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Research Article

Attracting and Quasi-Invariant Sets of Cohen-Grossberg Neural Networks with Time Delay in the Leakage Term under Impulsive Perturbations

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A class of impulsive Cohen-Grossberg neural networks with time delay in the leakage term is investigated. By using the method of $\mathscr{M}$-matrix and the technique of delay differential inequality, the attracting and invariant sets of the networks are obtained. The results in this paper extend and improve the earlier publications. An example is presented to illustrate the effectiveness of our conclusion.

1. Introduction

Cohen-Crossberg neural network model, which is initially proposed by Cohen and Grossberg [1] in 1983, has been found successful applications in many fields such as pattern recognition, parallel computing, associative memory, signal and image processing, and combinatorial optimization. Hence, there has been increasing interest in studying the stability and asymptotic behavior of this model with delays, impulses, and unique equilibrium, and many significant results have been obtained (see, e.g., [2–6]). However, the equilibrium point sometimes does not exist in many real physical systems, so it is an interesting subject to discuss the attracting and invariant sets of the neural networks [7, 8].

On the other hand, a leakage delay, which is the time delay in the leakage term and a factor affecting the stability of the system, has attracted considerable attentions (see, e.g., [9–13]). However, to the best of our knowledge, so far there are few results on the attracting and invariant sets of the Cohen-Grossberg neural networks with leakage delay. Motivated by the above discussion, in this paper, we investigate the attracting and quasi-invariant sets of a class of impulsive Cohen-Grossberg neural networks with leakage delay. By using the method of $\mathscr{M}$-matrix and the technique of delay differential inequality, the attracting and invariant sets of the addressed networks are obtained. The results in this paper extend and improve the earlier publications. An example is presented to illustrate the effectiveness of our conclusion.

2. Model Description and Preliminaries

Let R^n be the space of n-dimensional real column vectors, $\mathcal{N} \triangleq \{1, 2, \ldots, n\}$, $R_+ \triangleq [0, +\infty)$, $N \triangleq \{1, 2, \ldots\}$, and $R^{m\times n}$ denotes the set of $m \times n$ real matrices. Usually, E denotes an $n \times n$ unit matrix. For $A, B \in R^{m\times n}$ or $A, B \in R^n$, the notation $A \geq B$ ($A \leq B$, $A > B$, $A < B$) means that each pair of corresponding elements of A and B satisfies the inequality "$\geq$ ($\leq$, $>$, $<$)". Particularly, A is called a nonnegative matrix if $A \geq 0$, and $z \in R^n$ is called a positive vector if $z \geq 0$.

Let $\tau > 0$; for $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T : R \to R^n$, we define

$$\begin{gathered} [x(t)]^+ = \left(\left|x_1(t)\right|, \left|x_2(t)\right|, \ldots, \left|x_n(t)\right|\right)^T, \\ \left[x_i(t)\right]_\tau = \sup_{-\tau\leq s\leq 0} \left\{x_i(t+s)\right\}, \\ [x(t)]_\tau = \left(\left[x_1(t)\right]_\tau, \left[x_2(t)\right]_\tau, \ldots, \left[x_n(t)\right]_\tau\right)^T, \\ [x(t)]_\tau^+ = \left[[x(t)]^+\right]_\tau . \end{gathered} \tag{1}$$

$C[X, Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y. Particularly, let $C \triangleq C[[-\tau, 0], R^n]$.

$PC^1 \triangleq \{\phi : [-\tau, 0] \to R^n$ is continuous and with continuous derivative everywhere except at finite number of point t at which $\phi(t^+), \phi(t^-), \dot{\phi}(t^+)$, and $\dot{\phi}(t^-)$ exist and $\phi(t^+) = \phi(t)$, $\dot{\phi}(t^+) = \dot{\phi}(t)$, where $\dot{\phi}$ denotes the derivative of $\phi\}$. PC^1 is a space of piecewise right-hand continuous functions with the norm $\|\phi\| = \sup_{-\tau \le s \le 0} |\phi(s)|$, $\phi \in PC^1$, where $|\cdot|$ is a norm in R^n.

$PC[[t_0, \infty), R^{m\times n}] \triangleq \{\psi : [t_0, \infty) \to R^{m\times n} \mid \psi(t)$ is continuous at $t \ne t_k$, $\psi(t_k^+)$ and $\psi(t_k^-)$ exist, $\psi(t_k) = \psi(t_k^+)$, for $k \in N\}$.

In this paper, we consider the following Cohen-Grossberg neural networks with impulses and time delays:

$$\begin{aligned} \dot{x}_i(t) = -\alpha_i(t, x(t)) \Bigg(& \beta_i(x_i(t-\sigma)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \\ & - \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau_{ij}(t))) + I_i \Bigg), \\ & \qquad t \ge t_0,\ t \ne t_k, \\ \Delta x_{ik} &= x_i(t_k^+) - x_i(t_k^-) \\ &= h_{ik}(x_1(t_k^-), x_2(t_k^-), \dots, x_n(t_k^-)), \quad k \in N, \\ x_i(t_0 + s) &= \phi_i(s), \quad -\tau < s \le 0,\ i = 1, 2, \dots, n, \end{aligned} \tag{2}$$

where n corresponds to the number of units in a neural network; $x_i(t)$ corresponds to the state of the ith unit at time t; f_j and g_j are the activation functions of the jth unit; $\tau_{ij}(t)$ denotes the transmission delay and satisfies $0 \le \tau_{ij}(t) \le r$ (r is a constant); $\sigma > 0$ is the leakage delay. Consider $\tau = \max\{\sigma, r\}$; $\alpha_i(t, x(t)) > 0$ represents the amplification function of the ith neuron; $\beta_i(x_i(t))$ is the behaved function at time t. Consider $(s) = (\phi_1(s), \dots, \phi_n(s))^T \in PC^1[[-\tau, 0], R^n]$. The fixed impulsive moments t_k ($k \in N$) satisfy $\tau < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$.

Definition 1 (see [14]). A function $x(t) : [t_0 - \tau, \infty) \to R^n$ is said to be a solution of (2) through (t_0, ϕ), if $x(t) \in PC[[t_0, \infty), R^n]$ as $t \ge t_0$, and satisfies (2) with the initial condition

$$x(t_0 + s) = \phi(s), \quad s \in [-\tau, 0],\ \phi \in PC^1. \tag{3}$$

Throughout the paper, we always assume that, for any $\phi \in PC^1$, system (2) has at least one solution through (t_0, ϕ), denoted by $x(t, t_0, \phi)$ or $x_t(t_0, \phi)$ (simply $x(t)$ and x_t if no confusion should occur), where $x_t(t_0, \phi) = x(t + s, t_0, \phi) \in PC$, $s \in [-\tau, 0]$.

Definition 2 (see [7]). The set $S \subset PC^1$ is called a positive invariant set of (2), if for any initial value $\phi \in S$ we have the solution $x_t(t_0, \phi) \in S$ for $t \ge t_0$.

Definition 3 (see [7]). The set $S \subset PC^1$ is called a quasi-invariant set of (2), if there exist a matrix $W \ge 0$ and a vector $b \ge 0$ such that, for any $\phi \in S$, there exists a vector z such that the solution $x(t) = x_t(t_0, \phi)$ of (2) satisfies $[x(t)]_\tau^+ \le Wz + b$, $t \ge t_0$, as $[\phi]_\tau^+ \le z$. Obviously, the set S is an invariant set of (2) if $W = E$ and $b = 0$.

Definition 4 (see [7]). The set $S \subset PC^1$ is called a global attracting set of (2), if for any initial value $\phi \in PC^1$ the solution $x_t(t_0, \phi)$ converges to S as $t \to +\infty$. That is,

$$\operatorname{dist}(x_t, S) \longrightarrow 0, \quad t \longrightarrow +\infty, \tag{4}$$

where $\operatorname{dist}(\varphi, S) = \inf_{\psi\in S} \operatorname{dist}(\varphi, \psi)$, $\operatorname{dist}(\varphi, \psi) = \sup_{s\in[-\tau,0]} |\varphi(s) - \psi(s)|$ for $\varphi \in PC^1$.

Definition 5 (see [8]). The zero solution of (2) is said to be globally exponentially stable if for any solution $x(t, t_0, \phi)$ there exist constants $\lambda > 0$ and $\kappa \ge 1$ such that $|x(t, t_0, \phi)| \le \kappa \|\phi\| e^{-\lambda(t-t_0)}$, $t \ge t_0$.

Definition 6 (see [15]). Let the matrix $D = (d_{ij})_{n\times n}$ have nonpositive off-diagonal elements (i.e., $d_{ij} \le 0$, $i \ne j$); then each of the following conditions is equivalent to the statement that D is a nonsingular $\mathscr{M}$-matrix.

(i) All the leading principle minors of D are positive.

(ii) $D = C - G$ and $\rho(C^{-1}G) < 1$, where $G \ge 0$ and $C = \operatorname{diag}\{c_1, \dots, c_n\}$.

(iii) The diagonal elements of D are all positive and there exists a positive vector d such that $Dd > 0$ or $D^T d > 0$.

For a nonsingular matrix $D \in R^{n\times n}$, we denote $\Omega_M(D) \triangleq \{z \in R^n, z > 0 \mid Dz > 0\}$.

Lemma 7 (see [14]). *For a nonsingular $\mathscr{M}$-matrix D, $\Omega_M(D)$ is nonempty, and for any $z_1, z_2 \in \Omega_M(D)$ we have*

$$k_1 z_1 + k_2 z_2 \in \Omega_M(D) \quad \forall k_1, k_2 > 0. \tag{5}$$

So $\Omega_M(D)$ is a cone without conical surface in R^n. We call it an "$\mathscr{M}$-cone."

Lemma 8 (see [16]). *Let $t_0 \le b \le +\infty$ and $u(t) \in [[t_0, b), R^n]$ satisfy*

$$\begin{aligned} D^+[u(t)]^+ \le R(t, u(t)) \Big\{ & P[u(t)]^+ \\ & + Q[u(t)]_\tau^+ + \widehat{I} \Big\}, \quad t \in [t_0, b), \\ u(t_0 + s) \in PC&, \quad s \in [-\tau, 0], \end{aligned} \tag{6}$$

where $P = (p_{ij})_{n\times n}$, $p_{ij} \ge 0$, $(i \ne j)$, $Q = (q_{ij})_{n\times n} \ge 0$, $\widehat{I} = (\widehat{I}_1, \dots, \widehat{I}_n)^T \ge 0$, $R(t, u) = \operatorname{diag}(R_1(t, u), \dots, R_n(t, u))$, and $R_i(t, u) \in C[[t_0, b) \times R_n, R_+]$, $i \in \mathscr{N}$. Suppose that $-(P + Q)$ is a nonsingular $\mathscr{M}$-matrix.

(1) If the initial condition satisfies

$$[u(t)]^+ \le x^*, \quad t_0 - \tau \le t \le t_0, \tag{7}$$

where $x^* = (x_1^*, \ldots, x_n^*)^T = -d(P+Q)^{-1}\widehat{I}$, $d \geq 1$, *then* $[u(t)]^+ \leq x^*$, *for* $t \geq t_0$.

(2) Suppose that there exist a scalar $\lambda > 0$ *and a vector* $z = (z_1, \ldots, z_n)^T > 0$ *such that*

$$\left(\lambda E + P + Qe^{\lambda H\tau}\right) z \leq 0, \quad (8)$$

where

$$H = \sup_{t \geq t_0} \max_{(s,[u]^+) \in [t-\tau,t]\times(0,x^*]} \widehat{R}(s, u(s)) < \infty,$$
$$0 \leq \widehat{R}(t, u(t)) \leq \min_{1\leq i\leq n} \{R_i(t, u(t))\}. \quad (9)$$

If the initial condition satisfies

$$[u(t)]^+ \leq ze^{-\lambda \int_{t_0}^{t} \widehat{R}(s,u(s))ds} - (P+Q)^{-1}\widehat{I}, \quad t_0 - \tau \leq t \leq t_0, \quad (10)$$

then

$$[u(t)]^+ \leq ze^{-\lambda \int_{t_0}^{t} \widehat{R}(s,u(s))ds} - (P+Q)^{-1}\widehat{I}, \quad t \geq t_0. \quad (11)$$

3. Main Results

In this paper, we always suppose the following.

(A1) $\alpha_i(t, x(t)) \in C[[t_0, \infty) \times R^n, [0, \overline{\alpha}_i]]$, where $\overline{\alpha}_i > 0$ is a constant, $i \in \mathscr{N}$.

(A2) $\beta_i(\cdot)$ is differentiable, and there exist constants $\beta_i', \overline{\beta}_i > 0$ such that $0 < \beta_i' < \dot{\beta}_i(t) < \overline{\beta}_i$, $i \in \mathscr{N}$, for any $t \in [t_0, +\infty)$.

(A3) $f_i(\cdot)$ and $g_i(\cdot)$ are Lipschitz continuous; that is, there exist constants k_i and l_i such that, for any $x_1, x_2 \in R$,

$$\left|f_i(x_1) - f_i(x_2)\right| \leq k_i \left|x_1 - x_2\right|,$$
$$\left|g_i(x_1) - g_i(x_2)\right| \leq l_i \left|x_1 - x_2\right|. \quad (12)$$

(A4) $-(\widehat{P} + \widehat{Q})$ is a nonsingular $\mathscr{M}$-matrix, where

$$\widehat{P} = \left(\widehat{p}_{ij}\right)_{n\times n}, \qquad \widehat{Q} = \left(\widehat{q}_{ij}\right)_{n\times n},$$
$$\widehat{p}_{ii} = -\beta_i' + \left|a_{ii}\right| k_i, \qquad \widehat{p}_{ij} = \left|a_{ij}\right| k_j \,(i \neq j),$$
$$\widehat{q}_{ii} = \sigma\overline{\alpha}_i\overline{\beta}_i^2 + \left(1 + \sigma\overline{\alpha}_i\overline{\beta}_i\right)\left|b_{ii}\right| l_i + \sigma\overline{\alpha}_i\overline{\beta}_i\left|a_{ii}\right| k_i, \quad (13)$$
$$\widehat{q}_{ij} = \left(1 + \sigma\overline{\alpha}_i\overline{\beta}_i\right)\left|b_{ij}\right| l_j + \sigma\overline{\alpha}_i\overline{\beta}_i\left|a_{ij}\right| k_j.$$

(A5) $[x + H_k(x)]^+ \leq \Gamma_k[x]^+$, $k \in N$, for any $x \in R^n$, where $H_k(\cdot) = (h_{1k}(\cdot), \ldots, h_{nk}(\cdot))^T$, $\Gamma_k = (\gamma_{ij}^{(k)})_{n\times n} \geq 0$.

(A6) For $z \in \Omega_M(-\widehat{P} - \widehat{Q})$,

$$\Gamma_k z \leq \mu_k z,$$
$$\Gamma_k\left(-\widehat{P} - \widehat{Q}\right)^{-1} I^* \leq \nu_k\left(-\widehat{P} - \widehat{Q}\right)^{-1} I^*, \quad (14)$$

where $I^* = (I_1^*, I_1^*, \ldots, I_n^*)^T$, $I_i^* = (1 + \sigma\overline{\alpha}_i\overline{\beta}_i)(|\beta_i(0)| + \sum_{j=1}^n(|a_{ij}||f_j(0)| + |b_{ij}||g_j(0)|) + |I_i|) + \sigma m_i\overline{\beta}_i$, and μ_k, $\nu_k \geq 1$ satisfy

$$\ln \mu_k \leq \lambda \int_{t_{k-1}}^{t_k} \widehat{\alpha}(s, x(s))\, ds, \quad \nu = \sum_{k=1}^{n} \ln \nu_k < \infty, \quad k \in N, \quad (15)$$

where $\widehat{\alpha}(s, x(s)) \triangleq \min_{1\leq i\leq n}\{\alpha_i(s, x(s))\}$.

Theorem 9. *Assume that (A1)–(A6) hold. Then, for any* $\widehat{d} \geq 1$, *the set* $S_{\widehat{d}} = \{\phi \in PC^1 \mid [\phi]_\tau^+ \leq \widehat{d}(-\widehat{P} - \widehat{Q})^{-1} I^*\}$ *is a quasi-invariant set of (2).*

Proof. Combining with the middle value theorem, from (2) we can get

$$\begin{aligned}
\frac{dx_i(t)}{dt} &= -\alpha_i(t, x(t))\left(\beta_i(x_i(t-\sigma)) - \sum_{j=1}^n a_{ij} f_j\left(x_j(t)\right)\right.\\
&\qquad\left. - \sum_{j=1}^n b_{ij} g_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) + I_i\right)\\
&= \alpha_i(t, x(t))\left(-\beta_i(x_i(t-\sigma)) + \beta_i(x_i(t))\right.\\
&\qquad - \beta_i(x_i(t)) + \sum_{j=1}^n a_{ij} f_j\left(x_j(t)\right)\\
&\qquad\left. + \sum_{j=1}^n b_{ij} g_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) - I_i\right)\\
&= \alpha_i(t, x(t))\left(\sigma\dot{\beta}_i(x_i(t - (1-\theta)\sigma))\right.\\
&\qquad \times \dot{x}_i(t - (1-\theta)\sigma)\\
&\qquad - \beta_i(x_i(t)) + \sum_{j=1}^n a_{ij} f_i(x_i(t))\\
&\qquad\left. + \sum_{j=1}^n b_{ij} g_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) - I_i\right),
\end{aligned} \quad (16)$$

where $0 < \theta < 1$.

Case 1. Let us first consider $t_0 \leq t \leq t_0 + (1-\theta)\sigma$. In this case $\dot{x}_i(t - (1-\theta)\sigma) = \dot{\phi}_i(t - (1-\theta)\sigma) = \dot{\phi}_i(s)$, $s \in [-\tau, 0]$. Notice that $\phi \in PC^1$; there exist $m_i > 0$ such that $[\dot{\phi}_i(s)]_\tau \leq m_i$, for all $i \in \mathscr{N}$.

From (A1)–(A4) and (16), we have

$$\begin{aligned} D^{+}\left|x_i(t)\right| &= \operatorname{sgn}\left(x_i(t)\right)\frac{dx_i(t)}{dt} \\ &\leq \alpha_i(t,x(t))\left[-\beta_i'\left|x_i(t)\right| + \sum_{j=1}^{n}\left|a_{ij}\right|k_j\left|x_j(t)\right| \right. \\ &\quad + \sum_{j=1}^{n}\left|b_{ij}\right|l_j\left|x_j(t)\right|_{\tau} \\ &\quad + \left(\left|\beta_i(0)\right| + \sum_{j=1}^{n}\left(\left|a_{ij}\right|\left|f_j(0)\right| \right.\right. \\ &\quad \left.\left. + \left|b_{ij}\right|\left|g_j(0)\right|\right)\right) \\ &\quad \left. + \left|I_i\right| + \sigma\overline{\beta}m_{ii}\right] \\ &\leq \alpha_i(t,x(t))\left(\sum_{j=1}^{n}\widehat{p}_{ij}\left|x_j(t)\right| \right. \\ &\quad \left. + \sum_{j=1}^{n}\widehat{q}_{ij}\left|x_j(t)\right|_{\tau+(1-\theta)\sigma} + I_i^{*}\right). \end{aligned} \tag{17}$$

Hence,

$$\begin{aligned} D^{+}[x(t)]^{+} \leq \alpha(t,x(t))\left(\widehat{P}[x(t)]^{+} + \widehat{Q}[x(t)]^{+}_{\tau+(1-\theta)\sigma} + I^{*}\right), \\ t_0 \leq t \leq t_0 + (1-\theta)\sigma. \end{aligned} \tag{18}$$

Case 2. Let us consider $t \geq t_0 + (1-\theta)\sigma$. From (16), we get

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -\alpha_i(t,x(t)) \\ &\quad \times\left\{\sigma\dot{\beta}_i\left(x_i(t-(1-\theta)\sigma)\right.\right. \\ &\quad \times\left[-\alpha_i\left(x_i(t-(1-\theta)\sigma)\right)\right. \\ &\quad \times\left(\beta_i\left(x_i(t-(2-\theta)\sigma)\right)\right) \\ &\quad - \sum_{j=1}^{n}a_{ij}f_j\left(x_j(t-(1-\theta)\sigma)\right) \\ &\quad - \sum_{j=1}^{n}b_{ij}g_j\left(x_j(t-(1-\theta)\sigma)\right. \\ &\quad \left.\left. -\tau_{ij}(t-(1-\theta)\sigma)\right) + I_i\right)\Big] \\ &\quad - \beta_i(x_i(t)) + \sum_{j=1}^{n}a_{ij}f_j\left(x_j(t)\right) \\ &\quad \left. + \sum_{j=1}^{n}b_{ij}g_j\left(x_j\left(t-\tau_{ij}(t)\right)\right) - I_i\right\}. \end{aligned} \tag{19}$$

Then from (A1)–(A4), we have

$$\begin{aligned} D^{+}\left|x_i(t)\right| &= \operatorname{sgn}\left(x_i(t)\right)\frac{dx_i(t)}{dt} \\ &\leq \alpha_i(t,x(t)) \\ &\quad \times\left\{\sigma\overline{\beta}_i\left[\overline{\alpha}_i\left(\overline{\beta}_i\left|x_i(t)\right|_{\tau+(1-\theta)\sigma} + \left|\beta_i(0)\right|\right.\right.\right. \\ &\quad + \sum_{j=1}^{n}\left|a_{ij}\right|\left(k_j\left|x_j(t)\right|_{\tau+(1-\theta)\sigma}\right. \\ &\quad \left. + \left|f_j(0)\right|\right) \\ &\quad + \sum_{j=1}^{n}\left|b_{ij}\right|\left(l_j\left|x_j(t)\right|_{\tau+(1-\theta)\sigma}\right. \\ &\quad \left. + \left|g_j(0)\right|\right) \\ &\quad \left.\left. + \left|I_i\right|\right)\right] \\ &\quad - \beta_i'\left|x_i(t)\right| + \left|\beta_i(0)\right| \\ &\quad + \sum_{j=1}^{n}\left|a_{ij}\right|\left(k_j\left|x_j(t)\right| + \left|f_j(0)\right|\right) \\ &\quad + \sum_{j=1}^{n}\left|b_{ij}\right|l_j\left|(x_j(t)\right|_{\tau+(1-\theta)\sigma} \\ &\quad \left. + \left|g_j(0)\right|) + \left|I_i^{*}\right|\right\} \\ &= \alpha_i(t,x(t))\left(\sum_{j=1}^{n}\widehat{p}_{ij}\left|x_j(t)\right| \right. \\ &\quad \left. + \sum_{j=1}^{n}\widehat{q}_{ij}\left|x_j(t)\right|_{\tau+(1-\theta)\sigma} + I_i^{*}\right). \end{aligned} \tag{20}$$

From (17) and (20), we get

$$D^+[x(t)]^+ \le \alpha(t,x(t))\left(\widehat{P}[x(t)]^+ + \widehat{Q}[x(t)]^+_{\tau+(1-\theta)\sigma} + I^*\right),\quad t \in [t_k, t_{k+1}),\quad k \in N. \tag{21}$$

For the initial conditions $x(t_0+s) = \phi(s)$, $s \in [-\tau, 0]$, where $\phi(s) \in S_{\widehat{d}}$, we have

$$[x(t)]^+ \le -\widehat{d}\left(\widehat{P}+\widehat{Q}\right)^{-1} I^*,\quad t_0-\tau \le t \le t_0. \tag{22}$$

By (21) and Lemma 8, we have

$$[x(t)]^+ \le -\widehat{d}\left(\widehat{P}+\widehat{Q}\right)^{-1} I^*,\quad t_0 \le t \le t_1. \tag{23}$$

Suppose that for all $m = 1, 2, \ldots, k$ the inequalities

$$[x(t)]^+ \le -\nu_0 \ldots \nu_{m-1}\widehat{d}\left(\widehat{P}+\widehat{Q}\right)^{-1} I^*,\quad t_{m-1} \le t < t_m, \tag{24}$$

hold, where $\nu_0 = 1$. Then, from (A5) and (A6),

$$\begin{aligned}[x(t_k)]^+ &= [x(t_k^-) + H_k(x(t_k^-))]^+ \le \Gamma_k [x(t_k^-)]^+ \\ &\le \Gamma_k\left(\nu_0\cdots\nu_{k-1}\widehat{d}\left(-\widehat{P}-\widehat{Q}\right)^{-1} I^*\right) \\ &\le \nu_0\cdots\nu_k\widehat{d}\left(-\widehat{P}-\widehat{Q}\right)^{-1} I^*.\end{aligned} \tag{25}$$

This, together with $\nu_k \ge 1$, leads to

$$[x(t)]^+ \le \nu_0\cdots\nu_k\widehat{d}\left(-\widehat{P}-\widehat{Q}\right)^{-1} I^*,\quad \text{for } t \in [t_k-\tau, t_k]. \tag{26}$$

On the other hand,

$$\begin{aligned}D^+[x(t)]^+ \le \alpha(t,x(t))&\left(\widehat{P}[x(t)]^+ + \widehat{Q}[x(t)]^+_{\tau+(1-\theta)\sigma}\right.\\ &\left.+\ \nu_0\cdots\nu_k I^*\right),\quad t \ne t_k.\end{aligned} \tag{27}$$

It follows from (A4), (26), (27), and Lemma 8 that

$$[x(t)]^+ \le \nu_0\cdots\nu_k\widehat{d}\left(-\widehat{P}-\widehat{Q}\right)^{-1} I^* \quad \text{for } t \in [t_k, t_{k+1}). \tag{28}$$

By the induction, we can conclude that

$$[x(t)]^+ \le \nu_0\cdots\nu_k\widehat{d}\left(-\widehat{P}-\widehat{Q}\right)^{-1} I^*,\quad t \in [t_{k-1}, t_k),\ k \in N. \tag{29}$$

From (A6),

$$\nu_0\cdots\nu_k \le e^{\nu}, \tag{30}$$

and we have

$$[x(t)]^+ \le e^{\nu}\widehat{d}\left(-\widehat{P}-\widehat{Q}\right)^{-1} I^*,\quad t \in [t_0, t_k),\ k \in N. \tag{31}$$

This implies that the conclusion holds and the proof is complete. □

Theorem 10. *Assume that (A1)–(A6) hold. Then the set* $S = \{\phi \in PC^1 \mid [\phi]^+_\tau \le e^{\nu}(-\widehat{P}-\widehat{Q})^{-1}I^*\}$ *is a global attracting set of (2).*

Proof. By a similar proof of Theorem 9, we can get (21) and (31).

By continuity, we can find $\lambda > 0$ and an enough small $\varepsilon > 0$ such that

$$\left((\lambda+\varepsilon)E + \widehat{P} + \widehat{Q}e^{\lambda H(\tau+\sigma)}\right)z < 0, \tag{32}$$

where

$$\begin{aligned}H &\triangleq \sup_{t\ge t_0}\ \max_{s\in[t-\tau,t],x\in[0,x^*]} \widehat{\alpha}(s,x(s)) < \infty,\\ \widehat{\alpha}(s,x(s)) &\triangleq \min_{1\le i\le n}\{\alpha_i(s,x(s))\},\\ x^* &= -e^{2\nu}\left(-\widehat{P}-\widehat{Q}\right)^{-1} I^*.\end{aligned} \tag{33}$$

For the initial conditions $x(t_0+s) = \phi(s)$, $s \in [-\tau, 0]$, where $\phi(s) \in PC^1$, we have

$$[x(t)]^+ \le \kappa_0 z,\quad \kappa_0 = \frac{\|\phi\|}{\min_{1\le i\le n}\{z_i\}},\ t_0-\tau \le t \le t_0, \tag{34}$$

and so

$$[x(t)]^+ \le \kappa_0 z e^{-(\lambda+\varepsilon)\int_{t_0}^t \widehat{\alpha}(s,x(s))ds} - \left(\widehat{P}+\widehat{Q}\right)^{-1} I^*,\quad t_0-\tau \le t \le t_0. \tag{35}$$

By the induction and Lemma 8, we can conclude that

$$\begin{aligned}[x(t)]^+ \le\ &\mu_0\cdots\mu_k\kappa_0 z e^{-(\lambda+\varepsilon)\int_{t_0}^t \widehat{\alpha}(s,x(s))ds}\\ &+ \nu_0\cdots\nu_k\left(-\widehat{P}-\widehat{Q}\right)^{-1} I^*,\quad t \in [t_{k-1}, t_k),\ k \in N.\end{aligned} \tag{36}$$

From (A6), we conclude that

$$\begin{aligned}[x(t)]^+ &\le e^{\lambda(t_1-t_0)}\cdots e^{\lambda(t_{k-1}-t_{k-2})}\kappa_0 z e^{-(\lambda+\varepsilon)\int_{t_0}^t \widehat{\alpha}(s,x(s))ds}\\ &\quad + \nu_0\cdots\nu_k\left(-\widehat{P}-\widehat{Q}\right)^{-1} I^*\\ &\le \kappa_0 z e^{-\varepsilon\int_{t_0}^t \widehat{\alpha}(s,x(s))ds}\\ &\quad + e^{\nu}\left(-\widehat{P}-\widehat{Q}\right)^{-1} I^*,\quad t \in [t_0, t_k),\ k \in N.\end{aligned} \tag{37}$$

This implies that the conclusion holds and the proof is complete. □

Theorem 11. *Assume that (A1)–(A5) with* $\Gamma_k = E$ *hold. Then* $S = \{\phi \in PC^1 \mid [\phi]^+_\tau \le (-\widehat{P}-\widehat{Q})^{-1}I^*\}$ *is a positive invariant set and also a global attracting set of (2).*

Proof (straightforward). Obviously, if $\alpha(0) = 0$, then $x(t) = 0$ is a solution of (2). □

Corollary 12. *Assume that (A1)–(A5) with $\alpha_i(t, x(t)) \geq \delta > 0$, $I^* = 0$ hold. If*

$$\ln \mu_k \leq \lambda (t_k - t_{k-1}), \tag{38}$$

where $\mu_k \geq 1$ satisfy $\Gamma_k z \leq \mu_k z$, $k \in N$, then the zero solution of (2) is globally exponentially stable.

Remark 13. If $\sigma = 0$, and $H_k(\cdot) \equiv 0$, then from Theorems 9 and 10 we can get Theorem 5.1 and Corollary 5.1 in [16].

Remark 14. If $\sigma = 0$, and $H_k(\cdot) \equiv 0$, then from Corollary 12 we can get Corollary 3.1 in [5].

4. Illustrative Example

Consider the Cohen-Grossberg neural networks (2) with the following parameters, activation functions, amplification functions, behaved functions, and delay functions ($n = 2$, $i, j = 1, 2$):

$$\begin{gathered} a_{11} = 0.2, \quad a_{12} = 0.1, \quad a_{21} = 0.2, \quad a_{22} = 0.3, \\ b_{11} = 0.2, \quad b_{12} = 0.3, \quad b_{21} = 0.2, \quad b_{22} = 0.5, \\ J_1 = \frac{1}{2} = J_2, \quad \sigma = 0.2, \\ \alpha_1 (t, x(t)) = \max \left\{ \frac{1}{2}, \left| \sin (x_1 (t)) \right| \right\}, \\ \alpha_2 (t, x(t)) = \max \left\{ \frac{1}{2}, \left| \sin (2x_2 (t)) \right| \right\} \\ \beta_1 (x_1 (t)) = 3x_1 (t), \quad \beta_2 (x_2 (t)) = 2x_2 (t), \\ f_1 (x_1) = g_1 (x_1) = \frac{1}{2} \left(\left| x_1 + 1 \right| - \left| x_1 - 1 \right| \right) \\ f_2 (x_2) = g_2 (x_2) = x_2, \tau_{ij} (t) = \left| \sin (jt) \right|, \\ \Gamma_k = e^{1/2^k} E. \end{gathered} \tag{39}$$

Obviously, $k = l = 1$, $H = 1/2$, $\mu_k = \nu_k = e^{1/2^k}$, and $\nu = 1$. The parameters of condition (A4) are as follows:

$$\begin{gathered} \widehat{P} = \begin{pmatrix} -2.8 & 0.1 \\ 0.2 & -1.7 \end{pmatrix}, \quad \widehat{Q} = \begin{pmatrix} 2.04 & 0.24 \\ 0.16 & 1.12 \end{pmatrix}, \\ -\left(\widehat{P} + \widehat{Q} \right) = \begin{pmatrix} 0.76 & -0.34 \\ -0.36 & 0.58 \end{pmatrix}. \end{gathered} \tag{40}$$

We can easily observe that $-(\widehat{P} + \widehat{Q})$ is a nonsingular $\mathscr{M}$-matrix and

$$\Omega_M \left(-\widehat{P} - \widehat{Q} \right) = \left\{ (z_1, z_2)^T > 0 \mid \frac{18}{29} z_1 < z_2 < \frac{38}{17} z_1 \right\}. \tag{41}$$

Let $z = (1, 1)^T \in \Omega_M(-\widehat{P} - \widehat{Q})$ and $\lambda = 0.082$ which satisfies the inequality

$$\left(\lambda E + \widehat{P} + \widehat{Q} e^{\lambda H (\tau + \sigma)} \right) z = (-0.223, -1.123)^T < 0. \tag{42}$$

Clearly, all conditions of Theorems 9 and 10 are satisfied. So $S = \{\phi \in PC^1 \mid [\phi]_\tau^+ \leq (-\widehat{P} - \widehat{Q})^{-1} I^*\} = (31.89, 32.92)^T$ is a quasi-invariant set and also a global attracting set of (2).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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