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Research Article

Powers of Convex-Cyclic Operators

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A bounded operator T on a Banach space X is convex cyclic if there exists a vector x such that the convex hull generated by the orbit $\{T^n x\}_{n\geq 0}$ is dense in X. In this note we study some questions concerned with convex-cyclic operators. We provide an example of a convex-cyclic operator T such that the power T^n fails to be convex cyclic. Using this result we solve three questions posed by Rezaei (2013).

1. Introduction and Main Results

Throughout this paper we denote by $L(X)$ the algebra of all bounded linear operators on a real or complex infinite dimensional Banach space X. An operator $T \in L(X)$ is said to be cyclic if there exists a vector $x \in X$ (later called cyclic vector for T) such that the linear span of the orbit

$$\text{linear span}\left(\{T^n x : n \in \mathbb{N}\}\right) \tag{1}$$

is dense in X. If the orbit $\mathrm{Orb}(T,x) := \{T^n x : n \in \mathbb{N}\}$ is dense itself, without the help of the linear span, then T is called hypercyclic and x is called hypercyclic for T. In the midway stand several notions studied by different authors. For instance, the operator T is said to be supercyclic if the projective orbit is dense in X. We refer to the books [1, 2] and references therein for further information on hypercyclic operators.

When we sometimes abusively say that a polynomial $p(z)$ is a convex polynomial, what we really mean is that $p(z) = t_0 + t_1 z + t_2 z^2 + \cdots + t_n z^n$, $t_i \in \mathbb{R}$, $i = 0, \ldots, n$, and $t_0 + t_1 + \cdots + t_n = 1$. We will focus our attention on the notion of convex cyclicity introduced by Rezaei in [3]. An operator T is said to be convex cyclic if there exists a vector $x \in X$ such that the real convex hull of the orbit (denoted by $\mathrm{co}(\mathrm{Orb}(T,x))$)

$$\mathrm{co}\left(\mathrm{Orb}\left(T,x\right)\right) = \left\{p\left(T\right)x : p \text{ convex polynomial}\right\} \tag{2}$$

is dense in X.

In [3] are characterized the convex-cyclic matrices in finite dimension, and the author develops the main properties in the infinite dimensional setting.

A result by Ansari [4] states that if T is a hypercyclic operator then T^n is also hypercyclic; this fact is not true for cyclic operators. In this paper we show that Ansari's result fails also for convex-cyclic operators, solving a question posed in [3].

Another result proved by Bourdon and Feldman on hypercyclic operators says that if the orbit of a vector is somewhere dense, then it is dense (see [5]). From our previous counterexample we can construct a non-convex-cyclic operator T such that the $\mathrm{co}(\mathrm{Orb}(T,x))$ has nonempty interior. That is, Bourdon and Feldman's result is not true in the convex-cyclic setting. Finally we can construct a convex-cyclic operator T such that T is not weakly hypercyclic; that is, its orbit is not dense in the weak operator topology. The later examples solve Questions 5.5 and 5.6 in [3].

2. Powers of a Convex-Cyclic Operator

The first example of hypercyclic operator on Banach spaces was discovered by Rolewicz (see [6]). Throughout this section $\mathscr{B} = \ell_p$ $1 \leq p < \infty$ or c_0 of complex valued sequences. Rolewicz's operator μB with $|\mu| > 1$ is defined on $\mathscr{B}$ by

$$\mu B\left(x_0, x_1, \ldots, x_n, \ldots\right) = \mu\left(x_1, x_2, \ldots, x_n, \ldots\right), \tag{3}$$

where B denotes the backward shift operator.

Lemma 1. *Set $\alpha = e^{2\pi i/3}$ and $r_0 > 1$. For any $z_0 \in \mathbb{C}\setminus\{0\}$ there exist $k_0 \geq 0$ and a sequence of polynomials $p_k(z)$ such that*

(1) $p_k(z) = (t_{1,k} + t_{2,k}z + t_{3,k}z^2)z^k$ *for all* $k \geq k_0$;

(2) $t_{i,k} \in [0,1]$ *and* $t_{1,k} + t_{2,k} + t_{3,k} = 1$, $i \in \{1,2,3\}$ *and* $k \geq k_0$;

(3) $p_k(r_0\alpha) = z_0$, $k \geq k_0$.

Proof. Let us denote by $\mathcal{T}$ the triangle with vertices $\{1, r_0\alpha, r_0^2\alpha^2\}$. Since $|r_0\alpha| > 1$ and $0 \in \mathcal{T}$, there exists k_0 such that $z_k = z_0/(r_0\alpha)^k \in \mathcal{T}$ for all $k \geq k_0$. Then, there exist barycentric coordinates $t_{i,k} \in [0,1]$ $i = 1,2,3$ satisfying $t_{1,k} + t_{2,k}r_0\alpha + t_{3,k}(r_0\alpha)^2 = z_k$ and $t_{1,k} + t_{2,k} + t_{3,k} = 1$. Then, the polynomials $p_k(z) = (t_{1,k} + t_{2,k}z + t_{3,k}z^3)z^k$, for all $k \geq k_0$, yield the desired result. □

Lemma 2. *Let p_k be a sequence of polynomials satisfying Conditions (1)–(3) of Lemma 1. Then, there exists a G_δ dense subset $Z \subset \mathcal{B}$ of vectors such that $\{p_k(\mu B)x_0\}_{k\geq k_0}$ is dense in $\mathcal{B}$ for all $x_0 \in Z$.*

Proof. We will use some hypercyclicity criterion version for sequence of operators (see [2, Theorem 3.24]); that is, we will show the existence of two dense subsets X and Y and a sequence of mappings S_k such that

(i) $\lim_k p_k(\mu B)x = 0$ $\forall x \in X$;

(ii) $p_k(\mu B)S_k y = y$ $\forall y \in Y$;

(iii) $\lim_k S_k y = 0$ $\forall y \in Y$.

Let us consider the subsets

$$X = \operatorname{span}\{\operatorname{Ker}(\mu B - \lambda I) \ : |\lambda| < 1\},$$
$$Y = \{\operatorname{Ker}(\mu B - \lambda I) \ : \lambda \in \mathbb{R},\ 1 < \lambda < |\mu|\}, \tag{4}$$

which are dense in $\mathcal{B}$ (see [2, Example 3.2, page 70]).

If $x \in \operatorname{Ker}(\mu B - \lambda I)$ with $|\lambda| < 1$, then

$$\begin{aligned} p_k(\mu B)x &= \left(t_{1,k}(\mu B)^k + t_{2,k}(\mu B)^{k+1} + t_{3,k}(\mu B)^{k+2}\right)x \\ &= \left(t_{1,k}I + t_{2,k}(\mu B) + t_{3,k}(\mu B)^2\right)(\mu B)^k x \\ &\leq \text{const}|\lambda|^k, \end{aligned} \tag{5}$$

which goes to zero when $k \to \infty$; therefore, Condition (i) is fulfilled.

Denoting by $q_k(z) = t_{1,k} + t_{2,k}z + t_{3,k}z^2$, since $t_{1,k}$, $t_{2,k}$, $t_{3,k}$ are barycentric coordinates of a triangle, then $q_k(\lambda)$ lies in the degenerate triangle with vertices $\{1, \lambda, \lambda^2\}$, in particular $q_k(\lambda) \geq 1$.

Let us take $y \in \operatorname{Ker}(\mu B - \lambda I)$ with $\lambda \in \mathbb{R}$ and $1 < \lambda < |\mu|$, and let us define the mapping S_k on y as

$$S_k y = \frac{1}{\lambda^k q_k(\lambda)} y \tag{6}$$

and we extend linearly S_k on Y. Clearly $S_k y \to 0$ as $k \to \infty$ for all $y \in Y$ and $p_k(\mu B)S_k y = y$ for all $y \in Y$. Thus, by the hypercyclicity criterion there exists a G_δ dense subset of vectors $x_0 \in \mathcal{B}$ such that $\{p_k(\mu B)x_0\}_{k\geq k_0}$ is dense in $\mathcal{B}$. □

Now, let us prove the main result of this section, which solves Question 5.6 in [3].

Theorem 3. *The operator $T = r_0\alpha I_{\mathbb{C}} \oplus \mu B$ is convex cyclic on $\mathbb{C} \oplus \mathcal{B}$; however T^3 is not.*

Proof. If p is a polynomial, then $p(T) = p(r_0\alpha) \oplus p(\mu B)$. Let us observe that the first coordinate of the powers of $(T^3)^n$ are only real numbers. Take $x = \sum_{n=0}^{\infty} x_0 e_0 \in \mathbb{C} \oplus \mathcal{B}$. If $f^\star$ is the projection on the first coordinate,

$$\left\{f^\star\left(\operatorname{co}\left(\operatorname{Orb}\left(T^3, x\right)\right)\right) = t x_0,\ \forall t \geq r_0\right\} \tag{7}$$

which is not dense in $\mathbb{C}$. Therefore, T^3 is not convex cyclic.

Now, let us prove that T is a convex-cyclic operator using a direct application of the Baire category theorem (see, for instance, [2, Theorem 1.57]). Thus T is convex cyclic if for any nonempty open subsets $U, V \subset \mathbb{C} \oplus \mathcal{B}$, there exists a convex polynomial $p(z)$ such that $p(T)(U) \cap V \neq \emptyset$.

Indeed, let $U = G_1 \times W_1$ and $V = G_2 \times W_2$ open subsets of $\mathbb{C}\oplus\mathcal{B}$, where $G_i \subset \mathbb{C}$ and $W_i \subset \mathcal{B}$, $i = 1, 2$, are nonempty open subsets. Let $z_1 \in G_1$ and $z_2 \in G_2$ with $z_1 z_2 \neq 0$. Set $z_0 = z_2/z_1$ and $p_k(z)$ the sequence of polynomials which guarantees Lemma 1. Hence we have $p_k(r_0\alpha) = z_2/z_1$ and therefore $p_k(r_0\alpha)z_1 = z_2$ (this fact will imply that $p_k(T)$ acting on G_1 will intersect G_2). Now we apply Lemma 2 and we obtain a G_δ dense subset $Z \subset \mathcal{B}$ of hypercyclic vectors for the sequence$\{p_k(\mu B)\}$. Thus there exist $x_0 \in W_1$ and a subsequence $\{n_k\}$ such that $p_{n_k}(\mu B)x_0 \in W_2$. Therefore $p_{n_k}(T)(U) \cap V \neq \emptyset$, which yields the desired result. □

Remark 4. If we take $\alpha = e^{2\pi i/n}$ with $n \geq 4$, using similar arguments as in Theorem 3, we can show that $T = r_0\alpha \oplus \mu B$ is convex cyclic on $\mathbb{C} \oplus \mathcal{B}$ but T^n is not (if $n = 4$ the operator T is convex cyclic but T^2 is not).

Remark 5. If we consider the Rolewicz operator on real spaces ℓ^p, $1 \leq p < \infty$ or c_0, then we can get that the operator $T = -r_0 \oplus \mu B$ $(r_0 > 1)$ is convex cyclic on $\mathbb{R}\oplus\ell^p$ $1 \leq p < \infty$ or $\mathbb{R} \oplus c_0$, but clearly T^2 is not. Lemma 1 can be adapted clearly to the real case. Now it is well known that if we consider Rolewicz's operator on real spaces ℓ^p, $1 \leq p < \infty$ or c_0, then its complexification can be identified with the same operator on the corresponding spaces of complex sequences. With some slight modification in the proof of Corollary 2.51 in [2] we can obtain that Lemma 2 continues being true on real spaces. The rest of the proof is straightforward.

Remark 6. Another difference between hypercyclic operators and convex-cyclic operators is the following: hypercyclic operators are invariant under unimodular multiplications (see [7]); that is, if T is hypercyclic, then λT is also with $|\lambda| = 1$. However this is not true for convex-cyclic operators; the previous counterexample $T = r_0\alpha I_{\mathbb{C}} \oplus \mu B$ is convex cyclic; however $\overline{\alpha}T$ is not.

Now, let $\alpha = e^{2\pi i/3}$. Let us consider the operator $T = \alpha I_{\mathbb{C}} \oplus \mu B$, that is, the same operator provided by Theorem 3 but without the multiplier factor r_0. Then, an easy check shows that the set $S = \{p_k(\alpha) : p_k(z) \text{ convex polynomial}\}$ is contained in the unit disk. Moreover, the set S has nonempty interior in $\mathbb{C}$; for instance, S contains the triangle $\mathcal{T}$ with vertices $\{1, \alpha, \alpha^2\}$. Using the arguments of Theorem 3 we can find a vector $x_0 \in \mathscr{B}$ such that the convex orbit

$$\{p(T)(1 \oplus x_0) : p \text{ convex polynomial}\} \quad (8)$$

is dense in $\mathcal{T} \oplus \mathscr{B}$. Therefore the convex orbit has nonempty interior. However the operator T is not convex cyclic. This solves Question 5.5 in Rezaei's paper.

Proposition 7. *Bourdon and Feldman's result fails for convex-cyclic operators.*

The adjoint of the operator T in Theorem 3 has an eigenvalue; therefore, T cannot be weakly hypercyclic. This solves Question 5.4 in [3].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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