{P}_{U_n}(T_n \geq k_n) + \hat{P}_{U_n}\left(\max_{0 \leq i \leq k_n} |\hat{X}_i| \geq \frac{5n}{\rho\lambda_1}\right) \\
&\quad \leq \hat{P}_{U_n}\left(\hat{Y}_{k_n} \cdot e_1 + \sum_{i=0}^{k_n-1} \hat{d}_{U_n}(X_i) \cdot e_1 \leq \frac{n}{\lambda_1}\right) \qquad (16) \\
&\qquad + \hat{P}_{U_n}\left(\max_{0 \leq i \leq k_n} |\hat{Y}_i| \geq \frac{5n}{\rho\lambda_1} - \left|\sum_{i=0}^{k_n-1} \hat{d}_{U_n}(X_i)\right|\right) \\
&\quad \leq \hat{P}_{U_n}\left(\hat{Y}_{k_n} \cdot e_1 \leq -\frac{n}{\lambda_1}\right) + \hat{P}_{U_n}\left(\max_{0 \leq i \leq k_n} |\hat{Y}_i| \geq \frac{n}{\rho\lambda_1}\right),
\end{aligned}
$$

where in the last inequality we used the fact that

$$\rho\lambda \le \hat{d}_{U_n} \le 2\lambda.$$

The lemma follows by observing that $(\hat{Y}_j)_{j\ge 0}$ is a martingale with bounded increments and by applying the Azuma–Hoeffding inequality to (16). $\square$

4. Regenerations. In this section, we will construct a 1-dependent regeneration structure for perturbed RWRE that satisfies (i). Recall that we assume (without loss of generality) that $\ell = e_1$.

4.1. *Harnack inequality and its application.* Let a be a nonnegative function on $\mathbb{Z}^d \times \mathbb{Z}^d$ such that for any x, $a(x, y) > 0$ only if x and y are neighbors, that is, $|x - y| = 1$, denoted $x \sim y$. We also assume that

$$\sum_y a(x, y) = 1 \qquad \forall x \in \mathbb{Z}^d.$$

Define the linear operator L_a acting on the set of functions on $\mathbb{Z}^d$ by

$$L_a f(x) = \sum_y a(x, y)\big(f(y) - f(x)\big).$$

Set

$$b(x) = \sum_y a(x, y)(y - x) \quad \text{and} \quad b_0 = \sup |b|.$$

We assume that L_a is uniformly elliptic with constant $\kappa \in (0, \frac{1}{2d}]$. That is,

$$a(x, y) \ge \kappa \qquad \text{for any } x, y \text{ such that } x \sim y.$$

For $r > 0$, $x \in \mathbb{R}^d$, let $B_r(x) = \{z \in \mathbb{Z}^d : |z - x| < r\}$. We also write $B_r(o)$ as B_r.

The following Harnack estimate is due to Kuo and Trudinger [15], Theorem 3.1. See also the Appendix of [11] for a detailed proof.

THEOREM 15 (Harnack inequality). *Let u be a nonnegative function on B_R, $R > 1$. If*

$$L_a u = 0$$

in B_R, then for any $\sigma \in (0, 1)$ with $R(1 - \sigma) > 1$, we have

$$\max_{B_{\sigma R}} u \le C \min_{B_{\sigma R}} u,$$

where C is a positive constant depending on d, κ, σ and $b_0 R$.

With the Harnack inequality, we have the following.

LEMMA 16. *Assume* (i). *There exists a constant* $c_1 \in (0,1]$ *such that for* $\lambda \in (0,1)$, $x \in \mathbb{Z}^d$ *and* $\mathcal{P}$*-almost every* (ω,ξ),

$$P^x_{\omega^\lambda}(X_{T_1} = \cdot) \ge c_1 P^{x+0.5e_1/\lambda_1}_{\omega^\lambda}(X_{T_{0.5}} = \cdot | T_{0.5} < T_{-0.5}). \tag{17}$$

PROOF. For any $x \in \mathbb{Z}^d$ and $k \in \frac{1}{2}\mathbb{Z}$, recall the definition of $\mathcal{H}^x_k$ in (14). Fix $w \in \mathcal{H}^x_1$. Then the function

$$f(z) := P^z_{\omega^\lambda}\bigl(X_\cdot \text{ visits } \mathcal{H}^x_1 \text{ for the first time at } w\bigr)$$

satisfies

$$L_{\omega^\lambda} f(z) = 0$$

for all $z \in \{y : (y-x)\cdot e_1 < 1/\lambda_1\}$. By Theorem 15 (in this case $a = \omega^\lambda$, $R = 0.5/\lambda_1$ and $b_0 \le \lambda$), there exists a constant C_2 such that for any $y, z \in \mathcal{H}^x_{0.5}$ with $|z-y| \le 0.5/\lambda_1$,

$$f(z) \ge C_2 f(y). \tag{18}$$

Hence, for any $z \in \mathcal{H}^x_{0.5}$ such that $|z - (x + 0.5e_1/\lambda_1)| < C_0/\lambda_1$ (recall that C_0 is the constant in Proposition 13), we have

$$f(z) \ge C_2^{2C_0} f(x + 0.5e_1/\lambda_1). \tag{19}$$

Therefore,

$$\begin{aligned} P^x_{\omega^\lambda}(X_{T_1} = w) &\ge \sum_{|y-x|<C_0/\lambda} P^x_{\omega^\lambda}(X_{T_{0.5}} = y) P^y_{\omega^\lambda}(X_{T_{0.5}} = w) \\ &\overset{(19)}{\ge} C P^x_{\omega^\lambda}\bigl(|X_{T_{0.5}} - x| < C_0/\lambda_1\bigr) P^{x+0.5e_1/\lambda_1}_{\omega^\lambda}(X_{T_{0.5}} = w) \\ &\ge c_1 P^{x+0.5e_1/\lambda_1}_{\omega^\lambda}(X_{T_{0.5}} = w | T_{0.5} < T_{-0.5}), \end{aligned}$$

where in the last inequality we used Proposition 13 and [Proposition 11(b)]

$$P^{x+0.5e_1/\lambda_1}_{\omega^\lambda}(T_{0.5} < T_{-0.5}) > C. \qquad \square$$

4.2. *Construction of the regeneration times*. In this subsection, we will construct regeneration times that allow the path to backtrack at most distance $1/\lambda$ in direction e_1 after each regeneration. The main difficulty is to decouple the parts before and after a regeneration in such a way that they are "almost independent." Our main observation is that (by Lemma 16) the hitting probability $P^x_{\omega^\lambda}(X_{T_1} = \cdot)$ to the next level dominates [in the sense of (17)] a "good" probability measure

$$\mu^x_{\omega^\lambda,1}(\cdot) := P^{x+0.5e_1/\lambda_1}_{\omega^\lambda}(X_{T_{0.5}} = \cdot | T_{0.5} < T_{-0.5}), \tag{20}$$

which is independent of environment to the left of level $\mathcal{H}^x_0$. Hence, the hitting probability can be decomposed as

$$P^x_{\omega^\lambda}(X_{T_1} = \cdot) = \beta \mu^x_{\omega^\lambda,1}(\cdot) + (1-\beta)\mu^x_{\omega^\lambda,0}(\cdot),$$

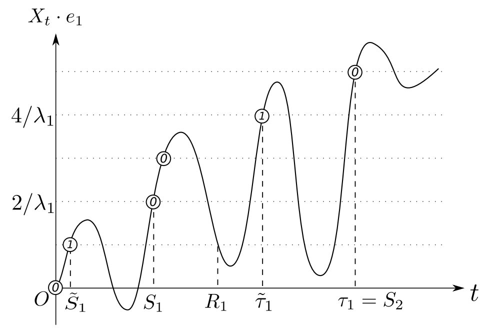

FIG. 1. *In this picture*, $K = 2$, $X_{\tau_1} = 5/\lambda_1$, $M_1 = 4/\lambda_1$.

where (recall that c_1 is the constant in Lemma 16)

$$\beta := c_1/2 \quad \text{and} \quad \mu^x_{\omega^\lambda,0}(\cdot) := \big[P^x_{\omega^\lambda}(X_{T_1} = \cdot) - \beta\mu^x_{\omega^\lambda,1}(\cdot)\big]/(1-\beta).$$

Note that by (17), both $\mu^x_{\omega^\lambda,1}$ and $\mu^x_{\omega^\lambda,0}$ are probability measures on $\mathcal{H}^x_1$. This suggests us to use a coin-tossing trick to decouple the paths and define the regenerations, which we explain as follows.

For any $\mathcal{O} \in \sigma(X_1, X_2, \dots, X_{T_1})$, $x \in \mathbb{Z}^d$ and $i \in \{0, 1\}$, put

$$\nu^x_{\omega^\lambda,i}(\mathcal{O}) := \sum_y \big[i\mu^x_{\omega^\lambda,1}(y) + (1-i)\mu^x_{\omega^\lambda,0}(y)\big] P^x_{\omega^\lambda}(\mathcal{O} | X_{T_1} = y). \tag{21}$$

Let $(\varepsilon_i)_{i=1}^\infty \in \{0, 1\}^{\mathbb{N}}$ be i.i.d. Bernoulli random variables with law Q_β:

$$Q_\beta(\varepsilon_i = 1) = \beta \quad \text{and} \quad Q_\beta(\varepsilon_i = 0) = 1 - \beta.$$

Intuitively, whenever the walker visits a new level $\mathcal{H}_i$, $i \ge 0$, we make him flip a coin ε_i. If $\varepsilon_i = 0$ (or 1), he then walks following the law $\nu_{\omega^\lambda,0}$ (or $\nu_{\omega^\lambda,1}$) until he reaches the $(i+1)$th level. The regeneration time τ_1 is defined to be the first time of visiting a new level $\mathcal{H}_k$ such that the outcome ε_{k-1} of the previous coin-tossing is "1" and the path will never backtrack to level $\mathcal{H}_{k-1}$ in the future. See Figure 1.

We now give the formal definition of the regeneration times.

We sample the sequence $\varepsilon := (\varepsilon_i)_{i=1}^\infty$ according to the product measure Q_β and fix it. Then we define a new law $P_{\omega^\lambda,\varepsilon}$ on the paths, by the following steps (see Figure 2):

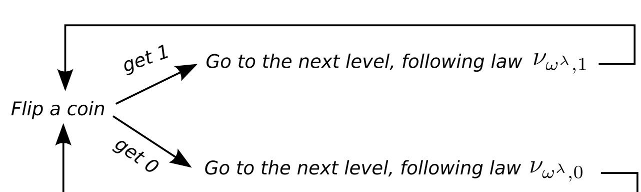

FIG. 2. *The law* $\bar{P}_{\omega^\lambda,\varepsilon}$ *for the walks.*

- *Step* 1. For $x \in \mathbb{Z}^d$, set

$$P^x_{\omega^\lambda,\varepsilon}(X_0 = x) = 1.$$

- *Step* 2. Suppose the $P^x_{\omega^\lambda,\varepsilon}$-law for paths of length $\le n$ is defined. For any path $(x_i)_{i=0}^{n+1}$ with $x_0 = x$, define

$$\begin{aligned} &P^x_{\omega^\lambda,\varepsilon}(X_{n+1} = x_{n+1}, \dots, X_0 = x_0) \\ &\qquad := P^x_{\omega,\varepsilon}(X_I = x_I, \dots, X_0 = x_0)\nu^{x_I}_{\omega^\lambda,\varepsilon_J}(X_{n+1-I} = x_{n+1}, \dots, X_1 = x_{I+1}), \end{aligned}$$

  where

$$J = \max\{j \ge 0 : \mathcal{H}^{x_0}_j \cap \{x_i, 0 \le i \le n\} \ne \varnothing\}$$

  is the highest level visited by $(x_i)_{i=0}^n$ and

$$I = \min\{0 \le i \le n : x_i \in \mathcal{H}^{x_0}_J\}$$

  is the hitting time to the Jth level.
- *Step* 3. By induction, the law $P^x_{\omega^\lambda,\varepsilon}$ is well defined for paths of all lengths.

Note that a path sampled by $P^x_{\omega^\lambda,\varepsilon}$ is not a Markov chain, but the law of $X_\cdot$ under

$$\bar{P}^x_{\omega^\lambda} := Q_\beta \otimes P^x_{\omega^\lambda,\varepsilon}$$

coincides with $P^x_{\omega^\lambda}$. That is,

$$\bar{P}^x_{\omega^\lambda}(X_\cdot \in \cdot) = P^x_{\omega^\lambda}(X_\cdot \in \cdot). \tag{22}$$

We denote by $\bar{\mathbb{P}}_\lambda := \mathcal{P} \otimes \bar{P}_{\omega^\lambda}$ the law of the triple $(\omega, \varepsilon, X_\cdot)$. Expectations with respect to $\bar{P}^x_{\omega^\lambda}$ and $\bar{\mathbb{P}}_\lambda$ are denoted by $\bar{E}^x_{\omega^\lambda}$ and $\bar{\mathbb{E}}_\lambda$, respectively.

Next, for a path $(X_n)_{n\ge 0}$ sampled according to $P^o_{\omega^\lambda,\varepsilon}$, we will define the regeneration times. See Figure 3 for an illustration.

To be specific, put $S_0 = 0$, $M_0 = 0$, and define inductively

$$\begin{aligned} S_{k+1} &= \inf\{T_{n+1} : n/\lambda_1 \ge M_k \text{ and } \varepsilon_n = 1\}, \\ R_{k+1} &= S_{k+1} + T_{-1} \circ \theta_{S_{k+1}}, \\ M_{k+1} &= X_{S_{k+1}} \cdot e_1 + N \circ \theta_{S_{k+1}}/\lambda_1, \qquad k \ge 0. \end{aligned}$$

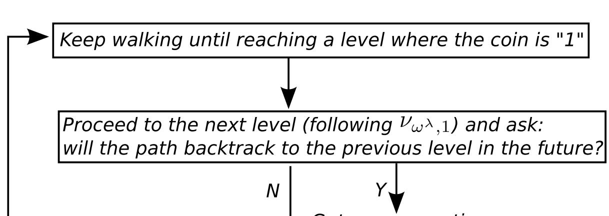

FIG. 3. *The definition of a regeneration time.*

Here, θ_n denotes the time shift of the path, that is, $\theta_n X. = (X_{n+i})_{i=0}^{\infty}$, and

$$N := \inf\{n : n/\lambda_1 > (X_i - X_0) \cdot e_1 \text{ for all } i \le T_{-1}\}. \tag{23}$$

Set

$$K := \inf\{k \ge 1 : S_k < \infty, R_k = \infty\},$$

$$\tau_1 := S_K \quad \text{and} \quad \tau_{k+1} = \tau_k + \tau_1 \circ \theta_{\tau_k}.$$

We call $(\tau_k)_{k\ge1}$ *regeneration times*.

4.3. *The renewal property of the regenerations.* The regeneration times possess good renewal properties:

1. Set $\tau_0 = 0$. For $k \ge 0$, define

$$\tilde{S}_{k+1} := \inf\{T_n : n/\lambda \ge M_k \text{ and } \varepsilon_n = 1\},$$

$$\tilde{\tau}_1 := \tilde{S}_K \quad \text{and set} \quad \tilde{\tau}_{k+1} := \tau_k + \tilde{\tau}_1 \circ \theta_{\tau_k}.$$

Namely, $\tilde{\tau}_k$ is the hitting time to the previous level of X_{τ_k}. Conditioning on $X_{\tilde{\tau}_k} = x$, the law of X_{τ_k} is $\mu^x_{\omega^\lambda,1}$, which is independent (under the environment measure $\mathcal{P}$) of $\sigma(\zeta_y : y \cdot e_1 \le x \cdot e_1)$. Moreover, after time τ_k, the path will never visit $\{y : y \cdot e_1 \le x \cdot e_1\}$. Therefore, $\tau_{k+1} - \tau_k$ is independent of what happened before τ_{k-1} and the inter-regeneration times form a 1-dependent sequence.

2. Since $(X_{\tilde{\tau}_{k+1}} - X_{\tau_k})_{k\ge1}$ are i.i.d. and $(X_{\tau_{k+1}} - X_{\tilde{\tau}_{k+1}}) \cdot e_1 = 1/\lambda_1$, the inter-regeneration distances $((X_{\tau_{k+1}} - X_{\tau_k}) \cdot e_1)_{k\ge1}$ are i.i.d.

3. From the construction, we see that a regeneration occurs after roughly a geometric number of levels. Thus, we expect $(X_{\tau_{k+1}} - X_{\tau_k}) \cdot e_1 \sim c/\lambda$ and (by Theorem 12) $\tau_{k+1} - \tau_k \sim c/\lambda^2$.

The above properties will be verified in Lemma 17, Proposition 18 and Corollary 20.

We introduce the σ-field

$$\mathcal{G}_k := \sigma\big(\tilde{\tau}_k, (X_i)_{i\le\tilde{\tau}_k}, (\zeta_y)_{y\cdot e_1 \le X_{\tilde{\tau}_k}\cdot e_1}\big)$$

and set

$$p_\lambda := E_{\mathcal{P}}\Big[\sum_y \mu_{\omega^\lambda,1}(y) P^y_{\omega^\lambda}(T_{-1} = \infty)\Big]. \tag{24}$$

LEMMA 17. *For any appropriate measurable sets B_1, B_2 and any event*

$$B := \{(X_i)_{i\ge0} \in B_1, (\zeta_y)_{y\cdot e_1 > -1/\lambda_1} \in B_2\},$$

we have for $k \ge 1$,

$$\bar{\mathbb{P}}_\lambda(B \circ \bar{\theta}_{\tau_k} | \mathcal{G}_k) = E_{\mathcal{P}}\Big[\sum_y \mu_{\omega^\lambda,1}(y) \bar{P}^y_{\omega^\lambda}\big(B \cap \{T_{-1} = \infty\}\big)\Big] \Big/ p_\lambda,$$

where $\bar{\theta}_n$ is the time-shift defined by

$$B \circ \bar{\theta}_n = \{(X_i)_{i \geq n} \in B_1, (\zeta_y)_{(y - X_n)\cdot e_1 > -1/\lambda_1} \in B_2\}.$$

PROOF. First, we consider the case $k = 1$. Let ϑ^n denote the shift of the ε-coins, that is, $\vartheta^n \varepsilon_\cdot = (\varepsilon_i)_{i \geq n}$. For any $A \in \mathcal{G}_1$,

$$\begin{aligned}
&\bar{\mathbb{P}}_\lambda(B \circ \bar{\theta}_{\tau_1} \cap A)\\
&\quad = E_{\mathcal{P}\otimes Q_\beta}\Biggl[\sum_{k\geq 1, x} P_{\omega^\lambda,\varepsilon}\bigl(A \cap \{\tilde{S}_k < \infty, R_k = \infty, X_{\tilde{S}_k} = x\} \cap B \circ \bar{\theta}_{S_k}\bigr)\Biggr]\\
&\quad = E_{\mathcal{P}\otimes Q_\beta}\Biggl[\sum_{k\geq 1, x, y} P_{\omega^\lambda,\varepsilon}\bigl(A \cap \{\tilde{S}_k < \infty, X_{\tilde{S}_k} = x\}\bigr) \nu^x_{\omega^\lambda,1}(X_{T_1} = x + y)\\
&\qquad\qquad \times P^{x+y}_{\omega^\lambda,\vartheta^{k+1}\varepsilon}\bigl(B \cap \{T_{-1} = \infty\}\bigr)\Biggr].
\end{aligned}$$

Note that in the last equality,

$$P_{\omega^\lambda,\varepsilon}\bigl(A \cap \{\tilde{S}_k < \infty, X_{\tilde{S}_k} = x\}\bigr)$$

is $\sigma((\varepsilon_i)_{i\leq k}, (\zeta_z)_{(z-x)\cdot e_1 \leq 0})$-measurable, whereas

$$\nu^x_{\omega^\lambda,1}(X_{T_1} = x + y) P^{x+y}_{\omega^\lambda,\vartheta^{k+1}\varepsilon}\bigl(B \cap \{T_{-1} = \infty\}\bigr)$$

is $\sigma((\varepsilon_i)_{i\geq k+1}, (\zeta_z)_{(z-x)\cdot e_1 > 0})$-measurable for $y \in \mathcal{H}^x_1$. Hence they are independent under $\mathcal{P} \otimes Q_\beta$ and we have

$$\begin{aligned}
&\bar{\mathbb{P}}_\lambda(B \circ \bar{\theta}_{\tau_1} \cap A)\\
&\quad = \sum_{k\geq 1} \bar{\mathbb{P}}_\lambda\bigl(A \cap \{\tilde{S}_k < \infty\}\bigr) E_P\Biggl[\sum_y \nu_{\omega^\lambda,1}(X_{T_1} = y) \bar{P}^y_{\omega^\lambda}\bigl(B \cap \{T_{-1} = \infty\}\bigr)\Biggr].
\end{aligned} \tag{25}$$

Substituting B with the set of all events, we get

$$\bar{\mathbb{P}}_\lambda(A) = \sum_{k\geq 1} \bar{\mathbb{P}}_\lambda\bigl(A \cap \{\tilde{S}_k < \infty\}\bigr) E_{\mathcal{P}}\Biggl[\sum_y \mu_{\omega^\lambda,1}(y) \bar{P}^y_{\omega^\lambda}(T_{-1} = \infty)\Biggr]. \tag{26}$$

Equalities (25) and (26) yield

$$\bar{\mathbb{P}}_\lambda(B \circ \bar{\theta}_{\tau_1} | A) = \frac{E_{\mathcal{P}}[\sum_y \mu_{\omega^\lambda,1}(y) \bar{P}^y_{\omega^\lambda}(B \cap \{T_{-1} = \infty\})]}{E_{\mathcal{P}}[\sum_y \mu_{\omega^\lambda,1}(y) \bar{P}^y_{\omega^\lambda}(T_{-1} = \infty)]}.$$

The lemma is proved for the case $k = 1$. The general case $k > 1$ follows by induction. □

We say that a sequence of random variables $(Y_i)_{i\in\mathbb{N}}$ is *m-dependent* ($m \in \mathbb{N}$) if

$$\sigma(Y_i; 1 \leq i \leq n) \quad \text{and} \quad \sigma(Y_j; j > n + m) \qquad \text{are independent } \forall n \in \mathbb{N}.$$

The law of large numbers and central limit theorem also hold for a stationary m-dependent sequence with finite means and variances, see [5], Theorem 5.2. The following proposition is an immediate consequence of Lemma 17.

PROPOSITION 18. *Under $\bar{\mathbb{P}}_\lambda$, $(X_{\tau_{n+1}} - X_{\tau_n})_{n\ge 1}$ and $(\tau_{n+1} - \tau_n)_{n\ge 1}$ are stationary* 1*-dependent sequences. Furthermore, for all $n \ge 1$, $(X_{\tau_{n+1}} - X_{\tau_n}, \tau_{n+1} - \tau_n)$ has law*

$$\begin{aligned}
&\bar{\mathbb{P}}_\lambda(X_{\tau_{n+1}} - X_{\tau_n} \in \cdot, \tau_{n+1} - \tau_n \in \cdot) \\
&\quad = E_{\mathcal{P}}\Big[\sum_y \mu_{\omega^\lambda,1}(y) \bar{P}^y_{\omega^\lambda}(X_{\tau_1} \in \cdot, \tau_1 \in \cdot, T_{-1} = \infty)\Big] \Big/ p_\lambda.
\end{aligned}$$

PROOF. For $k \ge 0$, let

$$\mathcal{F}_k = \sigma(\tau_k, X_1, \dots, X_{\tau_k}).$$

Then, for $n \ge 1$, $\mathcal{F}_{n-1} \subset \mathcal{G}_n$ and

$$\begin{aligned}
&\bar{\mathbb{P}}_\lambda(X_{\tau_{n+1}} - X_{\tau_n} \in \cdot, \tau_{n+1} - \tau_n \in \cdot | \mathcal{F}_{n-1}) \\
&\quad = \bar{\mathbb{E}}_\lambda\big[\bar{\mathbb{P}}_\lambda(X_{\tau_{n+1}} - X_{\tau_n} \in \cdot, \tau_{n+1} - \tau_n \in \cdot | \mathcal{G}_n) | \mathcal{F}_{n-1}\big].
\end{aligned}$$

By Lemma 17, the proposition is proved. $\square$

4.4. *Moment estimates.* We will show that the typical values of $e_1 \cdot (X_{\tau_{k+1}} - X_{\tau_k})$ and $\tau_{k+1} - \tau_k$, $(k \ge 0)$ are $C/(\beta\lambda)$ and $C/(\beta\lambda^2)$, respectively.

THEOREM 19. *Assume* (i). *There exists a constant $c > 0$ such that*

$$\bar{\mathbb{E}}_\lambda\big[\exp(c\beta\lambda X_{\tau_1})\big] < C$$

for all $\lambda \in (0, \lambda_0)$ and $\beta \in (0, 1)$.

PROOF. Our proof contains several steps.

1. For $0 \le k \le K - 1$, set

$$L_{k+1} = \inf\{n \ge \lambda_1 M_k : \varepsilon_n = 1\} - \lambda_1 M_k + 1.$$

Then L_1 is the number of coins tossed to get the first “1” and

$$X_{S_1} \cdot e_1 = L_1/\lambda_1. \tag{27}$$

Since $(L_i)_{i\ge 1}$ depends only on the coins $(\varepsilon_i)_{i\ge 0}$, it is easily seen that they are i.i.d. geometric with parameter β. Hence, for $i \ge 1$, $s \in (0, 1)$,

$$\bar{E}_{\omega^\lambda}\big[e^{s\beta L_i}\big] = \frac{\beta e^{s\beta}}{1 - (1-\beta)e^{s\beta}} \le \frac{1}{1-s}. \tag{28}$$

Moreover, for $1 \le k \le K - 1$,

$$(X_{S_{k+1}} - X_{S_k}) \cdot e_1 = N \circ \theta_{S_k} + L_{k+1}/\lambda_1.$$

2. We will show that

$$\bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{\tau_1}\cdot e_1)\big] = \sum_{k\ge 1}\bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{S_k}\cdot e_1), S_k<\infty\big]p_\lambda. \tag{29}$$

By the definition of X_{τ_1},

$$\begin{aligned}&\bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{\tau_1}\cdot e_1)\big]\\ &\quad = \sum_{k\ge 1,x,y} E_{\mathcal{P}}\big[\bar{E}_{\omega^\lambda}\big[\exp(s\beta\lambda_1 X_{\tilde{S}_k}\cdot e_1 + s\beta), \tilde{S}_k<\infty, X_{\tilde{S}_k}=x\big]\\ &\qquad\qquad\qquad\qquad \times \mu^x_{\omega^\lambda,1}(x+y)P^{x+y}_{\omega^\lambda}(T_{-1}=\infty)\big].\end{aligned}$$

Since

$$\bar{E}_{\omega^\lambda}\big[\exp(s\beta\lambda_1 X_{\tilde{S}_k}\cdot e_1 + s\beta), \tilde{S}_k<\infty, X_{\tilde{S}_k}=x\big]$$

is $\sigma(\zeta_z : z\cdot e_1 < x\cdot e_1)$-measurable, and $\mu^x_{\omega^\lambda,1}(x+y)P^{x+y}_{\omega^\lambda}(T_{-1}=\infty)$ is $\sigma(\zeta_z : z\cdot e_1 \ge x\cdot e_1)$-measurable, they are independent under $\mathcal{P}$. Therefore,

$$\begin{aligned}&\bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{\tau_1}\cdot e_1)\big]\\ &\quad = \sum_{k\ge 1,x,y} \bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{\tilde{S}_k}\cdot e_1 + s\beta), \tilde{S}_k<\infty, X_{\tilde{S}_k}=x\big]\\ &\qquad\qquad \times E_{\mathcal{P}}\big[\mu^x_{\omega^\lambda,1}(x+y)P^{x+y}_{\omega^\lambda}(T_{-1}=\infty)\big].\end{aligned} \tag{30}$$

Equation (29) follows.

3. Next, we will show that for $k\ge 1$,

$$\begin{aligned}&\bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{S_k}\cdot e_1), S_k<\infty\big]\\ &\quad \le \left(\frac{A(s,\lambda,\beta)}{1-s}\right)^{k-1}\bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{S_1}\cdot e_1), S_1<\infty\big],\end{aligned} \tag{31}$$

where

$$A(s,\lambda,\beta) := E_{\mathcal{P}}\Big[\sum_y \mu_{\omega^\lambda,1}(y)E^y_{\omega^\lambda}\big[e^{s\beta N}, T_{-1}<\infty\big]\Big].$$

By definition,

$$\begin{aligned}&\bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{S_{k+1}}\cdot e_1), S_{k+1}<\infty\big]\\ &\quad = \bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{S_k}\cdot e_1 + s\beta N\circ\theta_{S_k} + s\beta L_{k+1}), S_k<\infty, T_{-1}\circ\theta_{S_k}<\infty\big].\end{aligned}$$

Noting that L_{k+1} is independent of $\sigma\{R_k, X_1,\dots,X_{R_k}\}$, we get

$$\begin{aligned}&\bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{S_{k+1}}\cdot e_1), S_{k+1}<\infty\big]\\ &\quad = \bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{S_k}\cdot e_1 + s\beta N\circ\theta_{S_k}), S_k<\infty, T_{-1}\circ\theta_{S_k}<\infty\big]\\ &\qquad \times \bar{\mathbb{E}}_\lambda\big[e^{s\beta L_{k+1}}\big]\\ &\quad \overset{(28)}{\le} \bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{S_k}\cdot e_1 + s\beta N\circ\theta_{S_k}), S_k<\infty, T_{-1}\circ\theta_{S_k}<\infty\big]/(1-s).\end{aligned} \tag{32}$$

Further, by the same argument as in (30),

$$\begin{aligned}
&\bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{S_k}\cdot e_1 + s\beta N\circ\theta_{S_k}), S_k<\infty, T_{-1}\circ\theta_{S_k}<\infty\big]\\
&\quad=\sum_{x,y}\bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 x\cdot e_1 + s\beta), \tilde{S}_k<\infty, X_{\tilde{S}_k}=x\big]\\
&\qquad\quad\times E_{\mathcal{P}}\big[\mu^x_{\omega^\lambda,1}(x+y)E^{x+y}_{\omega^\lambda}\big[e^{s\beta N}, T_{-1}<\infty\big]\big]\\
&\quad=\bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{S_k}\cdot e_1), S_k<\infty\big]A(s,\lambda,\beta).
\end{aligned}$$

Combining the above equality and (32), ineqaulity (31) follows by induction.

4. By (29) and (31), we have

$$\bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{\tau_1}\cdot e_1)\big]\le p_\lambda\bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{S_1}\cdot e_1), S_1<\infty\big]\sum_{k=0}^\infty\left(\frac{A(s,\lambda,\beta)}{1-s}\right)^k.$$

Since $p_\lambda\le 1$, and [by (27) and (28)]

$$\bar{\mathbb{E}}_\lambda\big[\exp(s\beta\lambda_1 X_{S_1}\cdot e_1), S_1<\infty\big]=\bar{\mathbb{E}}_\lambda\big[e^{s\beta L_1}\big]\le\frac{1}{1-s},$$

to prove Theorem 19, we only need that when $s>0$ is small enough,

$$A(s,\lambda,\beta)<C<1. \tag{33}$$

For any $m\in\mathbb{N}$,

$$\begin{aligned}
A(s,\lambda,\beta)&\le e^{s\beta m}\mathbb{P}_\lambda(T_{-1}<\infty)\\
&\qquad+\sum_{n=m}^\infty e^{s\beta n}E_{\mathcal{P}}\Big[\sum_y\mu_{\omega^\lambda,1}(y)P^y_{\omega^\lambda}(N=n, T_{-1}<\infty)\Big].
\end{aligned}$$

Hence [note that $\mathbb{P}_\lambda(T_{-1}<\infty)\le e^{-\rho/2}$], to prove (33), it suffices to show

$$E_{\mathcal{P}}\Big[\sum_y\mu_{\omega^\lambda,1}(y)P^y_{\omega^\lambda}(N=n, T_{-1}<\infty)\Big]<Ce^{-cn}. \tag{34}$$

5. Recall the definition of N in (23):

$$\begin{aligned}
&E_{\mathcal{P}}\Big[\sum_y\mu_{\omega^\lambda,1}(y)P^y_{\omega^\lambda}(N=n, T_{-1}<\infty)\Big]\\
&\quad\le CE_{\mathcal{P}}\Big[\sum_y P_{\omega^\lambda}(X_{T_1}=y)P^y_{\omega^\lambda}(N=n, T_{-1}<\infty)\Big]\\
&\quad\le CE_{\mathcal{P}}\Big[\sum_{y,z} P_{\omega^\lambda}(X_{T_1}=y)P^y_{\omega^\lambda}(X_{T_{n-1}}=z)P^z_{\omega^\lambda}(T_{-n-1}<T_1)\Big]\\
&\quad=CE_{\mathcal{P}}\Big[\sum_z P_{\omega^\lambda}(X_{T_n}=z)P^z_{\omega^\lambda}(T_{-n-1}<T_1)\Big].
\end{aligned}$$

For $k \geq 0$, let $z_k := (ne_1 + ke_2)/\lambda_1$ and

$$A_k := \left\{z \in \mathcal{H} : \frac{k}{\lambda_1} \leq \left|X_{T_n} - \frac{n}{\lambda_1}e_1\right| < \frac{k+1}{\lambda_1}\right\}.$$

Then by the Harnack inequality, for any $x \in A_k$,

$$P^x_{\omega^\lambda}(T_{-n-1} < T_1) \leq C_2 P^{z_k}_{\omega^\lambda}(T_{-n-1} < T_1),$$

where C_2 is the constant in (18). Hence,

$$\begin{aligned}
&E_{\mathcal{P}}\left[\sum_z P_{\omega^\lambda}(X_{T_n} = z) P^z_{\omega^\lambda}(T_{-n-1} < T_1)\right] \\
&\quad \leq \mathbb{P}_\lambda\left(\left|X_{T_n} - \frac{n}{\lambda_1}e_1\right| \geq \frac{n}{\lambda_1}\right) \\
&\qquad + C_2 \sum_{k=0}^{n-1} E_{\mathcal{P}}\left[\sum_{z \in A_k} P_{\omega^\lambda}(X_{T_n} = z) P^{z_k}_{\omega^\lambda}(T_{-n-1} < T_1)\right] \\
&\quad \overset{\text{Lemma 14}}{\leq} Ce^{-cn} + C_2 E_{\mathcal{P}}\left[\sum_{k=0}^{n-1} P^{z_k}_{\omega^\lambda}(T_{-n-1} < T_1)\right] \\
&\quad = Ce^{-cn} + C_2 n \mathbb{P}_\lambda(T_{-n-1} < T_1) \leq Ce^{-cn}.
\end{aligned}$$

Inequality (34) is proved. $\square$

COROLLARY 20. *Assume* (i). *For all* $n \geq 0$, $\lambda \in (0, \lambda_0)$ *and* $\beta \in (0, 1)$,

$$\bar{\mathbb{E}}_\lambda\left[\exp\left(c_4 \beta \lambda (X_{\tau_{n+1}} - X_{\tau_n}) \cdot e_1\right)\right] < C, \tag{35}$$

$$\bar{\mathbb{P}}_\lambda\left(\beta \lambda_1^2 (\tau_{n+1} - \tau_n) > t\right) \leq Ce^{-c\sqrt{t}} \qquad \forall t > 0. \tag{36}$$

Here, $c_4 > 0$ *is a constant.*

PROOF. 1. First, we consider the case $n = 0$, $\tau_{n+1} - \tau_n = \tau_1$. Ineqaulity (35) is the conclusion of Theorem 19. To prove (36), note that for any $m \in \mathbb{N}$,

$$\bar{\mathbb{P}}_\lambda\left(\beta \lambda_1^2 \tau_1 > t\right) \leq \bar{\mathbb{P}}_\lambda(X_{\tau_1} \cdot e_1 \geq m/\lambda_1) + \bar{\mathbb{P}}_\lambda\left(\beta \lambda_1^2 T_m > t\right). \tag{37}$$

By Theorem 19,

$$\bar{\mathbb{P}}_\lambda(X_{\tau_1} \cdot e_1 \geq m/\lambda_1) \leq Ce^{-c\beta m}.$$

By Proposition 11 and Theorem 12,

$$\begin{aligned}
\bar{\mathbb{P}}_\lambda\left(\beta \lambda_1^2 T_m > t\right) &\leq \bar{\mathbb{P}}_\lambda(T_m > T_{-m}) + \bar{\mathbb{P}}_\lambda\left(\beta \lambda_1^2 \tilde{T}_m > t\right) \\
&\leq Ce^{-cm} + Ce^{-ct/(\beta m)}.
\end{aligned}$$

Coming back to (37), we get

$$\bar{\mathbb{P}}_\lambda\big(\beta\lambda_1^2\tau_1 > t\big) \le Ce^{-c\beta m} + Ce^{-ct/(\beta m)} \qquad \forall m \in \mathbb{N}.$$

Ineqaulity (36) (for $n = 0$) follows by letting $m = \lfloor \sqrt{t}/\beta \rfloor$.

2. Next, we will prove (35) for $n \ge 1$. By Lemma 17,

$$\begin{aligned}
&\bar{\mathbb{E}}_\lambda\big[\exp\big(s\beta\lambda_1(X_{\tau_{n+1}} - X_{\tau_n})\cdot e_1\big)\big] \\
&\quad = E_{\mathcal{P}}\Big[\sum_y \mu_{\omega^\lambda,1}(y)\bar{E}^y_{\omega^\lambda}\big[\exp(s\beta\lambda_1 X_{\tau_1}\cdot e_1), T_{-1} = \infty\big]\Big]\Big/ p_\lambda \\
&\quad \le E_{\mathcal{P}}\Big[\sum_y \mu_{\omega^\lambda,1}(y)\bar{E}^y_{\omega^\lambda}\big[\exp(s\beta\lambda_1 X_{\tau_1}\cdot e_1)\big]\Big]\Big/ p_\lambda.
\end{aligned}$$

By the same argument as in (29) and (31), we get

$$\begin{aligned}
&E_{\mathcal{P}}\Big[\sum_y \mu_{\omega^\lambda,1}(y)\bar{E}^y_{\omega^\lambda}\big[\exp(s\beta\lambda_1 X_{\tau_1}\cdot e_1)\big]\Big] \\
&\quad = \sum_{k\ge 1} E_{\mathcal{P}}\Big[\sum_y \mu_{\omega^\lambda,1}(y)\bar{E}^y_{\omega^\lambda}\big[\exp(s\beta\lambda_1 X_{S_k}\cdot e_1), S_k < \infty\big]\Big] p_\lambda,
\end{aligned}$$

and

$$\begin{aligned}
&E_{\mathcal{P}}\Big[\sum_y \mu_{\omega^\lambda,1}(y)\bar{E}^y_{\omega^\lambda}\big[\exp(s\beta\lambda_1 X_{S_k}\cdot e_1), S_k < \infty\big]\Big] \\
&\quad \le \left(\frac{A(s,\lambda,\beta)}{1-s}\right)^{k-1} E_{\mathcal{P}}\Big[\sum_y \mu_{\omega^\lambda,1}(y)\bar{E}^y_{\omega^\lambda}\big[\exp(s\beta\lambda_1 X_{S_1}\cdot e_1), S_1 < \infty\big]\Big] \\
&\quad \overset{(27),(28)}{\le} \frac{A(s,\lambda,\beta)^{k-1}}{(1-s)^k} \qquad \forall k \ge 1.
\end{aligned}$$

Therefore,

$$\bar{\mathbb{E}}_\lambda\big[\exp\big(s\beta\lambda_1(X_{\tau_{n+1}} - X_{\tau_n})\cdot e_1\big)\big] \le \sum_{k\ge 1}\frac{A(s,\lambda,\beta)^{k-1}}{(1-s)^k}.$$

By (34), ineqaulity (35) is proved.

3. Finally, we will prove (36) for $n \ge 1$. Similar to (37), for any $m \in \mathbb{N}$,

$$\begin{aligned}
&\bar{\mathbb{P}}_\lambda\big(\beta\lambda_1^2(\tau_{n+1} - \tau_n) > t\big) \\
&\quad \le \bar{\mathbb{P}}_\lambda\big((X_{\tau_{n+1}} - X_{\tau_n})\cdot e_1 \ge m/\lambda_1\big) + \bar{\mathbb{P}}_\lambda\big(\beta\lambda_1^2 T_m \circ \theta_{\tau_n} > t\big).
\end{aligned} \tag{38}$$

By (35),

$$\bar{\mathbb{P}}_\lambda\big((X_{\tau_{n+1}} - X_{\tau_n})\cdot e_1 \ge m/\lambda_1\big) \le Ce^{-cm\beta} \qquad \forall m \in \mathbb{N}. \tag{39}$$

By Lemma 17,

$$\begin{aligned}
&\bar{\mathbb{P}}_\lambda\big(\beta\lambda_1^2 T_m \circ \theta_{\tau_n} > t\big) \\
&\quad = \sum_y E_{\mathcal{P}}\big[\mu_{\omega^\lambda,1}(y)\bar{P}^y_{\omega^\lambda}\big(\beta\lambda_1^2 T_m > t, T_{-1} = \infty\big)\big]/p_\lambda \\
&\quad \le \sum_y E_{\mathcal{P}}\big[\mu_{\omega^\lambda,1}(y)\bar{P}^y_{\omega^\lambda}\big(\beta\lambda_1^2 \tilde{T}_m > t\big)\big]/p_\lambda.
\end{aligned}$$

Applying Theorem 12 to the above inequality, we have

$$\bar{\mathbb{P}}_\lambda\big(\beta\lambda_1^2 T_m \circ \theta_{\tau_n} > t\big) \le C e^{-ct/(m\beta)}/p_\lambda \qquad \forall m \in \mathbb{N}. \tag{40}$$

Combining (38), (39) and (40) and letting $m = \lfloor \sqrt{t}/\beta \rfloor$,

$$\bar{\mathbb{P}}_\lambda\big(\beta\lambda_1^2(\tau_{n+1} - \tau_n) > t\big) \le C e^{-c\sqrt{t}}/p_\lambda.$$

It remains to show that

$$p_\lambda > C > 0 \qquad \forall \lambda \in (0, \lambda).$$

By (20) and (24),

$$\begin{aligned}
p_\lambda &= E_{\mathcal{P}}\big[P^{0.5e_1/\lambda_1}_{\omega^\lambda}\big(T_{-0.5} = \infty | T_{0.5} < T_{-0.5}\big)\big] \\
&= E_{\mathcal{P}}\big[P_{\omega^\lambda}\big(T_{-0.5} = \infty | T_{0.5} < T_{-0.5}\big)\big] \\
&\ge \mathbb{P}_\lambda(T_{-0.5} = \infty) \overset{\text{Proposition 11}}{\ge} C\big(1 - e^{-\rho/4}\big).
\end{aligned}$$

Our proof of (36) is complete. $\square$

By Corollary 20, we conclude that for any $p \ge 1, k \ge 0$, there exists a constant $C(p) < \infty$ such that

$$\bar{\mathbb{E}}_\lambda\big[\big(\beta\lambda_1^2 \tau_1 \circ \theta_{\tau_k}\big)^p\big] < C(p) \tag{41}$$

and

$$\bar{\mathbb{E}}_\lambda\big[\big(\beta\lambda_1 X_{\tau_1} \circ \theta_{\tau_k}\big)^p\big] < C(p). \tag{42}$$

Moreover, by the law of large numbers,

$$v_\lambda \overset{\bar{\mathbb{P}}_\lambda\text{-a.s.}}{=} \lim_{n\to\infty} \frac{X_{\tau_n} \cdot e_1}{\tau_n} = \frac{\bar{\mathbb{E}}_\lambda[X_{\tau_2} - X_{\tau_1}]}{\bar{\mathbb{E}}_\lambda[\tau_2 - \tau_1]}.$$

On the other hand,

$$\begin{aligned}
&\bar{\mathbb{E}}_\lambda\big[(X_{\tau_2} - X_{\tau_1}) \cdot e_1\big] \ge 1/\lambda_1, \\
&\qquad\qquad v_\lambda \cdot e_1 = \lambda E_{\mathcal{Q}_\lambda}\big[d(\xi)\big] \cdot e_1 \le 2\lambda.
\end{aligned}$$

Hence,

$$\bar{\mathbb{E}}_\lambda[\tau_2 - \tau_1] \ge \frac{1/\lambda_1}{2\lambda} \ge C/\lambda^2. \tag{43}$$

5. Proof of Theorem 1. Let $\alpha_n = \alpha_n(\beta, \lambda) := \bar{\mathbb{E}}_\lambda \tau_n$. Note that by (41) and (43),

$$\frac{Cn}{\lambda^2} \le \alpha_n \le \frac{C'n}{\lambda^2}. \tag{44}$$

LEMMA 21. *Assume* (i). *Let f be a function that satisfies* (1). *Then for $\beta > 0$ and $\lambda \in (0, 1/N_f)$,*

$$\left| E_{\mathcal{Q}_\lambda} f - \frac{1}{\bar{\mathbb{E}}_\lambda \tau_n} \bar{\mathbb{E}}_\lambda \left[\sum_{i=0}^{\tau_n} f(\bar{\zeta}_i) \right] \right| \le C \|f\|_\infty / n \qquad \textit{for all } n \in \mathbb{N}.$$

PROOF. The lemma is trivial when $n = 1$, so we only consider $n \ge 2$. Recall that $\tau_0 = 0$. For $k \ge 0$, set

$$Z_k = Z_k(f) = \sum_{i=\tau_k}^{\tau_{k+1}-1} f(\bar{\zeta}_i).$$

Since $N_f \le \frac{1}{\lambda} \le \bar{\mathbb{E}}_\lambda [X_{\tau_2} - X_{\tau_1}]$, we see that $(Z_k)_{k \ge 0}$ is an 1-dependent sequence. On one hand,

$$\frac{1}{\bar{\mathbb{E}}_\lambda \tau_n} \bar{\mathbb{E}}_\lambda \left[\sum_{i=0}^{\tau_n} f(\bar{\zeta}_i) \right] = \frac{(n-1)\bar{\mathbb{E}}_\lambda Z_1 + \bar{\mathbb{E}}_\lambda Z_0}{(n-1)\bar{\mathbb{E}}_\lambda [\tau_2 - \tau_1] + \bar{\mathbb{E}}_\lambda \tau_1}. \tag{45}$$

On the other hand, since the $\mathbb{P}_\lambda$-law of $\bar{\zeta}_n$ converges weakly to $\mathcal{Q}_\lambda$, by (22),

$$\mathcal{Q}_\lambda f = \lim_{n \to \infty} \bar{\mathbb{E}}_\lambda \left[\frac{1}{n} \sum_{i=0}^{n-1} f(\bar{\zeta}_i) \right].$$

Hence, by the law of large numbers,

$$\mathcal{Q}_\lambda f = \frac{\bar{\mathbb{E}}_\lambda [Z_1]}{\bar{\mathbb{E}}_\lambda [\tau_2 - \tau_1]}. \tag{46}$$

The lemma follows by combining (45), (46) and using the moment bounds (41) and (43). $\square$

LEMMA 22. *Assume* (i). *Let f be a function that satisfies* (1). *Then*

$$\frac{1}{\alpha_n} \bar{\mathbb{E}}_\lambda \left| \sum_{i=0}^{\tau_n} f(\bar{\zeta}_i) - \sum_{i=0}^{\alpha_n} f(\bar{\zeta}_i) \right| \le \frac{C \|f\|_\infty}{\sqrt{n}} \qquad \forall n \in \mathbb{N}, \lambda \in (0, 1/N_f).$$

PROOF. Noting that the left-hand side is less than

$$\frac{\|f\|_\infty}{\alpha_n} \bar{\mathbb{E}}_\lambda |\tau_n - \alpha_n| \le \frac{\|f\|_\infty}{\alpha_n} \sqrt{\operatorname{Var} \tau_n},$$

by (44), it suffices to show

$$\operatorname{Var}\tau_n \le Cn/\lambda^4. \tag{47}$$

Indeed, by the inequality $(a+b)^2 \le 2(a^2+b^2)$, we have

$$\begin{aligned}\operatorname{Var}\tau_n &= \operatorname{Var}\left[\sum_{k=0}^{n-1}(\tau_k - \tau_{k-1})\right] \\ &\le 2\left(\operatorname{Var}\left[\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(\tau_{2k+1} - \tau_{2k})\right] + \operatorname{Var}\left[\sum_{k=1}^{\lfloor n/2\rfloor}(\tau_{2k} - \tau_{2k-1})\right]\right).\end{aligned}$$

Since $(\tau_{2k+1} - \tau_{2k})_{k\ge 0}$ and $(\tau_{2k} - \tau_{2k-1})_{k\ge 1}$ are i.i.d. sequences, we conclude that

$$\operatorname{Var}\tau_n \le 2\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\operatorname{Var}[\tau_{2k+1} - \tau_{2k}] + 2\sum_{k=1}^{\lfloor n/2\rfloor}\operatorname{Var}[\tau_{2k} - \tau_{2k-1}] \overset{(41)}{\le} Cn/\lambda^4.$$

This completes the proof of (47). $\square$

PROOF OF THEOREM 6. Since the left-hand side is uniformly bounded (by $2\|f\|_\infty$) for all t, the case $t < \lambda^2\alpha_1 \le C$ is trivial. For $t \ge \lambda^2\alpha_1$, we let $n = n(t,\lambda) \ge 1$ be the integer that satisfies

$$\alpha_n \le \frac{t}{\lambda^2} < \alpha_{n+1}.$$

Since

$$\begin{aligned}&\left|\frac{\lambda^2}{t}\bar{\mathbb{E}}_\lambda \sum_{i=0}^{\lceil t/\lambda^2\rceil} f(\bar\zeta_i) - \frac{1}{\alpha_n}\bar{\mathbb{E}}_\lambda \sum_{i=0}^{\alpha_n} f(\bar\zeta_i)\right| \\ &\quad\le \left|\frac{\lambda^2}{t}\bar{\mathbb{E}}_\lambda\left[\sum_{i=0}^{\lceil t/\lambda^2\rceil} f(\bar\zeta_i) - \sum_{i=0}^{\alpha_n} f(\bar\zeta_i)\right]\right| + \left|\left(\frac{\lambda^2}{t} - \frac{1}{\alpha_n}\right)\bar{\mathbb{E}}_\lambda \sum_{i=0}^{\alpha_n} f(\bar\zeta_i)\right| \\ &\quad\le \|f\|_\infty\left(\frac{\lambda^2}{t}(\alpha_{n+1} - \alpha_n) + \left(\frac{1}{\alpha_n} - \frac{1}{\alpha_{n+1}}\right)\alpha_n\right) \\ &\quad\overset{(44)}{\le} C\|f\|_\infty/n.\end{aligned}$$

Theorem 6 follows by combining Lemma 21, Lemma 22 and the above inequality. $\square$

PROOF OF THEOREM 1. Since the space Ω of the environment is compact under the product topology, it suffices to show that

$$\lim_{\lambda\to 0} \mathcal{Q}_\lambda f = \mathcal{Q} f$$

for all f that satisfies (1). The above equality follows immediately from Theorems 5 and 6. $\square$

6. Proof of Theorem 2. Let us recall the regeneration structure defined by Sznitman and Zerner [23]. For a path $(X_n)_{n\geq 0}$, we call $t > 0$ a *renewal time*[2] in the direction ℓ if

$$X_m \cdot \ell < X_t \cdot \ell < X_n \cdot \ell$$

for all m, n such that $m < t < n$. For ballistic RWRE, the renewal times exist a.s. and have finite first moments. We let

$$T(1) < T(2) < \cdots$$

denote all the renewal times. Then $(X_{T(k+1)} - X_{T(k)}, T(k+1) - T(k))_{k\geq 1}$ is an i.i.d. sequence under $\mathbb{P}$.

LEMMA 23. *If the $\mathcal{P}$-law of ω satisfies Sznitman's* (T′) *condition*, *then there exists a constant $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0)$*:

(a) ω^λ *satisfies* (T′);
(b) $\mathbb{E}_\lambda[T(1)^2] < C$ *and* $\mathbb{E}_\lambda[(T(2) - T(1))^2] < C$.

PROOF. It is shown in [3], Theorem 1.6, that (T′) is equivalent to a polynomial ballisticity condition ($\mathcal{P}$). Note that ($\mathcal{P}$) only involves checking a strict inequality for some (finitely many) exit probabilities from a finite box (see [3], Definition 1.4). Hence, there exists $\lambda_0 > 0$ such that ($\mathcal{P}$) holds for all ω^λ, $\lambda \in [0, \lambda_0)$, with the same constants in the upper bounds of [3], Definition 3.2. We have proved (a). Furthermore, by [22], Proposition 3.1 and [3], Theorem 1.6, ($\mathcal{P}$) implies that the regeneration time has finite moments. Therefore, the second moments of $T(1)$ and $T(2) - T(1)$ (under $\mathbb{P}_\lambda$) can be bounded by the same constant [since they are deduced from the same ($\mathcal{P}$) condition] for all $\lambda \in [0, \lambda_0)$. (b) is proved. □

THEOREM 24. *Assume that ω satisfies Sznitman's* (T′) *condition. If f satisfies* (1), *then for any $t \geq 1$, $n \in \mathbb{N}$ and $\lambda \in [0, \lambda_0)$*,

$$\mathbb{E}_\lambda\left[\left(\sum_{i=0}^{n}(f(\bar{\zeta}_i) - \mathcal{Q}_\lambda f)\right)^2\right] \leq C N_f \|f\|_\infty^2 n.$$

To prove this theorem, we need two lemmas.

LEMMA 25. *Assume that ω satisfies Sznitman's* (T′) *condition. If f satisfies* (1), *then for any $\lambda \in [0, \lambda_0)$*,

$$\mathbb{E}_\lambda\left[\left[\sum_{i=0}^{T(n)}(f(\bar{\zeta}_i) - \mathcal{Q}_\lambda f)\right]^2\right] \leq C N_f \|f\|_\infty^2 n.$$

[2]It is usually called a *regeneration time* in the RWRE literature. But we use a different name to distinguish with the regeneration structure defined in Section 4.

PROOF. For $k \geq 0$, let

$$Z_k = Z_k(f) := \sum_{i=T(k)}^{T(k+1)-1} \big(f(\bar{\zeta}_i) - \mathcal{Q}_\lambda f\big).$$

Then $(Z_k)_{k \geq N_f}$ is a N_f-dependent and stationary sequence. Moreover, for $k \geq N_f$,

$$\mathbb{E}_\lambda Z_k = 0.$$

Hence, for $n > N_f$,

$$\begin{aligned}
&\mathbb{E}_\lambda\left[\left(\sum_{k=N_f}^{n-1} Z_k\right)^2\right] \\
&\qquad = \sum_{k=N_f}^{n} E_\lambda[Z_k^2] + 2\sum_{j=N_f}^{n-N_f}\sum_{k=j+1}^{N_f} E_\lambda[Z_j Z_k] \\
&\qquad \leq 3nN_f E_\lambda[Z_{N_f}^2] \leq CnN_f\|f\|_\infty^2.
\end{aligned}$$

Noting that

$$\mathbb{E}_\lambda\left[\left(\sum_{k=0}^{N_f-1} Z_k\right)^2\right] \leq \|f\|_\infty^2 \mathbb{E}_\lambda[T(N_f)^2] \leq CN_f\|f\|_\infty^2,$$

our proof is complete. □

LEMMA 26. *Let $\alpha_n = \alpha(n,\lambda) = \mathbb{E}_\lambda T(n)$. Assume that ω satisfies Sznitman's* (T′) *condition. If f satisfies* (1), *then for any $\lambda \in [0, \lambda_0')$,*

$$\mathbb{E}_\lambda\left[\left(\sum_{i=0}^{T(n)} f(\bar{\zeta}_i) - \sum_{i=0}^{\alpha_n} f(\bar{\zeta}_i)\right)^2\right] \leq C\|f\|_\infty^2 n.$$

PROOF.

$$\begin{aligned}
\mathbb{E}_\lambda\left[\left(\sum_{i=0}^{T(n)} f(\bar{\zeta}_i) - \sum_{i=0}^{\alpha_n} f(\bar{\zeta}_i)\right)^2\right] &\leq \|f\|_\infty^2 \mathbb{E}_\lambda\big[\big(T(n)-\alpha_n\big)^2\big] \\
&\leq \|f\|_\infty^2 \sum_{i=0}^{n-1} \mathrm{Var}\big[T(i+1) - T(i)\big].
\end{aligned}$$

By Lemma 23(b), the lemma follows. □

PROOF OF THEOREM 24. Set

$$\tilde{f}(\zeta) := f(\zeta) - \mathcal{Q}f.$$

By Lemmas 25 and 26, for any $m \in \mathbb{N}$,

$$\mathbb{E}_\lambda\left[\left(\sum_{i=0}^{\alpha_m} \tilde{f}(\bar{\zeta}_i)\right)^2\right] \le C N_f \|f\|_\infty^2 m.$$

For $n \ge 1$, we let $m = m(s, \lambda) \ge 0$ be the integer that satisfies

$$\alpha_m \le n < \alpha_{m+1}.$$

Thus,

$$\begin{aligned}
&\mathbb{E}_\lambda\left[\left(\sum_{i=0}^{n} \tilde{f}(\bar{\zeta}_i)\right)^2\right] \\
&\quad \le 2\mathbb{E}_\lambda\left[\left(\sum_{i=0}^{\alpha_m} \tilde{f}(\bar{\zeta}_i)\right)^2\right] + 8\|f\|_\infty^2(\alpha_{n+1} - \alpha_n) \le C N_f \|f\|_\infty^2 n. \qquad \square
\end{aligned}$$

PROOF OF THEOREM 2. Recall the definitions of $G(\cdot,\cdot)$ and $a(\zeta, e)$ in Section 2. Since

$$\big((A_n, B_n)\big)_{n\ge 1} := \left(\left(\sum_{i=T(n)}^{T(n+1)-1} \tilde{f}(\zeta_i), \sum_{i=T(n)}^{T(n+1)-1} a(\zeta_i, \Delta X_i)\right)\right)_{n\ge 1}$$

is an N_f-dependent (under $\mathbb{P}$) stationary sequence with zero means, by Lemma 23 and the CLT for m-dependent sequences [5], Theorem 5.2, we conclude that as $n \to \infty$, the $\mathbb{P}$-law of $(\frac{1}{\sqrt{n}} A_{\lfloor \cdot n\rfloor}, \frac{1}{\sqrt{n}} B_{\lfloor \cdot n\rfloor})$ converges weakly to a Brownian motion in $\mathbb{R}^2$. Moreover, by the same argument as in [21], Theorem 4.1,

$$\left(\lambda \sum_{i=0}^{\lfloor t/\lambda^2\rfloor} \tilde{f}(\bar{\zeta}_i), \lambda \sum_{i=0}^{\lfloor t/\lambda^2\rfloor} a(\bar{\zeta}_i, \Delta X_i)\right)_{t\ge 0} \tag{48}$$

converges weakly (under $\mathbb{P}$, as $\lambda \to 0$) to a Brownian motion $(\tilde{N}_t, N_t)$ in $\mathbb{R}^2$. On the other hand, by (9) and Theorem 24,

$$\begin{aligned}
&\mathbb{E}\left[\left(\exp\big(G(t/\lambda^2, \lambda)\big)\lambda \sum_{i=0}^{t/\lambda^2} \tilde{f}(\bar{\zeta}_i)\right)^{3/2}\right] \\
&\quad \le \big(\mathbb{E}\big[\exp\big(6G(t/\lambda^2, \lambda)\big)\big]\big)^{1/4}\left(\mathbb{E}\left[\lambda^2\left(\sum_{i=0}^{t/\lambda^2} \tilde{f}(\bar{\zeta}_i)\right)^2\right]\right)^{3/4} \qquad (49)\\
&\quad \le C e^{ct} N_f \|f\|_\infty^{3/2}.
\end{aligned}$$

Therefore, by the invariance principle (48) and uniform integrability (49),

$$\begin{aligned}
\lim_{\lambda\to 0}\lambda\mathbb{E}_\lambda\Bigg[\sum_{i=0}^{t/\lambda^2}\big(f(\bar\zeta_i)-\mathcal{Q}f\big)\Bigg] &= \lim_{\lambda\to 0}\mathbb{E}\Bigg[\exp\big(G(t/\lambda^2,\lambda)\big)\lambda\sum_{i=0}^{t/\lambda^2}\tilde f(\bar\zeta_i)\Bigg]\\
&\overset{(5)}{=} E\big[\tilde N_t\exp\big(N_t - EN_t^2/2\big)\big]\\
&= t\operatorname{Cov}(N_1,\tilde N_1) := t\Lambda(f).
\end{aligned}\tag{50}$$

Setting $U_j=\sum_{k=T(j+N_f)}^{T(j+N_f+1)-1}\tilde f(\bar\zeta_k)$ and $V_j=\sum_{k=T(j+N_f)}^{T(j+N_f+1)-1}a(\bar\zeta_k,\Delta X_k)$, Λ also has the expression

$$\begin{aligned}
\Lambda(f) &= \lim_{n\to\infty}\mathbb{E}\Bigg[\sum_{i=0}^{T(n)}\tilde f(\bar\zeta_i)\sum_{i=0}^{T(n)}a(\bar\zeta_i,\Delta X_i)\Bigg]\Big/\mathbb{E}\big[T(n)\big]\\
&= \Bigg(\mathbb{E}[U_1V_1]+\sum_{i=1}^{N_f}\mathbb{E}[U_1V_{1+i}+U_{1+i}V_1]\Bigg)\Big/\mathbb{E}\big[T(2)-T(1)\big].
\end{aligned}\tag{51}$$

Therefore,

$$\begin{aligned}
&\left|\frac{\mathcal{Q}_\lambda f-\mathcal{Q}f}{\lambda}-\operatorname{Cov}(N_1,\tilde N_1)\right|\\
&\qquad\le\frac{1}{\lambda}\mathbb{E}_\lambda\left|\frac{\lambda^2}{t}\sum_{i=0}^{t/\lambda^2}f(\bar\zeta_i)-\mathcal{Q}_\lambda f\right|+\left|\frac{\lambda}{t}\mathbb{E}_\lambda\Bigg[\sum_{i=0}^{t/\lambda^2}\big(f(\bar\zeta_i)-\mathcal{Q}f\big)\Bigg]-\operatorname{Cov}(N_1,\tilde N_1)\right|.
\end{aligned}$$

Letting first $\lambda\to 0$ and then $t\to\infty$, we obtain [by Theorem 24 and (50)]

$$\lim_{\lambda\to 0}\frac{\mathcal{Q}_\lambda f-\mathcal{Q}f}{\lambda}=\Lambda(f).\tag{52}$$

Theorem 2 is proved. $\square$

REMARK 27. 1. By (51), for any f that satisfies (1),

$$\big|\Lambda(f)\big|\le CN_f\|f\|_\infty.$$

2. By the same argument as in [4], one can obtain a quenched invariance principle for (48).

7. The derivative of the speed and Einstein relation. In this section, we will apply Theorems 1 and 2 to derive the derivative of the speed.

PROOF OF COROLLARY 3. Note that

$$v_\lambda=\mathcal{Q}_\lambda\big[d\big(\omega^\lambda\big)\big]=\mathcal{Q}_\lambda\big[d(\omega)\big]+\lambda\mathcal{Q}_\lambda\big[d(\xi)\big]$$

and

$$v_0 = \mathcal{Q}\big[d(\omega)\big].$$

Thus,

$$\frac{v_\lambda - v_0}{\lambda} = \mathcal{Q}_\lambda\big[d(\xi)\big] + \frac{\mathcal{Q}_\lambda[d(\omega)] - \mathcal{Q}[d(\omega)]}{\lambda}.$$

Therefore, by Theorem 1 [recall that $\Lambda = 0$ in case (i)] and Theorem 2,

$$\lim_{\lambda\to 0} \frac{v_\lambda - v_0}{\lambda} = \mathcal{Q}\big[d(\xi)\big] + \Lambda\big(d(\omega)\big).$$

[Here, we write $\Lambda(f) := (\Lambda f_1, \dots, \Lambda f_d)$ for a function $f = (f_1, \dots, f_d) : \Omega \to \mathbb{R}^d$.] Corollary 3 is proved. $\square$

PROOF OF PROPOSITION 4. The existence of the speed is proved in Proposition 8. When ω is balanced and $\xi(x, e) = \omega(x, e) e \cdot \ell$, it is straightforward to check that $\mathcal{Q}(d(\xi)) = D\ell$. $\square$

REMARK 28. 1. For case (ii), with Corollary 3, we can also write the derivative of the speed at $\lambda > 0$:

$$\frac{\mathrm{d}v_\lambda}{\mathrm{d}\lambda} = \mathcal{Q}_\lambda d(\xi) + \Lambda_\lambda\big(d(\omega^\lambda)\big),$$

where Λ_λ is as Λ in (51), with ω, $\mathbb{E}$ and $\mathcal{Q}$ replaced by ω^λ, $\mathbb{E}_\lambda$ and $\mathcal{Q}_\lambda$, respectively. It is not hard (by considering the Radon–Nikodym derivative) to obtain

$$\lim_{\lambda\to 0} \Lambda_\lambda f = \Lambda f.$$

So $\mathrm{d}v_\lambda/\mathrm{d}\lambda$ is continuous at $\lambda = 0$ and hence also continuous for $\lambda \in [0, \lambda_0)$.

2. For case (i),

$$\frac{\mathrm{d}v_\lambda}{\mathrm{d}\lambda} = \mathcal{Q}_\lambda d(\xi) + \lambda\Lambda_\lambda\big(d(\xi)\big).$$

Λ_λ can also be expressed in terms of the regeneration times defined in Section 4. Moreover, using Lebowitz–Rost's argument and the moment estimates of the regenerations, it is not hard to obtain $|\lambda\Lambda_\lambda(d(\xi))| \le C$. But it is not clear whether $\lim_{\lambda\to 0} \lambda\Lambda_\lambda(d(\xi)) = 0$, that is, $\mathrm{d}v_\lambda/\mathrm{d}\lambda$ is also continuous at $\lambda = 0$.

3. By (51), we get

$$\Lambda(\text{Constant}) = 0.$$

Hence when the original environment is deterministic, Corollary 3 agrees with Sabot's result [18]. [Note that when ω and ξ are independent, $\mathcal{Q}(\xi \in \cdot) = \mathcal{P}(\xi \in \cdot)$.]

8. Questions.

1. Is the Einstein relation still true for balanced environment without the uniform ellipticity assumption? (Recall that the quenched invariance principle for random walks in i.i.d. balanced random environment is proved for elliptic environment [12] and "genuinely d-dimensional" environment [4].)
2. In case (i), is $\mathrm{d}v_\lambda/\mathrm{d}\lambda$ continuous at $\lambda = 0$? Further, is v_λ an analytic function of λ?
3. Does the Einstein relation hold for a random environment with zero-speed but is not balanced, for example, RWRE with cut points [6]?
4. We expect Theorem 1 to be true for general random environment (with an ergodic stationary measure for the environment viewed from the particle process) with general perturbations. But it is not clear how this can be proved.

APPENDIX: PROOF OF THEOREM 12

The idea of our proof is the following. Since the drift ω^λ at each point is of size $c\lambda$, the "worst case" is that all the drifts $d(\theta^x\omega^\lambda)$ point toward the level $\{z : z \cdot e_1 = 0\}$. Hence, we only need to work on the "worst case" to get the upper bound. To this end, we couple X_i with a slow chain Y_i on $\mathbb{Z}^+$, which is defined by

$$Y_0 = |X_0 \cdot e_1|,$$
$$Y_{i+1} - Y_i = \begin{cases} 0, & \text{if } X_{i+1} \cdot e_1 - X_i \cdot e_1 = 0, \\ 1, & \text{if } X_{i+1} \cdot e_1 \neq X_i \cdot e_1 \text{ and } Y_i = 0, \\ B_i(X_i), & \text{if } X_{i+1} \cdot e_1 - X_i \cdot e_1 = 1 \text{ and } Y_i \neq 0, \\ -1, & \text{if } X_{i+1} \cdot e_1 - X_i \cdot e_1 = -1 \text{ and } Y_i \neq 0, \end{cases}$$

where $(B_i(x))_{i\in\mathbb{N},x\in\mathbb{Z}^d}$ are independent Bernoulli random variables [which are independent of $(X_j, Y_j)_{0\le j\le i}$] such that

$$P\big(B_i(x) = 1\big) = \frac{(1 - \lambda/\kappa)}{2p(x)}$$

and

$$P\big(B_i(x) = -1\big) = 1 - \frac{(1 - \lambda/\kappa)}{2p(x)},$$

where[3]

$$p(x) := P^x_{\omega^\lambda}(X_1 \cdot e_1 = 1 | X_1 \cdot e_1 \neq 0).$$

---

[3] Note that by the uniform ellipticity assumption,

$$p(x) = \frac{\omega^\lambda(x, e_1)}{\omega^\lambda(x, e_1) + \omega^\lambda(x, -e_1)} \ge \frac{1 - \lambda/\kappa}{2}.$$

That is, Y_i reflects at the origin and moves only when $X_i \cdot e_1$ changes. When $|X_i \cdot e_1|$ decreases and $Y_i \neq 0$, $Y.$ moves left. When $|X_i \cdot e_1|$ increases and $Y_i \neq 0$, $Y.$ flips a coin $B_i(X_i)$ to decide where to move. $(Y_i)_{i\geq 0}$ has the following good properties:

1. $|X_i \cdot e_1| - Y_i$ is always a nonnegative even integer. Hence,

$$\tilde{T}_n \leq \tilde{S}_n, \tag{53}$$

where

$$\tilde{S}_n := \left\{ i \geq 0 : Y_i = \frac{n}{\lambda_1} \right\}.$$

Moreover,

$$\begin{aligned} &P(Y_{i+1} - Y_i = \pm 1 | X_j, Y_j, 0 \leq j \leq i) \\ &\quad = \frac{1 \mp \lambda/\kappa}{2} \big(\omega^\lambda(X_i, e_1) + \omega^\lambda(X_i, -e_1)\big) \qquad \text{if } Y_i > 0 \end{aligned} \tag{54}$$

and $P(Y_{i+1} - Y_i = 1 | X_j, Y_j, 0 \leq j \leq i) = 1$ if $Y_i = 0$.

2. $(Y_i)_{i\geq 0}$ is not a Markov chain. But if we set $t_0 = 0$, $t_{i+1} = \inf\{n > t_i : X_n \neq X_{t_i}\}$, then

$$Z_i := Y_{t_i}, \qquad i \geq 0$$

is a nearest-neighbor random walk on $\mathbb{Z}^+$ that satisfies

$$P(Z_{i+1} - Z_i = \pm 1 | Z_j, j \leq i) = \frac{1 \mp \lambda/\kappa}{2} \qquad \text{if } Z_i > 0 \tag{55}$$

and $P(Z_{i+1} - Z_i = 1 | Z_j, j \leq i) = 1$ if $Z_i = 0$.

PROOF OF THEOREM 12. Let $Y_i, Z_i, \tilde{S}_i, i \geq 0$ be defined as above, by (53), it suffices to show that for some $s > 0$,

$$E\big[e^{s\lambda^2 \tilde{S}_n/n} | Y_0 = 0\big] < \infty.$$

By the same argument as in [20], Lemma 1.1, it is enough to show that for any $x \in \{0, 1, \dots, n/\lambda_1\}$,

$$E[\tilde{S}_n | Y_0 = x] \leq \frac{cn}{\lambda^2}. \tag{56}$$

Putting

$$S_n := \inf\{i \geq 0 : Z_i = n/\lambda_1\},$$

we have

$$\tilde{S}_n = \sum_{i=0}^{S_n - 1} t_{i+1}.$$

Since for every $i \geq 0$, t_{i+1} is a geometric random variable with success probability $\omega^\lambda(X_{t_i}, e_1) + \omega^\lambda(X_{t_i}, -e_1) \geq \kappa$, we can stochastically dominate $(t_i)_{i\geq 0}$ by a sequence of i.i.d. Geometric(κ) random variables $(G_i)_{i\geq 0}$ that are independent of S_n. Thus, for any $x \in \{0, 1, \dots, n/\lambda_1\}$,

$$\begin{aligned} E[\tilde{S}_n|Y_0 = x] &\leq E[S_n/\kappa|Z_0 = x] \\ &\leq E[S_n|Z_0 = 0]/\kappa. \end{aligned}$$

Therefore, to prove (56), we only need to show that

$$E[S_n|Z_0 = 0] \leq \frac{cn}{\lambda^2}. \tag{57}$$

With abuse of notation, we write $P(\cdot|Z_0 = x)$ and $E[\cdot|Z_0 = x]$ as $P^x(\cdot)$ and $E^x[\cdot]$, respectively.

Set

$$H_n = \inf\{i > 0 : Z_i \in \{0, n/\lambda_1\}\}.$$

Conditioning on the hitting time to the origin, we have

$$E^0[S_n] = 1 + E^1[H_n] + P^0(Z_{H_n} = 0)E^0[S_n]. \tag{58}$$

By (55), $Z_m - Z_0 - m\lambda/\kappa$ is a martingale for $0 \leq m \leq H_n$. Thus, by the optional stopping theorem, for any $x \in \{1, \dots, n/\lambda_1\}$,

$$E^x\left[Z_{H_n} - x - \frac{\lambda}{\kappa}H_n\right] = 0.$$

Hence,

$$E^1[H_n] \leq \kappa E^1[Z_{H_n}]/\lambda. \tag{59}$$

By (55) and Proposition 10, we get

$$c\lambda \leq P^1(Z_{H_n} = n/\lambda_1) \leq c'\lambda \tag{60}$$

and so

$$E^1[Z_{H_n}] \leq c\lambda \cdot n/\lambda_1 \leq cn.$$

This and (59) yield

$$E^1[H_n] \leq cE^1[Z_{H_n}]/\lambda \leq cn/\lambda.$$

It then follows by (58) and (60) that

$$E^0[S_n] = \frac{1 + E^1[H_n]}{P^0(Z_{H_n} = n/\lambda_1)} = \frac{1 + E^1[H_n]}{P^1(Z_{H_n} = n/\lambda_1)} \leq C\frac{n}{\lambda^2}.$$

Inequality (57) is proved. □

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