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Research Article

Stability Analysis of a Multigroup SEIR Epidemic Model with General Latency Distributions

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The global stability of a multigroup SEIR epidemic model with general latency distribution and general incidence rate is investigated. Under the given assumptions, the basic reproduction number $\mathfrak{R}_0$ is defined and proved as the role of a threshold; that is, the disease-free equilibrium P_0 is globally asymptotically stable if $\mathfrak{R}_0 \leq 1$, while an endemic equilibrium P^* exists uniquely and is globally asymptotically stable if $\mathfrak{R}_0 > 1$. For the proofs, we apply the classical method of Lyapunov functionals and a recently developed graph-theoretic approach.

1. Introduction

Mathematical models have become important tools in analyzing the spread and control of infectious diseases. The SIR model is one of the most popular ones in this field, for which the total population is subdivided into three compartments: susceptible, infectious, and removed. For some diseases, it is reasonable to include a latent (or exposed) class for those susceptible individuals who are infected with the disease but are not yet infectious, which leads to SEIR model [1–6]. Let $S(t)$, $E(t)$, $I(t)$, and $R(t)$ be the numbers of individuals in the susceptible, exposed, infectious, and removed compartments, respectively, with the total population $N(t) = S(t) + E(t) + I(t) + R(t)$. Suppose that $d > 0$ represents the constant recruitment rate and the natural mortality rate. Assuming mass action for the disease transmission and letting $\beta > 0$ denote the effective contact rate, the rate of change of $S(t)$ is

$$S'(t) = d - \beta S(t) I(t) - dS(t). \tag{1}$$

Taking into consideration a general exposed distribution, van den Driessche et al. [5] formulated and studied the following model:

$$\begin{aligned} S'(t) &= d - \beta S(t) I(t) - dS(t), \\ E(t) &= \int_0^t \beta S(u) I(u) e^{-d(t-u)} P(t-u)\,\mathrm{d}u, \\ R'(t) &= rI(t) - dR(t), \\ I(t) &= N - S(t) - E(t) - R(t), \end{aligned} \tag{2}$$

where $r \geq 0$ is the rate at which infective individuals recover. N is constant total populations. It is assumed in [5] that individuals rarely die of the disease and the disease-induced death is negligible, which ensures a constant population; that is, $N(t) = N \cdot P(t)$ denotes the probability (without taking death into account) that an exposed individual still remains in the exposed class t time units after entering the exposed class and it satisfies the following.

($\mathbf{A}_1$) $P : [0,\infty) \to [0,1]$ is nonincreasing, piecewise continuous with possibly finitely many jumps and satisfies $P(0_+) = 1$, $\lim_{t\to\infty} P(t) = 0$ with $\int_0^\infty P(u)du$ being positive and finite.

In fact, the integral term in model (2) is in the sense of Riemann-Stieltjes integrals; the second equation of (2) takes the following form:

$$\begin{aligned} E'(t) &= \beta S(t) I(t) - dE(t) \\ &\quad + \int_0^t \beta S(u) I(u) e^{-d(t-u)} d_t P(t-u)\,\mathrm{d}u, \end{aligned} \tag{3}$$

where $d_t P(t-u) = dP(t-u)/dt$. It follows from total population size N which is constant that the rate of change of I is governed by

$$I'(t) = -\int_0^t \beta S(u) I(u) e^{-d(t-u)} d_t P(t-u)\,du - (d+r) I(t). \quad (4)$$

Thus, model (2) can be written as the system

$$\begin{aligned}
S'(t) &= d - \beta S(t) I(t) - dS(t),\\
E'(t) &= \beta S(t) I(t) - dE(t)\\
&\quad + \int_0^t \beta S(u) I(u) e^{-d(t-u)} d_t P(t-u)\,du,\\
I'(t) &= -\int_0^t \beta S(u) I(u) e^{-d(t-u)} d_t P(t-u)\,du\\
&\quad - (d+r) I(t),\\
R'(t) &= rI(t) - dR(t).
\end{aligned} \quad (5)$$

Recently, a model of this type including the possibility of disease relapse has been proposed in [5, 6] to study the transmission and spread of some infectious diseases such as herpes, and its global dynamics have been completely investigated in [5, 7].

Heterogeneity in the host population can result from different contact modes such as those among children and adults for childhood diseases (e.g., measles and mumps) or different behaviors such as the numbers of sexual partners for some sexually transmitted infections (e.g., herpes and condyloma acuminatum). Taking into consideration different contact patterns, distinct number of sexual partners, or different geography and so forth, it is more proper to divide individual hosts into groups. Therefore, lots of multigroup models have been proposed in the literature to describe the transmission of infectious disease in heterogeneity environment (see [8–17] and references cited therein).

In multigroup epidemic models, a heterogeneous host population is divided into several homogeneous groups according to modes of transmission, contact patterns, or geographic distributions, so that within-group and intergroup interactions can be modeled separately. In this paper, we formulate a multigroup SEIR epidemic model with general exposed distribution and general incidence rates. The population is divided into n distinct groups ($n \geq 2$). For $1 \leq k \leq n$, the kth group is further partitioned into four compartments: susceptible, exposed, infectious, and recovered, whose numbers of individuals at time t are denoted by $S_k(t)$, $E_k(t)$, $I_k(t)$, and $R_k(t)$, respectively. Within the kth group, $\varphi_k(S_k)$ represents the growth rate of S_k, which includes both the production and the natural death of susceptible individuals.

In [18], Zhang et al. studied a multigroup SEIR epidemic model with general exposed distribution and general incidence rates. By using the well-known "linear chain trick," the authors reformulate the model into an equivalent ordinary differential equations system. The global stability results of equilibria are obtained by constructing suitable Lyapunov functionals for general incidence rate function $f_{kj}(S_k(t), I_j(t))$. In [19], Hattaf et al. introduced a general incidence rate $f(S,I)I$ in a delayed SIR epidemic model.

Motivated by these facts, in this paper, we incorporate the general incidence rate presented in [19] to the following system of differential and integral equations:

$$\begin{aligned}
S_k'(t) &= \varphi_k(S_k(t)) - \sum_{j=1}^n f_{kj}(S_k(t), I_j(t)) I_j(t),\\
E_k'(t) &\\
&= \sum_{j=1}^n f_{kj}(S_k(t), I_j(t)) I_j(t)\\
&\quad - \sum_{j=1}^n \int_0^t f_{kj}(S_k(u), I_j(u)) I_j(u) e^{-\delta_k(t-u)} g_k(t-u)\,du\\
&\quad - \delta_k E_k(t),\\
I_k'(t) &= \sum_{j=1}^n \int_0^t f_{kj}(S_k(u), I_j(u)) I_j(u) e^{-\delta_k(t-u)} g_k(t-u)\,du\\
&\quad - (\delta_k + \gamma_k) I_k(t),\\
R_k'(t) &= \gamma_k I_k(t) - \delta_k R_k(t),
\end{aligned} \quad (6)$$

where $g_j(t) = -P_j'(t)$, the nonlinear term $f_{kj}(S_k(t), I_j(t)) I_j(t)$ represents the cross-infection from group j to group k, δ_k denotes the natural death rates of exposed and infectious classes in the kth group, and γ_k denotes the production of the recovered individuals from infectious ones in the kth group. All constants δ_k, γ_k, $k = 1, 2, \ldots, n$, are assumed to be positive.

The organization of this paper is as follows; in the next section, we give some preliminaries of our main model. In Section 3, we prove the global asymptotic stability of the disease-free equilibrium P_0 for $\mathfrak{R}_0 \leq 1$ using the classical method of Lyapunov. The existence of endemic equilibrium is also proved. In Section 4, we prove global asymptotic stability of an endemic equilibrium P^* for $\mathfrak{R}_0 > 1$ using the graph-theoretic approach.

2. Preliminaries

Since the variables E_k and R_k do not appear in the first and third equations of (6), we can only consider the reduced system as follows:

$$\begin{aligned}
S_k'(t) &= \varphi_k(S_k(t)) - \sum_{j=1}^n f_{kj}(S_k(t), I_j(t)) I_j(t),\\
I_k'(t) &= \sum_{j=1}^n \int_0^t f_{kj}(S_k(u), I_j(u)) I_j(u) e^{-\delta_k(t-u)} g_k(t-u)\,du\\
&\quad - (\delta_k + \gamma_k) I_k(t).
\end{aligned} \quad (7)$$

The incidence function $f_{kj}(S_k, I_j)$ in (7) is assumed to be continuously differentiable in the interior of $\mathbb{R}_+^2$ and to satisfy the following hypotheses:

(S_1) $f_{kj}(0, I_j) = 0$, for all $I_j \geq 0$;

(S_2) $\partial f_{kj}(S_k, I_j)/\partial S_k > 0$, for all $S_k > 0$ and $I_j \geq 0$;

(S_3) $\partial f_{kj}(S_k, I_j)/\partial I_j \leq 0$, for all $S_k \geq 0$ and $I_j \geq 0$;

assume that the functions φ_k satisfy the following conditions:

(S_4) φ_k are local Lipschitz on $[0, \infty)$ with $\varphi_k(0) > 0$, and there is a unique positive solution $\xi = S_k^0$ for the equation $\varphi_k(\xi) = 0$; $\varphi_k(S_k) > 0$ for $0 \leq S_k < S_k^0$, and $\varphi_k(S_k) < 0$ for $S_k > S_k^0$.

Typical examples of $f_{kj}(S_k, I_j)$ satisfying (S_1)–(S_3) include common incidence functions such as

$$f_{kj}\left(S_k, I_j\right) = S_k I_j \quad [20, 2, 3], \qquad f_{kj}\left(S_k, I_j\right) = S_k^q I_j \quad [21], \qquad f_{kj}\left(S_k, I_j\right) = \frac{\eta S_k I_j}{1 + \theta S_k} \quad [1]. \tag{8}$$

The class of $\varphi_k(S_k)$ that satisfies (S_4) includes both $\lambda_k - d_k^S S_k$ and $\lambda_k - d_k^S S_k + r_k S_k(1 - S_k/N_k)$, which have been widely used in the literature of population dynamics [1, 8].

For model (7), the existence, uniqueness, and continuity of solutions follow from the theory for integrodifferential equations in [22]. It can be easily verified that every solution of (7) with nonnegative initial conditions remains nonnegative. It follows from (S_4) and the first equation in (7) that $S_k'(t) \leq \varphi_k(S_k(t))$, and thus

$$\limsup_{t \to \infty} S_k(t) \leq S_k^0, \quad \text{for } 1 \leq k \leq n. \tag{9}$$

From the biological significance, we only need to consider (7) in the following region:

$$\begin{aligned}\Gamma := \Big\{ &\left(S_1, I_1, S_2, I_2, \ldots, S_n, I_n\right) \\ &\in \mathbb{R}_+^{2n} : S_k,\ I_k \geq 0, S_k + I_k \leq S_k^0, 1 \leq k \leq n \Big\}.\end{aligned} \tag{10}$$

Indeed, one can easily verify that the set Γ is positively invariant with respect to (7).

It is clear that system (7) has a disease-free equilibrium $P_0 = (S_1^0, 0, S_1^0, 0, \ldots, S_n^0, 0)$ in Γ. Next, we will give some notations which will be useful for our main results.

Let

$$\begin{aligned} J(\xi) &= \int_\xi^\infty g_k(u)\, e^{-\delta_k u} \mathrm{d}u, \\ Q_k = J(0) &= \int_0^\infty g_k(u)\, e^{-\delta_k u} \mathrm{d}u. \end{aligned} \tag{11}$$

It can be verified that $Q_k \in (0, 1)$.

For finite time t, system (7) may not have an endemic equilibrium. If system (7) has an endemic equilibrium, the endemic equilibrium must satisfy the limiting system

$$\begin{aligned} S_k'(t) &= \varphi_k\left(S_k(t)\right) - \sum_{j=1}^n f_{kj}\left(S_k(t), I_j(t)\right) I_j(t), \\ I_k'(t) &= \sum_{j=1}^n \int_0^\infty f_{kj}\left(S_k(t-u), I_j(t-u)\right) \\ &\qquad \times I_j(t-u)\, e^{-\delta_k u} g_k(u)\, \mathrm{d}u \\ &\qquad - \left(\delta_k + \gamma_k\right) I_k(t). \end{aligned} \tag{12}$$

Since the limiting system (12) contains an infinite delay, its associated initial condition needs to be restricted in an appropriate fading memory space. For any $\sigma_k \in (0, \delta_k)$, define the following Banach space of fading memory type (see [23, 24] and references therein):

$$\begin{aligned} C_k = \Big\{ &\phi_k \in C\left((-\infty, 0], \mathbb{R}\right) : \phi_k(s)\, e^{\sigma_k s} \\ &\text{is uniformly continuous on } (-\infty, 0], \\ &\sup_{s \leq 0} \left|\phi_k(s)\right| e^{\sigma_k s} < \infty \Big\}, \\ Y_\Delta = \{&\phi_k \in C_k : \phi_k(s) \geq 0 \ \forall s \leq 0\} \end{aligned} \tag{13}$$

with norm $\|\phi\|_k = \sup_{s \leq 0} |\phi(s)| e^{\sigma_k s}$. Let $\psi_t \in C_i$ and $t > 0$ be such that $\psi_t(s) = \psi(t+s)$, $s \in (-\infty, 0]$.

Let $\phi_k, \psi_k \in C_k$ such that $\phi_k(s), \psi_k(s) \geq 0$ for all $s \in (-\infty, 0]$. We consider solutions of system (12), $(S_{1t}, I_{1t}, \ldots, S_{nt}, I_{nt})$, with initial conditions

$$\left(\phi_1, \psi_1, \phi_2, \psi_2, \ldots, \phi_n, \psi_n\right). \tag{14}$$

The standard theory of functional differential equations [24] implies $(S_{1t}, I_{1t}, \ldots, S_{nt}, I_{nt}) \in C_k$ for all $t > 0$. We study system (12) in the following phase space:

$$\mathbb{X}_{\mathbb{g}} = \prod_{k=1}^n \left(\mathbb{R} \times C_k\right). \tag{15}$$

It can be verified that solutions of (12) in $\mathbb{X}_{\mathbb{g}}$ with initial conditions (14) remain nonnegative.

An equilibrium $P^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$ in the interior of Γ is called an endemic equilibrium of system (12), where $S_k^*, I_k^* > 0$ satisfy the equilibrium equations

$$\begin{aligned} \varphi_k\left(S_k^*\right) &= \sum_{j=1}^n f_{kj}\left(S_k^*, I_j^*\right) I_j^*, \\ \sum_{j=1}^n f_{kj}\left(S_k^*, I_j^*\right) I_j^* Q_k &= \left(\delta_k + \gamma_k\right) I_k^*. \end{aligned} \tag{16}$$

Set $R_0 = \rho(M^0)$ to denote the special radius of the matrix M^0, where

$$M^0 = \left(\frac{f_{kj}(S_k^0, 0) Q_k}{\delta_k + \gamma_k}\right)_{n \times n}. \tag{17}$$

The parameter R_0 is defined as the basic reproduction number [25, 26]. Since it can be verified that system (7) satisfies conditions (A_1)–(A_5) of Theorem 2 of [26], we have the following lemma.

Lemma 1. *For system* (7)*, the disease-free equilibrium P_0 is locally asymptotically stable if $\mathfrak{R}_0 < 1$ while it is unstable if $\mathfrak{R}_0 > 1$.*

3. Global Stability of the Disease-Free Equilibrium

Theorem 2. *Assume that the functions φ_k and f_{kj} satisfy (S_1)–(S_4), and M^0 is irreducible.*

(i) *If $\mathfrak{R}_0 \le 1$, then P_0 is the unique equilibrium of system* (7)*, and P_0 is globally asymptotically stable in Γ.*

(ii) *If $\mathfrak{R}_0 > 1$, then P_0 is unstable and system* (7) *is uniformly persistent.*

Proof. It follows from the Perron-Frobenius theorem (see Theorem 2.1.4 in [27]) that the nonnegative irreducible matrix M^0 has a positive eigenvector $(\omega_1, \omega_2, \dots, \omega_n)$ such that

$$(\omega_1, \omega_2, \dots, \omega_n)\, \rho\left(M^0\right) = (\omega_1, \omega_2, \dots, \omega_n)\, M^0. \tag{18}$$

Now, we construct a Lyapunov functional

$$V_{P_0} = \sum_{k=1}^{n} \frac{\omega_k}{\delta_k + \gamma_k} I_k. \tag{19}$$

Differentiating V_{P_0} along the solution of system (7) and under (S_2) and (S_3), we obtain

$$\begin{aligned}
V'_{P_0} &= \sum_{k=1}^{n} \omega_k \Bigg[\frac{1}{\delta_k + \gamma_k} \times \sum_{j=1}^{n} \int_0^t f_{kj}\left(S_k(u), I_j(u)\right) I_j(u) \times e^{-\delta_k (t-u)} g_k(t-u)\, \mathrm{d}u - I_k(t) \Bigg] \\
&\le \sum_{k=1}^{n} \omega_k \left[\frac{1}{\delta_k + \gamma_k} \sum_{j=1}^{n} f_{kj}(S_k, 0)\, I_j(t)\, Q_k - I_k(t) \right] \\
&\le \sum_{k=1}^{n} \omega_k \left[\frac{1}{\delta_k + \gamma_k} \sum_{j=1}^{n} f_{kj}\left(S_k^0, 0\right) I_j(t)\, Q_k - I_k(t) \right] \\
&= (\omega_1, \omega_2, \dots, \omega_n) \left[M^0 I - I \right] \\
&= \left[\rho\left(M^0\right) - 1 \right] (\omega_1, \omega_2, \dots, \omega_n)\, I,
\end{aligned} \tag{20}$$

where $I = (I_1, I_2, \dots, I_n)^T$. Suppose that $\rho(M^0) < 1$. Then, $V'_{P_0} = 0$ if and only if $I = 0$. Suppose that $\rho(M^0) = 1$. Then, it follows from (20) that $V'_{P_0} = 0$ implies

$$\sum_{k=1}^{n} \omega_k \left[\frac{1}{\delta_k + \gamma_k} \sum_{j=1}^{n} f_{kj}(S_k, 0)\, I_j(t)\, Q_k \right] = \sum_{k=1}^{n} \omega_k I_k(t). \tag{21}$$

If $S_k \ne S_k^0$, then

$$\begin{aligned}
&\sum_{k=1}^{n} \omega_k \left[\frac{1}{\delta_k + \gamma_k} \sum_{j=1}^{n} f_{kj}(S_k, 0)\, I_j(t)\, Q_k \right] \\
&\quad \le \sum_{k=1}^{n} \omega_k \left[\frac{1}{\delta_k + \gamma_k} \sum_{j=1}^{n} f_{kj}\left(S_k^0, 0\right) I_j(t)\, Q_k \right] \\
&\quad \le (\omega_1, \omega_2, \dots, \omega_n)\, M^0 I \\
&\quad = (\omega_1, \omega_2, \dots, \omega_n)\, \rho\left(M^0\right) I \\
&\quad = (\omega_1, \omega_2, \dots, \omega_n)\, I,
\end{aligned} \tag{22}$$

which implies that (21) has only the trivial solution $I = 0$. Therefore, $V'_{P_0} = 0$ if and only if $I_k = 0$ or $S_k = S_k^0$ provided $\rho(M^0) = 1$. It can be verified that the only compact invariant subset of the set where $V'_{P_0} = 0$ is the singleton $\{P_0\}$. By LaSalle's Invariance Principle, P_0 is globally asymptotically stable in Γ if $\rho(M^0) \le 1$.

If $\mathfrak{R}_0 > 1$ and $I \ne 0$, it is easy to see that

$$\left[\rho\left(M^0\right) - 1 \right] (\omega_1, \omega_2, \dots, \omega_n)\, I > 0. \tag{23}$$

It follows from the continuity that $V'_{P_0} > 0$ holds in a small neighborhood of P_0. This implies that P_0 is unstable. Using a uniform persistence result from [28] and similar arguments as in [4, 10, 13, 16, 17], we know that, if $\mathfrak{R}_0 > 1$, the instability of P_0 implies the uniform persistence of (7) in Γ; that is, there exists a positive constant $\epsilon > 0$ such that

$$\liminf_{t \to \infty} S_k(t) \ge \epsilon, \quad \liminf_{t \to \infty} I_k(t) \ge \epsilon, \quad k = 1, 2, \dots, n. \tag{24}$$

The uniform persistence of system (7) together with the uniform boundedness of solutions in Γ, which follows from the positive invariance of Γ, implies the existence of an endemic equilibrium P^* in Γ (see Theorem 2.8.6 of [29] or Theorem D.3 of [30]). Summarizing the statements above, if $\mathfrak{R}_0 > 1$, system (7) is uniformly persistent and there exists at least one endemic equilibrium P^* in Γ. This completes the proof. □

4. Global Stability of an Endemic Equilibrium

Denote

$$H(u) = u - 1 - \ln u, \quad \forall u > 0. \tag{25}$$

Obviously, $H : \mathbb{R}^+ \to \mathbb{R}^+$ attains its strict global minimum at $u = 1$ and $H(1) = 0$.

To get the global stability of P^*, we make the following assumptions:

(S_5) $(\varphi_k(S_k) - \varphi_k(S_k^*))(S_k - S_k^*) \le 0$ for $S_k \ge 0$;

(S_6) $(\varphi_k(S_k) - \varphi_k(S_k^*))[f_{kk}(S_k, I_k^*) - f_{kk}(S_k^*, I_k^*)] < 0$ for $S_k \ne S_k^*$;

(S_7) $(((f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j) I_j)/(f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*) I_j^*)) - 1)(1 - ((f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*))/(f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j)))) \le 0$ for $S_k, I_j > 0$.

Theorem 3. *Assume that the functions φ_k and f_{kj} satisfy (S_1)–(S_7), and the matrix M^0 is irreducible. If $\mathfrak{R}_0 > 1$, then there is a unique endemic equilibrium P^* for system* (12), *and P^* is globally asymptotically stable in the interior of* Γ.

Proof. Define a Lyapunov functional as

$$V_{P^*} = Q_k \int_{S_k^*}^{S_k(t)} \frac{f_{kk}(\eta, I_k^*) - f_{kk}(S_k^*, I_k^*)}{f_{kk}(\eta, I_k^*)} d\eta + I_k^* H\left(\frac{I_k(t)}{I_k^*}\right) + V_+, \tag{26}$$

where

$$V_+ = \sum_{j=1}^n \int_0^\infty f_{kj}(S_k^*, I_j^*) I_j^* J(u) \times H\left(\frac{f_{kj}(S_k(t-u), I_j(t-u)) I_j(t-u)}{f_{kj}(S_k^*, I_j^*) I_j^*}\right) du. \tag{27}$$

First, we calculate the derivative of V_+; then, we have

$$\begin{aligned}
V_+' &= \sum_{j=1}^n \int_0^\infty f_{kj}(S_k^*, I_j^*) I_j^* J(u) \frac{d}{dt} \times H\left(\frac{f_{kj}(S_k(t-u), I_j(t-u)) I_j(t-u)}{f_{kj}(S_k^*, I_j^*) I_j^*}\right) du \\
&= -\sum_{j=1}^n \int_0^\infty f_{kj}(S_k^*, I_j^*) I_j^* J(u) \frac{d}{du} \times H\left(\frac{f_{kj}(S_k(t-u), I_j(t-u)) I_j(t-u)}{f_{kj}(S_k^*, I_j^*) I_j^*}\right) du \\
&= -\sum_{j=1}^n f_{kj}(S_k^*, I_j^*) I_j^* J(u) \times H\left(\frac{f_{kj}(S_k(t-u), I_j(t-u)) I_j(t-u)}{f_{kj}(S_k^*, I_j^*) I_j^*}\right)\Bigg|_{u=0}^{\infty} \\
&\quad + \sum_{j=1}^n \int_0^\infty f_{kj}(S_k^*, I_j^*) I_j^* \times H\left(\frac{f_{kj}(S_k(t-u), I_j(t-u)) I_j(t-u)}{f_{kj}(S_k^*, I_j^*) I_j^*}\right) dJ(u) \\
&= \sum_{j=1}^n Q_k f_{kj}(S_k^*, I_j^*) I_j^* H\left(\frac{f_{kj}(S_k(t), I_j(t)) I_j(t)}{f_{kj}(S_k^*, I_j^*) I_j^*}\right) \\
&\quad - \sum_{j=1}^n \int_0^\infty f_{kj}(S_k^*, I_j^*) I_j^* g_k(u) e^{-\delta_k u} \times H\left(\frac{f_{kj}(S_k(t-u), I_j(t-u)) I_j(t-u)}{f_{kj}(S_k^*, I_j^*) I_j^*}\right) du \\
&= \sum_{j=1}^n Q_k \left(f_{kj}(S_k(t), I_j(t)) I_j(t) - f_{kj}(S_k^*, I_j^*) I_j^* \times \ln \frac{f_{kj}(S_k(t), I_j(t)) I_j(t)}{f_{kj}(S_k^*, I_j^*) I_j^*}\right) \\
&\quad - \sum_{j=1}^n \int_0^\infty g_k(u) e^{-\delta_k u} \times \left[f_{kj}(S_k(t-u), I_j(t-u)) I_j(t-u) - f_{kj}(S_k^*, I_j^*) I_j^*. \times \ln \frac{f_{kj}(S_k(t-u), I_j(t-u)) I_j(t-u)}{f_{kj}(S_k^*, I_j^*) I_j^*}\right] du.
\end{aligned} \tag{28}$$

Calculating the time derivative of V_{P^*} along the solution of system (12), we have

$$\begin{aligned}
V_{P^*}' = Q_k &\left(1 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k(t), I_k^*)}\right) \times \left[\varphi_k(S_k(t)) - \sum_{j=1}^n f_{kj}(S_k(t), I_j(t)) I_j(t)\right] \\
&+ \left(1 - \frac{I_k^*}{I_k(t)}\right) \times \left[\sum_{j=1}^n \int_0^\infty f_{kj}(S_k(u), I_j(u)) I_j(u) e^{-\delta_k(t-u)} g_k(t-u) du - (\delta_k + \gamma_k) I_k(t)\right] + V_+'.
\end{aligned} \tag{29}$$

Using equilibrium equations (16), we have

$$\begin{aligned}
V'_{P^*} &= Q_k\left(\varphi_k\left(S_k(t)\right)-\varphi_k\left(S_k^*\right)\right)\left(1-\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}\left(S_k(t),I_k^*\right)}\right)\\
&\quad+\sum_{j=1}^{n}Q_k f_{kj}\left(S_k^*,I_j^*\right)I_j^*-\sum_{j=1}^{n}Q_k f_{kj}\left(S_k(t),I_j(t)\right)I_j(t)\\
&\quad-\sum_{j=1}^{n}Q_k f_{kj}\left(S_k^*,I_j^*\right)I_j^*\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}\left(S_k(t),I_k^*\right)}\\
&\quad+\sum_{j=1}^{n}Q_k f_{kj}\left(S_k(t),I_j(t)\right)I_j(t)\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}\left(S_k(t),I_k^*\right)}\\
&\quad+\sum_{j=1}^{n}\int_0^\infty f_{kj}\left(S_k(t-u),I_j(t-u)\right)\\
&\qquad\times I_j(t-u)e^{-\delta_k u}g_k(u)\,du\\
&\quad-\frac{I_k(t)}{I_k^*}\sum_{j=1}^{n}\int_0^\infty f_{kj}\left(S_k^*,I_j^*\right)I_j^*e^{-\delta_k u}g_k(u)\,du\\
&\quad-\frac{I_k^*}{I_k(t)}\sum_{j=1}^{n}\int_0^\infty f_{kj}\left(S_k(t-u),I_j(t-u)\right)\\
&\qquad\times I_j(t-u)e^{-\delta_k u}g_k(u)\,du\\
&\quad+\sum_{j=1}^{n}\int_0^\infty f_{kj}\left(S_k^*,I_j^*\right)I_j^*e^{-\delta_k u}g_k(u)\,du+V'_+.
\end{aligned}\tag{30}$$

Using V'_+, we rewrite (30) as

$$\begin{aligned}
V'_{P^*} &= Q_k\left(\varphi_k\left(S_k(t)\right)-\varphi_k\left(S_k^*\right)\right)\left(1-\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}\left(S_k(t),I_k^*\right)}\right)\\
&\quad+\sum_{j=1}^{n}Q_k f_{kj}\left(S_k^*,I_j^*\right)I_j^*\\
&\qquad\times\left[2-\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}\left(S_k(t),I_k^*\right)}\right.\\
&\qquad\quad+\frac{f_{kk}\left(S_k^*,I_k^*\right)f_{kj}\left(S_k(t),I_j(t)\right)I_j(t)}{f_{kk}\left(S_k(t),I_k^*\right)f_{kj}\left(S_k^*,I_j^*\right)I_j^*}\\
&\qquad\quad\left.-\frac{I_k(t)}{I_k^*}\right]\\
&\quad-\sum_{j=1}^{n}\int_0^\infty f_{kj}\left(S_k^*,I_j^*\right)I_j^*g_k(u)e^{-\delta_k u}\\
&\qquad\cdot\left[\frac{I_k^*f_{kj}\left(S_k(t-u),I_j(t-u)\right)I_j(t-u)}{I_k(t)f_{kj}\left(S_k^*,I_j^*\right)I_j^*}\right.\\
&\qquad\quad\left.-\ln\frac{f_{kj}\left(S_k(t-u),I_j(t-u)\right)I_j(t-u)}{f_{kj}\left(S_k(t),I_j(t)\right)I_j(t)}\right]du.
\end{aligned}\tag{31}$$

Therefore,

$$\begin{aligned}
&V'_{P^*}\\
&=Q_k\left(\varphi_k\left(S_k(t)\right)-\varphi_k\left(S_k^*\right)\right)\left(1-\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}\left(S_k(t),I_k^*\right)}\right)\\
&\quad-\sum_{j=1}^{n}Q_k f_{kj}\left(S_k^*,I_j^*\right)I_j^*\\
&\quad\times\left[H\left(\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}\left(S_k(t),I_k^*\right)}\right)\right.\\
&\qquad\left.+H\left(\frac{f_{kk}\left(S_k(t),I_k^*\right)f_{kj}\left(S_k^*,I_j^*\right)}{f_{kk}\left(S_k^*,I_k^*\right)f_{kj}\left(S_k(t),I_j(t)\right)}\right)\right]\\
&\quad+\sum_{j=1}^{n}Q_k f_{kj}\left(S_k^*,I_j^*\right)\\
&\quad\times\left(\frac{f_{kk}\left(S_k^*,I_k^*\right)f_{kj}\left(S_k(t),I_j(t)\right)I_j(t)}{f_{kk}\left(S_k(t),I_k^*\right)f_{kj}\left(S_k^*,I_j^*\right)I_j^*}-1\right)\\
&\quad\times\left(1-\frac{f_{kk}\left(S_k(t),I_k^*\right)f_{kj}\left(S_k^*,I_j^*\right)}{f_{kk}\left(S_k^*,I_k^*\right)f_{kj}\left(S_k(t),I_j(t)\right)}\right)\\
&\quad-\sum_{j=1}^{n}\int_0^\infty f_{kj}\left(S_k^*,I_j^*\right)I_j^*g_k(u)e^{-\delta_k u}\\
&\qquad\times H\left(\frac{I_k^*f_{kj}\left(S_k(t-u),I_j(t-u)\right)I_j(t-u)}{I_k(t)f_{kj}\left(S_k^*,I_j^*\right)I_j^*}\right)du\\
&\quad+\sum_{j=1}^{n}Q_k f_{kj}\left(S_k^*,I_j^*\right)\\
&\qquad\times I_j^*\left[\frac{I_j(t)}{I_j^*}-\frac{I_k(t)}{I_k^*}-\ln\frac{I_j(t)}{I_j^*}+\ln\frac{I_k(t)}{I_k^*}\right].
\end{aligned}\tag{32}$$

Furthermore, under (S_5)–(S_7), we have

$$\begin{aligned}
V'_{P^*} &\le \sum_{j=1}^{n}Q_k f_{kj}\left(S_k^*,I_j^*\right)\\
&\qquad\times I_j^*\left[\frac{I_j(t)}{I_j^*}-\frac{I_k(t)}{I_k^*}-\ln\frac{I_j(t)}{I_j^*}+\ln\frac{I_k(t)}{I_k^*}\right].
\end{aligned}\tag{33}$$

Obviously, the equalities in (33) hold if and only if $S_k = S_k^*$ and $I_k = I_k^*$, $k = 1, 2, \ldots, n$. Therefore, the functional $V = \sum_{k=1}^{n} v_k V_{P^*}$ as defined in Theorem 3.1 of [12] is a Lyapunov function for system (12). Using similar arguments as in [4, 8–13, 16, 17], one can show that the largest invariant subset where $V'_{P^*} = 0$ is the singleton $\{P^*\}$. By LaSalle's Invariance Principle, P^* is globally asymptotically stable in the interior of Γ. This completes the proof of Theorem 3. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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