

## Research Article

# The Stability of SI Epidemic Model in Complex Networks with Stochastic Perturbation

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We investigate a stochastic SI epidemic model in the complex networks. We show that this model has a unique global positive solution. Then we consider the asymptotic behavior of the model around the disease-free equilibrium and show that the solution will oscillate around the disease-free equilibrium of deterministic system when  $R_0 \leq 1$ . Furthermore, we derive that the disease will be persistent when  $R_0 > 1$ . Finally, a series of numerical simulations are presented to illustrate our mathematical findings. A new result is given such that, when  $R_0 \leq 1$ , with the increase of noise intensity the solution of stochastic system converging to the disease-free equilibrium is faster than that of the deterministic system.

## 1. Introduction

Epidemiology is the science to study the distribution of disease and influencing factors, so as to explore the etiology, clarify the popular rule of the disease, and formulate the countermeasures and measures for preventing, controlling, and eliminating the disease. Many mathematical models of diseases spreading help us to understand the propagation of diseases [1, 2]. The transmission of diseases can be influenced by many factors, such as the age and social structure of the population, the contact network among individuals, and the metapopulation characteristics. So it is difficult to establish an accurate epidemic model which is completely consistent with the real world. In recent years, a lot of compartmental epidemic models have been studied by many researchers [3–5], and complex networks also have been used to study the spread of diseases [6–17].

In this paper we consider an SI model with the birth and death in complex networks. As mentioned in the paper [6, 13], the birth and death do not affect the degree of nodes. Suppose  $S_k(t)$ ,  $I_k(t)$  are the number of the healthy and infected nodes with the degree  $k$  at time  $t$ ; the mean-field equations can be written as

$$\frac{dS_k}{dt} = b_k - \lambda k S_k \theta - d S_k,$$

$$\frac{dI_k}{dt} = \lambda k S_k \theta - (d + \epsilon) I_k, \quad (1)$$

where  $\theta = (1/\langle k \rangle) \sum_{k=1}^n \lambda k P(k)$ . For system (1), it can be written as the following form:

$$\begin{aligned} dS_k(t) &= \left( b_k - \frac{1}{\langle k \rangle} \sum_{j=1}^n \lambda k_j P(j) S_k(t) I_j(t) - d S_k(t) \right) dt, \\ dI_k(t) &= \left( \frac{1}{\langle k \rangle} \sum_{j=1}^n \lambda k_j P(j) S_k(t) I_j(t) - (d + \epsilon) I_k(t) \right) dt. \end{aligned} \quad (2)$$

We denote  $\beta_{kj} = (1/\langle k \rangle) \lambda k_j P(j)$ , so we obtain

$$\begin{aligned} dS_k(t) &= \left( b_k - \sum_{j=1}^n \beta_{kj} S_k(t) I_j(t) - d S_k(t) \right) dt, \\ dI_k(t) &= \left( \sum_{j=1}^n \beta_{kj} S_k(t) I_j(t) - (d + \epsilon) I_k(t) \right) dt. \end{aligned} \quad (3)$$

It always has the disease-free equilibrium  $E_0 = (S_1^0, 0, \dots, S_n^0, 0)$ , where  $S_k^0 = b_k/d$ ,  $k = 1, 2, \dots, n$ . If  $A = (\beta_{kj})_{n \times n}$  is irreducible and  $R_0 \leq 1$ , then  $E_0$  is globally stable in  $D$ , while

if  $R_0 > 1$ ,  $E_0$  is unstable and there is an endemic equilibrium  $E^* = (S_1^*, I_1^*, \dots, S_n^*, I_n^*)$  belonging to  $D$  which is globally asymptotically stable in  $D$ ; here

$$D = \left\{ (S_1, I_1, \dots, S_n, I_n) \in \mathbb{R}_+^{2n} : S_k, I_k \leq \frac{b_k}{d}, S_k + I_k \leq \frac{b_k}{d}, \right. \\ \left. k = 1, 2, \dots, n \right\}, \\ M_0 = M(S_1^0, \dots, S_n^0) = \begin{pmatrix} \beta_{k1} S_k^0 \\ d + \epsilon \end{pmatrix}_{n \times n}, \quad R_0 = \rho(M_0), \tag{4}$$

and  $\rho(M_0)$  denotes the spectral radius of  $M_0$ .

The deterministic models have some limitations in describing the spread of disease. The accident in the process of disease transmission can not be reflected by the deterministic models. This is because of the fact that the deterministic models ignore the effect of the environmental noise. In an ecosystem, the environmental noise is inevitably in the real world; thus stochastic models are more realistic. In the research of stochastic epidemic models, many researchers make a lot of contributions [17–26].

In this paper, we consider the following stochastic system:

$$dS_k(t) = \left( b_k - \sum_{j=1}^n \beta_{kj} S_k(t) I_j(t) - dS_k(t) \right) dt \\ + \sigma_{k1} S_k(t) dB_{k1}(t), \\ dI_k(t) = \left( \sum_{j=1}^n \beta_{kj} S_k(t) I_j(t) - (d + \epsilon) I_k(t) \right) dt \\ + \sigma_{k2} I_k(t) dB_{k2}(t), \tag{5}$$

where  $B_{ki}(t)$ ,  $k = 1, 2, \dots, n$ ,  $i = 1, 2$ , are independent standard Brownian motions with  $B_{ki}(0) = 0$ , and  $\sigma_{ki}^2 \geq 0$ ,  $k = 1, 2, \dots, n$ ,  $i = 1, 2$ , represent the intensities of  $B_{ki}(t)$ .

The remaining parts of this paper are as follows. In the next section we show the existence and uniqueness of a global positive solution of model (5). In Section 3, we analyze the asymptotic behavior around the disease-free equilibrium. In Section 4, we study the dynamic of system (5) around the endemic of the deterministic model. In Section 5, numerical simulations and conclusions are carried out.

## 2. Global Positive Solution

When we study a dynamical behavior, a global solution is important for the system. In this section we show that the solution of system (5) is global and nonnegative. As we know, for a stochastic differential equation, the coefficients of the equation are generally required to satisfy the linear growth condition and the local Lipschitz condition. It is a sufficient condition for a stochastic differential equation has a unique global (i.e., no explosion in a finite time) solution for any given initial value [27, 28]. Although the coefficients of system

(5) satisfy locally Lipschitz continuous, they are not satisfied with the linear growth condition, so the solution of system (5) may explode at a finite time. In this section, Lyapunov analysis method (mentioned in [29]) is used to show that the solution of system (5) is positive and global.

**Theorem 1.** *For any given initial value  $(S_1(0), I_1(0), \dots, S_n(0), I_n(0)) \in \mathbb{R}_+^{2n}$ , there is a unique positive solution  $(S_1(t), I_1(t), \dots, S_n(t), I_n(t))$  of model (5) on  $t \geq 0$  and the solution will remain in  $\mathbb{R}_+^{2n}$  with probability 1, namely,  $(S_1(t), I_1(t), \dots, S_n(t), I_n(t)) \in \mathbb{R}_+^{2n}$  for all  $t \geq 0$  a.s.*

*Proof.* Due to the fact that the coefficients of the system (5) are locally Lipschitz continuous, for any given initial value  $(S_1(0), I_1(0), \dots, S_n(0), I_n(0)) \in \mathbb{R}_+^{2n}$ , it has a unique local solution  $(S_1(t), I_1(t), \dots, S_n(t), I_n(t))$  on  $t \in [0, \tau_e)$ , where  $\tau_e$  is the explosion time [30]. If we show that  $\tau_e = \infty$  a.s., it suggests that this solution is global. Let  $l_0 > 0$  be sufficiently large so that  $S_k(0), I_k(0)$  ( $k = 1, 2, \dots, n$ ) all lie within the interval  $[1/l_0, l_0]$ . For each integer  $l \geq l_0$ , defining the stopping time

$$\tau_l = \inf \left\{ t \in [0, \tau_e) : \min \{ S_k(t), I_k(t), k = 1, \dots, n \} \leq \frac{1}{l} \right. \\ \left. \text{or } \max \{ S_k(t), I_k(t), k = 1, \dots, n \} \geq l \right\}, \tag{6}$$

we set  $\inf \emptyset = \infty$  (as usual  $\emptyset$  denotes the empty set). Obviously,  $\tau_l$  is increasing as  $l \rightarrow \infty$ . Set  $\tau_\infty = \lim_{l \rightarrow \infty} \tau_l$ ; therefore  $\tau_\infty \leq \tau_e$  a.s. If  $\tau_\infty = \infty$  a.s. is true, then  $\tau_e = \infty$  a.s. and  $(S_1(t), I_1(t), \dots, S_n(t), I_n(t)) \in \mathbb{R}_+^{2n}$  a.s. for  $t \geq 0$ . In other words, to complete the proof it is required to show that  $\tau_\infty = \infty$  a.s. If this statement is false, then there is a pair of constants  $T > 0$  and  $\epsilon \in (0, 1)$  such that  $P\{\tau_\infty \leq T\} > \epsilon$ . Thus there is an integer  $l_1 \geq l_0$ , such that

$$P\{\tau_l \leq T\} \geq \epsilon, \quad \forall l \geq l_1. \tag{7}$$

Define a  $C^2$ -function  $V : \mathbb{R}_+^{2n} \rightarrow \mathbb{R}_+$  as follows:

$$V(S_1, I_1, \dots, S_n, I_n) \\ = \sum_{k=1}^n [(S_k - 1 - \ln S_k) + (I_k - 1 - \ln I_k)]. \tag{8}$$

Applying Itô's formula, we obtain

$$d(S_1, I_1, \dots, S_n, I_n) \\ = \sum_{k=1}^n \left[ \left( 1 - \frac{1}{S_k} \right) dS_k + \frac{1}{2} \frac{1}{S_k^2} (dS_k)^2 + \left( 1 - \frac{1}{I_k} \right) dI_k \right. \\ \left. + \frac{1}{2} \frac{1}{I_k^2} (dI_k)^2 \right]$$

$$\begin{aligned}
 &= \sum_{k=1}^n \left[ \left( 1 - \frac{1}{S_k} \right) \left[ \left( b_k - \sum_{j=1}^n \beta_{kj} S_k I_j - d S_k \right) dt \right. \right. \\
 &\quad \left. \left. + \sigma_{k1} S_k(t) dB_{k1}(t) \right] \right. \\
 &\quad \left. + \frac{1}{2} \sigma_{k1}^2 dt + \left( 1 - \frac{1}{I_k} \right) \right. \\
 &\quad \left. \times \left[ \left( \sum_{j=1}^n \beta_{kj} S_k I_j - (d + \epsilon) I_k \right) dt + \sigma_{k2} I_k dB_{k2}(t) \right] \right. \\
 &\quad \left. + \frac{1}{2} \sigma_{k2}^2 dt \right] \\
 &= \sum_{k=1}^n \left[ b_k - \sum_{j=1}^n \beta_{kj} S_k I_j - d S_k - \frac{b_k}{S_k} + \sum_{j=1}^n \beta_{kj} S_k I_j + d \right. \\
 &\quad \left. + \frac{1}{2} \sigma_{k1}^2 + \sum_{j=1}^n \beta_{kj} S_k I_j - (d + \epsilon) I_k - \frac{\sum_{j=1}^n \beta_{kj} S_k I_j}{I_k} \right. \\
 &\quad \left. + (d + \epsilon) + \frac{1}{2} \sigma_{k2}^2 \right] dt \\
 &\quad + \sum_{k=1}^n [\sigma_{k1} (S_k - 1) dB_{k1}(t) + \sigma_{k2} I_k dB_{k2}(t)] \\
 &\leq \sum_{k=1}^n \left( b_k + 2d + \epsilon + \sum_{j=1}^n \beta_{kj} I_j + \frac{1}{2} (\sigma_{k1}^2 + \sigma_{k2}^2) \right) dt \\
 &\quad + \sum_{k=1}^n [\sigma_{k1} (S_k - 1) dB_{k1}(t) + \sigma_{k2} (I_k - 1) dB_{k2}(t)] \\
 &=: K dt + [\sigma_{k1} (S_k - 1) dB_{k1}(t) + \sigma_{k2} I_k dB_{k2}(t)]. \tag{9}
 \end{aligned}$$

We can now integrate both sides of (9) from 0 to  $\tau_l \wedge T$  and then take the expectations

$$\begin{aligned}
 &E [V(S_1(\tau_l \wedge T), I_1(\tau_l \wedge T), \dots, S_n(\tau_l \wedge T), I_n(\tau_l \wedge T))] \\
 &\leq V(S_1(0), I_1(0), \dots, S_n(0), I_n(0)) + E \left[ \int_0^{\tau_l \wedge T} K dt \right] \\
 &\leq V(S_1(0), I_1(0), \dots, S_n(0), I_n(0)) + KT. \tag{10}
 \end{aligned}$$

Let  $\Omega_l = \{\tau_l \leq T\}$  for  $l \geq l_1$  and, by (7),  $P(\Omega_l) \geq \epsilon$ . Note that, for every  $\omega \in \Omega_l$ , there is at least one of  $S_k(\tau_l, \omega)$  and  $I_k(\tau_l, \omega)$ ,  $k = 1, 2, \dots, n$ , that equals either  $l$  or  $1/l$ , and therefore  $V(S_1(\tau_l, \omega), I_1(\tau_l, \omega), \dots, S_n(\tau_l, \omega), I_n(\tau_l, \omega))$  is not less than either

$$l - 1 - \ln l \quad \text{or} \quad \frac{1}{l} - 1 - \ln \frac{1}{l} = \frac{1}{l} - 1 + \ln l. \tag{11}$$

Hence,

$$\begin{aligned}
 &V(S_1(\tau_l, \omega), I_1(\tau_l, \omega), \dots, S_n(\tau_l, \omega), I_n(\tau_l, \omega)) \\
 &\geq (l - 1 - \ln l) \wedge \left( \frac{1}{l} - 1 + \ln l \right). \tag{12}
 \end{aligned}$$

It then follows from (7) and (10) that

$$\begin{aligned}
 &V(S_1(0), I_1(0), \dots, S_n(0), I_n(0)) + KT \\
 &\geq E [1_{\Omega_l(\omega)} V(S_1(\tau_l, \omega), I_1(\tau_l, \omega), \dots, S_n(\tau_l, \omega), I_n(\tau_l, \omega))] \\
 &\geq \epsilon \left[ (l - 1 - \ln l) \wedge \left( \frac{1}{l} - 1 + \ln l \right) \right], \tag{13}
 \end{aligned}$$

where  $1_{\Omega_l(\omega)}$  is the indicator function of  $\Omega_l$ . Letting  $l \rightarrow \infty$ , we have that

$$\infty > V(S_1(0), I_1(0), \dots, S_n(0), I_n(0)) + KT \geq \infty \tag{14}$$

is a contradiction. So we must have  $\tau_\infty = \infty$ . Therefore, it implies  $S_k(t), I_k(t), k = 1, 2, \dots, n$ , will not explode in a finite time with probability one.  $\square$

### 3. Asymptotic Behavior around the Disease-Free Equilibrium

As mentioned in the Introduction,  $E_0 = (b_1/d, 0, \dots, b_n/d, 0)$  is the disease-free equilibrium of system (3), and when  $R_0 \leq 1$ ,  $E_0$  is globally stable, which means that the disease will be extinct in the limited time. In this section, we will study the asymptotic behavior around  $E_0$  of system (5).

**Lemma 2.** *If  $A$  is nonnegative and irreducible, then the spectral radius  $\rho(A)$  of  $A$  is a simple eigenvalue, and  $A$  has a positive eigenvector  $\omega = (\omega_1, \omega_2, \dots, \omega_n)$  corresponding to  $\rho(A)$ . Besides, if  $0 \leq A \leq B$ , then  $\rho(A) \leq \rho(B)$ . (This lemma can be found in [20].)*

**Theorem 3.** *Assume  $A = (\beta_{kj})_{n \times n}$  is irreducible. If  $R_0 \leq 1$  and the following condition is satisfied:*

$$\sigma_{k1}^2 \leq \frac{4}{3}d, \quad \sigma_{k2}^2 \leq 2(d + \epsilon), \tag{15}$$

then for any given initial value  $(S_1(0), I_1(0), \dots, S_n(0), I_n(0)) \in \mathbb{R}_+^{2n}$ , the solution of system (5) has the property

$$\begin{aligned}
 &\limsup_{t \rightarrow \infty} \frac{1}{t} E \int_0^t \left[ \left( S_k(r) - \frac{b_k}{d} \right)^2 + I_k^2(r) \right] dr \\
 &\leq \frac{3}{2} \sum_{k=1}^n \frac{a_k \sigma_{k1}^2 b_k^2}{d^2 K_1}, \tag{16}
 \end{aligned}$$

where

$$K_1 = \min \left\{ \frac{\omega_k \beta_{kk}}{(2d + \epsilon)(d + \epsilon)} \left( d - \frac{3}{4} \sigma_{k1}^2 \right), d + \epsilon - \frac{1}{2} \sigma_{k2}^2 \right\}. \tag{17}$$

*Proof.* First change the variables  $s_k = S_k - b_k/d, i_k = I_k$ ; then  $-b_k/d \leq s_k \leq 0, i_k \geq 0$  and system (5) can be written as

$$\begin{aligned}
 ds_k &= \left( -\sum_{j=1}^n \beta_{kj} \left( s_k + \frac{b_k}{d} \right) i_j - ds_k \right) dt \\
 &\quad + \sigma_{k1} \left( s_k + \frac{b_k}{d} \right) dB_{k1}(t), \\
 di_k &= \left( \sum_{j=1}^n \beta_{kj} \left( s_k + \frac{b_k}{d} \right) i_j - (d + \epsilon) i_k \right) dt + \sigma_{k2} i_k dB_{k2}(t).
 \end{aligned} \tag{18}$$

Let  $S^0 = (S_1^0, S_2^0, \dots, S_n^0)$ , where  $S_k^0 = b_k/d, k = 1, 2, \dots, n$ . Define

$$M(s) = \begin{bmatrix} \frac{\beta_{11} S_1^0}{d + \epsilon} & \frac{\beta_{12} (S_1^0 + s_1)}{d + \epsilon} & \dots & \frac{\beta_{1n} (S_1^0 + s_1)}{d + \epsilon} \\ \frac{\beta_{21} (S_2^0 + s_2)}{d + \epsilon} & \frac{\beta_{22} S_2^0}{d + \epsilon} & \dots & \frac{\beta_{2n} (S_2^0 + s_2)}{d + \epsilon} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_{n1} (S_n^0 + s_n)}{d + \epsilon} & \frac{\beta_{n2} (S_n^0 + s_n)}{d + \epsilon} & \dots & \frac{\beta_{nn} S_n^0}{d + \epsilon} \end{bmatrix}; \tag{19}$$

then it is nonnegative and irreducible. By Lemma 2, there is a positive eigenvector  $\omega = (\omega_1, \omega_2, \dots, \omega_n)$  of  $M(s)$  corresponding to  $\rho(M(s))$ , such that

$$(\omega_1, \omega_2, \dots, \omega_n) M(s) = (\omega_1, \omega_2, \dots, \omega_n) \rho(M(s)). \tag{20}$$

Define a  $C^2$ -function  $V: \mathbb{R}_+^{2n} \rightarrow \overline{\mathbb{R}}_+$  by

$$V(s_1, i_1, \dots, s_n, i_n) = \frac{1}{2} \sum_{k=1}^n a_k (s_k + i_k)^2 + \sum_{k=1}^n \frac{\omega_k}{d + \epsilon} i_k, \tag{21}$$

where  $a_k, k = 1, 2, \dots, n$ , are positive constants. Then the function  $V$  is positive definite, and

$$\begin{aligned}
 dV &= LVdt + \sum_{k=1}^n a_k (s_k + i_k) \\
 &\quad \times \left( \sigma_{k1} \left( s_k + \frac{b_k}{d} \right) dB_{k1}(t) + \sigma_{k2} i_k dB_{k2}(t) \right) \\
 &\quad + \sum_{k=1}^n \frac{\omega_k}{d + \epsilon} \sigma_{k2} i_k dB_{k2}(t),
 \end{aligned} \tag{22}$$

where

$$\begin{aligned}
 LV &= \sum_{k=1}^n a_k (s_k + i_k) [-ds_k - (d + \epsilon) i_k] \\
 &\quad + \frac{1}{2} \sum_{k=1}^n a_k \left[ \sigma_{k1}^2 \left( s_k + \frac{b_k}{d} \right)^2 + \sigma_{k2}^2 i_k^2 \right] \\
 &\quad + \sum_{k=1}^n \frac{\omega_k}{d + \epsilon} \left[ \sum_{j=1}^n \beta_{kj} \left( s_k + \frac{b_k}{d} \right) i_j - (d + \epsilon) i_k \right] \\
 &= -\sum_{k=1}^n a_k \left[ \left( d - \frac{1}{2} \sigma_{k1}^2 \right) s_k^2 + \left( d + \epsilon - \frac{1}{2} \sigma_{k2}^2 \right) i_k^2 \right] \\
 &\quad + \sum_{k=1}^n \sum_{j \neq k} \frac{\omega_k}{d + \epsilon} \beta_{kj} s_k i_k \\
 &\quad - \sum_{k=1}^n \left[ a_k (2d + \epsilon) - \frac{\omega_k}{d + \epsilon} \beta_{kk} \right] s_k i_k \\
 &\quad + \sum_{k=1}^n \sum_{j=1}^n \frac{\omega_k}{d + \epsilon} \beta_{kj} \frac{b_k}{d} i_k - \sum_{k=1}^n \omega_k i_k \\
 &\quad + \frac{1}{2} \sum_{k=1}^n a_k \sigma_{k1}^2 \left( 2s_k \frac{b_k}{d} + \frac{b_k^2}{d^2} \right).
 \end{aligned} \tag{23}$$

Choose  $a_k = \omega_k \beta_{kk} / (2d + \epsilon)(d + \epsilon), k = 1, 2, \dots, n$ ; then  $a_k (2d + \epsilon) - (\omega_k / (d + \epsilon)) \beta_{kk} = 0$ . And we note that

$$\begin{aligned}
 &\sum_{k=1}^n \sum_{j \neq k} \frac{\omega_k}{d + \epsilon} \beta_{kj} s_k i_k + \sum_{k=1}^n \sum_{j=1}^n \frac{\omega_k}{d + \epsilon} \beta_{kj} \frac{b_k}{d} i_k \\
 &= (\omega_1, \omega_2, \dots, \omega_n) \\
 &\quad \times \begin{bmatrix} \frac{\beta_{11} S_1^0}{d + \epsilon} & \frac{\beta_{12} (S_1^0 + s_1)}{d + \epsilon} & \dots & \frac{\beta_{1n} (S_1^0 + s_1)}{d + \epsilon} \\ \frac{\beta_{21} (S_2^0 + s_2)}{d + \epsilon} & \frac{\beta_{22} S_2^0}{d + \epsilon} & \dots & \frac{\beta_{2n} (S_2^0 + s_2)}{d + \epsilon} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_{n1} (S_n^0 + s_n)}{d + \epsilon} & \frac{\beta_{n2} (S_n^0 + s_n)}{d + \epsilon} & \dots & \frac{\beta_{nn} S_n^0}{d + \epsilon} \end{bmatrix} \\
 &\quad \times \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix} \\
 &= \omega M(s) i.
 \end{aligned} \tag{24}$$

Then

$$\begin{aligned}
 LV &= - \sum_{k=1}^n a_k \left[ \left( d - \frac{1}{2} \sigma_{k1}^2 \right) s_k^2 + \left( d + \epsilon - \frac{1}{2} \sigma_{k2}^2 \right) i_k^2 \right] \\
 &\quad - \omega i + \omega M(s) i + \frac{1}{2} \sum_{k=1}^n a_k \sigma_{k1}^2 \left( 2s_k \frac{b_k}{d} + \frac{b_k^2}{d^2} \right) \\
 &\leq - \sum_{k=1}^n a_k \left[ \left( d - \frac{3}{4} \sigma_{k1}^2 \right) s_k^2 + \left( d + \epsilon - \frac{1}{2} \sigma_{k2}^2 \right) i_k^2 \right] \\
 &\quad + \omega (M(s) - 1) i + \frac{3}{2} \sum_{k=1}^n a_k \sigma_{k1}^2 \frac{b_k^2}{d^2} \\
 &= - \sum_{k=1}^n a_k \left[ \left( d - \frac{3}{4} \sigma_{k1}^2 \right) s_k^2 + \left( d + \epsilon - \frac{1}{2} \sigma_{k2}^2 \right) i_k^2 \right] \\
 &\quad + \omega (\rho(M(s)) - 1) i + \frac{3}{2} \sum_{k=1}^n a_k \sigma_{k1}^2 \frac{b_k^2}{d^2},
 \end{aligned} \tag{25}$$

where the last equality is derived from (20). Since  $-b_k/d \leq s_k \leq 0$ , then  $0 \leq M(s) \leq M(S^0) = (S_k^0 \beta_{kj} / (d + \epsilon))_{n \times n} = M_0$ , and so  $\rho(M(s)) \leq \rho(M_0)$  according to Lemma 2. Besides,  $R_0 \leq 1$ , and then  $\rho(M(s)) \leq 1$ . Therefore

$$\begin{aligned}
 dV &\leq \left[ - \sum_{k=1}^n a_k \left[ \left( d - \frac{3}{4} \sigma_{k1}^2 \right) s_k^2 + \left( d + \epsilon - \frac{1}{2} \sigma_{k2}^2 \right) i_k^2 \right] \right. \\
 &\quad \left. + \frac{3}{2} \sum_{k=1}^n a_k \sigma_{k1}^2 \frac{b_k^2}{d^2} \right] dt \\
 &\quad + \sum_{k=1}^n a_k (s_k + i_k) \sigma_{k1} \left( s_k + \frac{b_k}{d} \right) dB_{k1}(t) \\
 &\quad + \sum_{k=1}^n \left( a_k (s_k + i_k) \sigma_{k2} + \frac{\omega_k}{d + \epsilon} \sigma_{k2} \right) i_k dB_{k2}(t).
 \end{aligned} \tag{26}$$

Integrating both sides of (26) from 0 to  $t$ , and taking expectation, yields

$$\begin{aligned}
 0 &\leq E [V(s_1(t), i_1(t), \dots, s_n(t), i_n(t))] \\
 &\leq E [V(s_1(0), i_1(0), \dots, s_n(0), i_n(0))] \\
 &\quad + E \int_0^t \left[ - \sum_{k=1}^n a_k \left[ \left( d - \frac{3}{4} \sigma_{k1}^2 \right) s_k^2(r) \right. \right. \\
 &\quad \left. \left. + \left( d + \epsilon - \frac{1}{2} \sigma_{k2}^2 \right) i_k^2(r) \right] \right. \\
 &\quad \left. + \frac{3}{2} \sum_{k=1}^n a_k \sigma_{k1}^2 \frac{b_k^2}{d^2} \right] dr,
 \end{aligned} \tag{27}$$

which implies

$$\begin{aligned}
 E \int_0^t \left[ \sum_{k=1}^n a_k \left[ \left( d - \frac{3}{4} \sigma_{k1}^2 \right) s_k^2(r) + \left( d + \epsilon - \frac{1}{2} \sigma_{k2}^2 \right) i_k^2(r) \right] \right] dr \\
 \leq E [V(s_1(0), i_1(0), \dots, s_n(0), i_n(0))] + \frac{3}{2} \sum_{k=1}^n a_k \sigma_{k1}^2 \frac{b_k^2}{d^2} t.
 \end{aligned} \tag{28}$$

Therefore

$$\begin{aligned}
 \limsup_{t \rightarrow \infty} \frac{1}{t} E \int_0^t \left[ \sum_{k=1}^n a_k \left[ \left( d - \frac{3}{4} \sigma_{k1}^2 \right) s_k^2(r) \right. \right. \\
 \left. \left. + \left( d + \epsilon - \frac{1}{2} \sigma_{k2}^2 \right) i_k^2(r) \right] \right] dr \\
 \leq \frac{3}{2} \sum_{k=1}^n a_k \sigma_{k1}^2 \frac{b_k^2}{d^2};
 \end{aligned} \tag{29}$$

that is,

$$\begin{aligned}
 \limsup_{t \rightarrow \infty} \frac{1}{t} E \int_0^t \left[ \sum_{k=1}^n a_k \left[ \left( d - \frac{3}{4} \sigma_{k1}^2 \right) \left( S_k(r) - \frac{b_k}{d} \right)^2 \right. \right. \\
 \left. \left. + \left( d + \epsilon - \frac{1}{2} \sigma_{k2}^2 \right) I_k^2(r) \right] \right] dr \\
 \leq \frac{3}{2} \sum_{k=1}^n a_k \sigma_{k1}^2 \frac{b_k^2}{d^2}.
 \end{aligned} \tag{30}$$

If we let

$$K_1 = \min \left\{ a_k \left( d - \frac{3}{4} \sigma_{k1}^2 \right), d + \epsilon - \frac{1}{2} \sigma_{k2}^2 \right\}, \tag{31}$$

then

$$\begin{aligned}
 \limsup_{t \rightarrow \infty} \frac{1}{t} E \int_0^t \left[ \left( S_k(r) - \frac{b_k}{d} \right)^2 + I_k^2(r) \right] dr \\
 \leq \frac{3}{2} \sum_{k=1}^n \frac{a_k \sigma_{k1}^2 b_k^2}{d^2 K_1},
 \end{aligned} \tag{32}$$

as the theorem is proved.  $\square$

*Remark 4.* From Theorem 3, we can get the conclusion that the solution of the stochastic system will oscillate around the disease-free equilibrium of the deterministic model; the values of  $\sigma_{k1}$  and  $\sigma_{k2}$  have bearing on the intensity of turbulence. If the stochastic perturbations become small, the solution of system (5) will be close to the disease-free equilibrium of system (3).

Besides, if  $\sigma_{k1} = 0$ , then  $E_0$  is also the disease-free equilibrium of system (5). From the proof of Theorem 3, we can obtain

$$LV \leq - \sum_{k=1}^n a_k \left[ \left( d - \frac{3}{4} \sigma_{k1}^2 \right) s_k^2 + \left( d + \epsilon - \frac{1}{2} \sigma_{k2}^2 \right) i_k^2 \right] \leq 0. \tag{33}$$

Therefore,  $E_0$  is globally asymptotically stable.

### 4. The Dynamic of System (5) around the Endemic of System (3)

In the deterministic model, if  $R_0 > 1$ , there exists the endemic equilibrium  $E^*$ . But  $E^*$  is not the endemic equilibrium of stochastic system (5), because there is no endemic equilibrium for the stochastic system (5). In fact, we still want to find the relation between the solution of stochastic system and  $E^*$ .

Given a weighted digraph  $(\mathcal{G}, A)$  with  $n$  vertices, where  $A = (a_{kj})_{n \times n}$  is the weight matrix, whose entry  $a_{kj}$  equals the weight of arc  $(j, k)$  if it exists, and 0 otherwise, the Laplacian matrix of  $A$  is defined as

$$L_A = \begin{bmatrix} \sum_{k \neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{k \neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum_{k \neq n} a_{nk} \end{bmatrix}. \tag{34}$$

Let  $c_k$  denote the cofactor of the  $k$ th diagonal element of  $L_A$ , and we have the following results.

**Theorem 5.** Assume  $A = (\beta_{kj})_{n \times n}$  is irreducible and  $R_0 > 1$ . For any given initial value  $(S_1(0), I_1(0), \dots, S_n(0), I_n(0)) \in \mathbb{R}_+^{2n}$ , the solution of system (5) has the property

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{k=1}^n \int_0^t [p_k d(S_k - S_k^*)^2 + m_k (d + \epsilon) (I_k - I_k^*)^2] ds \\ & \leq \sum_{k=1}^n \left[ \left( \frac{a \bar{c}_k b_k}{d} + \frac{(m_k + p_k) b_k^2}{d^2} \right) \sigma_{k1}^2 \right. \\ & \quad \left. + \left( \frac{(a + 1) \bar{c}_k b_k}{d} + \frac{m_k b_k^2}{d^2} \right) \sigma_{k2}^2 \right], \quad a.s., \end{aligned} \tag{35}$$

where  $E^* = (S_1^*, I_1^*, \dots, S_n^*, I_n^*)$  is the endemic equilibrium of system (3) and  $\bar{c}_k, k = 1, 2, \dots, n$ , denote the cofactor of the  $k$ th diagonal element of  $L_{\bar{A}} (\bar{A} = (\bar{\beta}_{kj})_{n \times n} = (\beta_{kj} S_k^* I_j^*)_{n \times n})$ , and  $a, m_k, p_k, k = 1, 2, \dots, n$ , are positive constants defined as in the proof.

*Proof.* Since  $E^*$  is the endemic equilibrium of system (3), we have

$$\sum_{j=1}^n \beta_{kj} S_k^* I_j^* + d S_k^* = b_k, \quad \sum_{j=1}^n \beta_{kj} S_k^* I_j^* = (d + \epsilon) I_k^*. \tag{36}$$

Define

$$\begin{aligned} V(S_1, I_1, \dots, S_n, I_n) & = a \sum_{k=1}^n \bar{c}_k \left( S_k - S_k^* - S_k^* \ln \frac{S_k}{S_k^*} + I_k - I_k^* - I_k^* \ln \frac{I_k}{I_k^*} \right) \end{aligned}$$

$$\begin{aligned} & + \sum_{k=1}^n \bar{c}_k \left( I_k - I_k^* - I_k^* \ln \frac{I_k}{I_k^*} \right) \\ & + \frac{1}{2} \sum_{k=1}^n m_k (S_k - S_k^* + I_k - I_k^*)^2 + \frac{1}{2} \sum_{k=1}^n p_k (S_k - S_k^*)^2 \\ & := aV_1 + V_2 + V_3 + V_4, \end{aligned} \tag{37}$$

where  $a, m_k, p_k, k = 1, 2, \dots, n$ , are positive constants to be determined later. From the property (1) of Lemma A.2 (see [20]), we know  $\bar{c}_k > 0, k = 1, 2, \dots, n$ . Hence  $V$  is positive definite. Let  $L$  be the generating operator of system (5). Then we get

$$\begin{aligned} LV_1 & = \sum_{k=1}^n \bar{c}_k \left( 1 - \frac{S_k^*}{S_k} \right) \left( b_k - \sum_{j=1}^n \beta_{kj} S_k I_j - d S_k \right) \\ & + \sum_{k=1}^n \frac{\bar{c}_k S_k^* \sigma_{k1}^2}{2} \\ & + \sum_{k=1}^n \bar{c}_k \left( 1 - \frac{I_k^*}{I_k} \right) \left( \sum_{j=1}^n \beta_{kj} S_k I_j - (d + \epsilon) I_k \right) \\ & + \sum_{k=1}^n \frac{\bar{c}_k I_k^* \sigma_{k2}^2}{2} \\ & = \sum_{k=1}^n \bar{c}_k \left[ b_k - d S_k - b_k \frac{S_k^*}{S_k} + \sum_{j=1}^n \beta_{kj} S_k^* I_j + d S_k^* \right. \\ & \quad \left. - (d + \epsilon) I_k + (d + \epsilon) I_k^* - \sum_{j=1}^n \beta_{kj} \frac{S_k I_j I_k^*}{I_k} \right. \\ & \quad \left. + \frac{1}{2} (S_k^* \sigma_{k1}^2 + I_k^* \sigma_{k2}^2) \right] \\ & = \sum_{k=1}^n \bar{c}_k \left[ \sum_{j=1}^n \bar{\beta}_{kj} + d S_k^* - d S_k - \sum_{j=1}^n \bar{\beta}_{kj} \frac{S_k^*}{S_k} - d S_k^* \frac{S_k}{S_k} \right. \\ & \quad \left. + \sum_{j=1}^n \bar{\beta}_{kj} \frac{I_j}{I_j^*} + d S_k^* - \sum_{j=1}^n \bar{\beta}_{kj} \frac{I_k}{I_k^*} - \sum_{j=1}^n \bar{\beta}_{kj} \frac{S_k I_j I_k^*}{I_k S_k^* I_j^*} \right. \\ & \quad \left. + \sum_{j=1}^n \bar{\beta}_{kj} + \frac{1}{2} (S_k^* \sigma_{k1}^2 + I_k^* \sigma_{k2}^2) \right] \\ & = \sum_{k=1}^n \bar{c}_k \left[ -d S_k^* \left( \frac{S_k^*}{S_k} + \frac{S_k}{S_k^*} - 2 \right) + \sum_{j=1}^n \bar{\beta}_{kj} \left( \frac{I_j}{I_j^*} - \frac{I_k}{I_k^*} \right) \right. \\ & \quad \left. + \sum_{j=1}^n \bar{\beta}_{kj} \left( 2 - \frac{S_k^*}{S_k} - \frac{S_k I_j I_k^*}{I_k S_k^* I_j^*} \right) \right. \\ & \quad \left. + \frac{1}{2} (S_k^* \sigma_{k1}^2 + I_k^* \sigma_{k2}^2) \right]. \end{aligned} \tag{38}$$

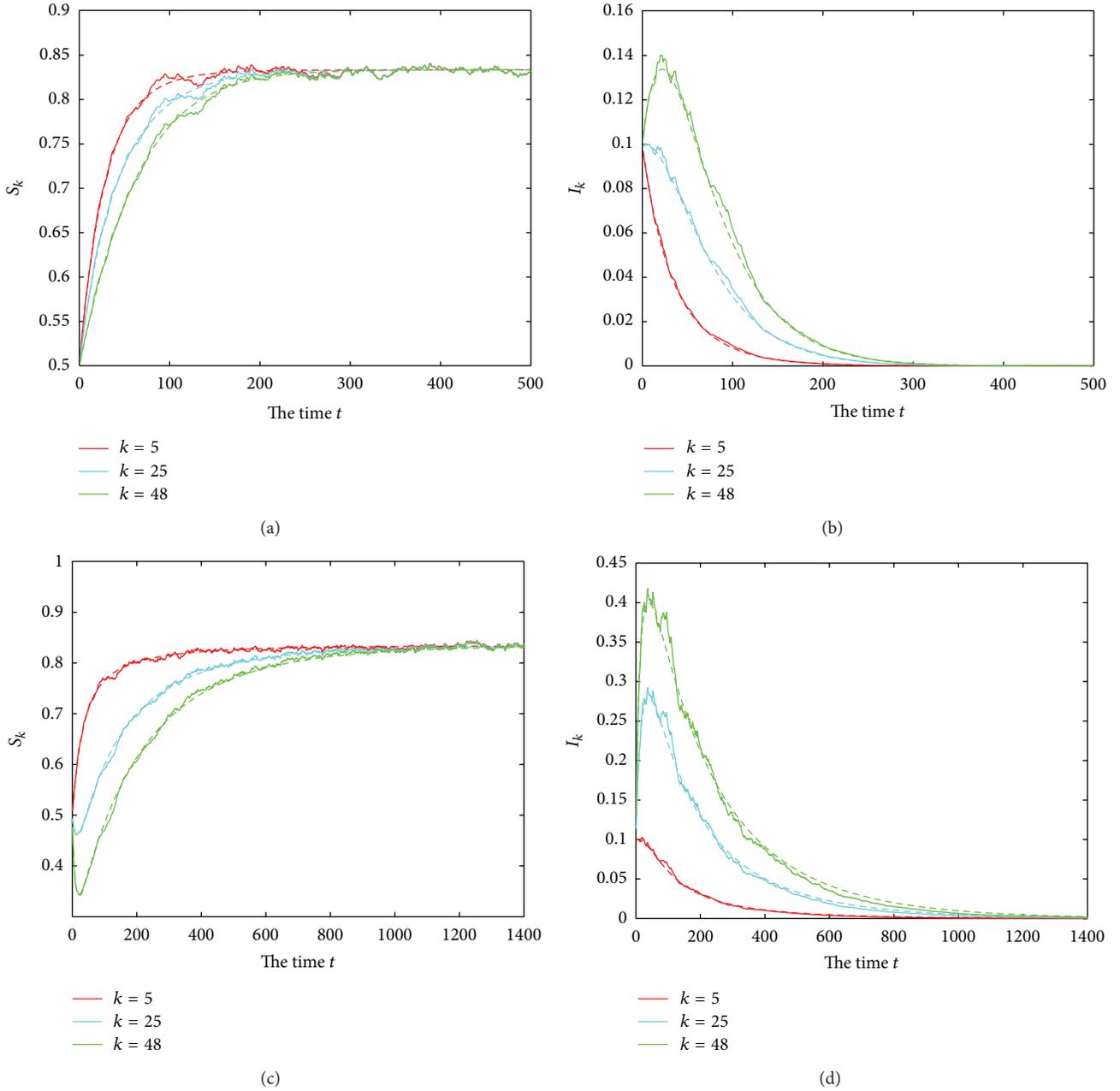


FIGURE 1:  $\lambda = 0.025, \sigma_{k1} = 0.005, \sigma_{k2} = 0.03, R_0 \leq 1$ . (a), (b)  $P(k) = m^k \exp(-m)/k!, m = 6, R_0 = 0.4704 \leq 1$ ; (c), (d)  $P(k) = 2m^2 k^{-3}, m = 3, R_0 = 0.9071 \leq 1$ .

$$\begin{aligned}
 LV_2 &= \sum_{k=1}^n \bar{c}_k \left( 1 - \frac{I_k^*}{I_k} \right) \left( \sum_{j=1}^n \beta_{kj} S_k I_j - (d + \epsilon) I_k \right) \\
 &\quad + \sum_{k=1}^n \frac{\bar{c}_k I_k^* \sigma_{k2}^2}{2} \\
 &= \sum_{k=1}^n \bar{c}_k \left[ \sum_{j=1}^n \beta_{kj} S_k I_j - (d + \epsilon) I_k - \frac{I_k^*}{I_k} \sum_{j=1}^n \beta_{kj} S_k I_j \right. \\
 &\quad \left. + (d + \epsilon) I_k^* + \frac{I_k^* \sigma_{k2}^2}{2} \right] \\
 &= \sum_{k=1}^n \bar{c}_k \left[ \sum_{j=1}^n \beta_{kj} S_k I_j - \sum_{j=1}^n \bar{\beta}_{kj} \frac{I_k}{I_k^*} - \sum_{j=1}^n \bar{\beta}_{kj} \frac{S_k I_j I_k^*}{I_k S_k^* I_j^*} \right. \\
 &\quad \left. + \sum_{j=1}^n \bar{\beta}_{kj} + \frac{I_k^* \sigma_{k2}^2}{2} \right]. \tag{39}
 \end{aligned}$$

$$\begin{aligned}
 LV_3 &= \sum_{k=1}^n m_k (S_k - S_k^* + I_k - I_k^*) (b_k - d S_k - (d + \epsilon) I_k) \\
 &\quad + \sum_{k=1}^n m_k \frac{\sigma_{k1}^2 S_k^2 + \sigma_{k2}^2 I_k^2}{2}
 \end{aligned}$$

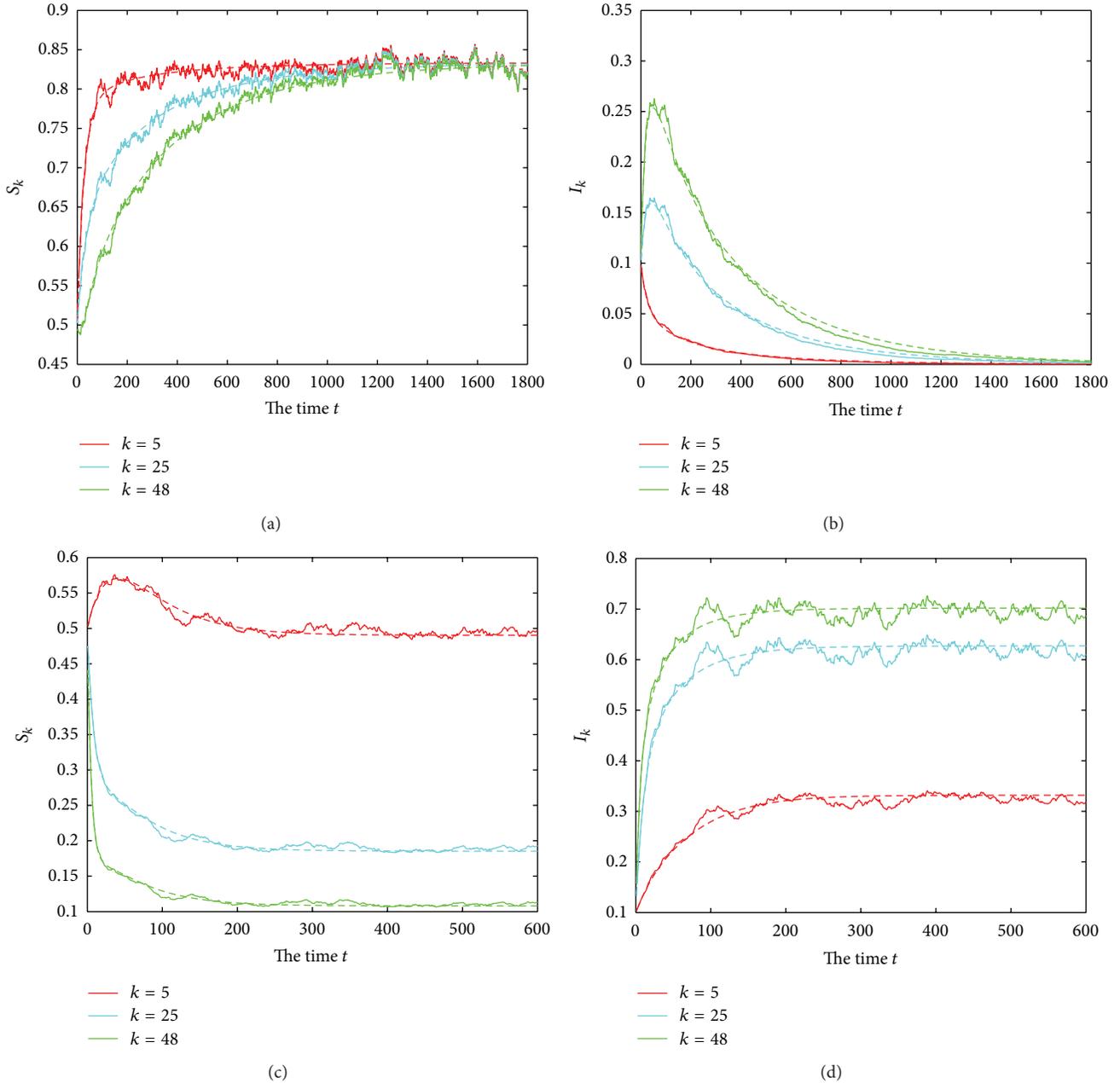


FIGURE 2:  $\lambda = 0.05, \sigma_{k1} = 0.01, \sigma_{k2} = 0.02$ . (a), (b)  $P(k) = m^k \exp(-m)/k!, m = 6, R_0 = 0.9409 \leq 1$ ; (c), (d)  $P(k) = 2m^2 k^{-3}, m = 3, R_0 = 1.8142 \geq 1$ .

$$\begin{aligned}
 &= \sum_{k=1}^n m_k (S_k - S_k^* + I_k - I_k^*) \\
 &\quad \times (-d(S_k - S_k^*) - (d + \epsilon)(I_k - I_k^*)) \\
 &\quad + \sum_{k=1}^n m_k \frac{\sigma_{k1}^2 S_k^2 + \sigma_{k2}^2 I_k^2}{2} \\
 &= \sum_{k=1}^n m_k \left[ -d(S_k - S_k^*)^2 - (d + \epsilon)(I_k - I_k^*)^2 \right. \\
 &\quad \left. - (2d + \epsilon)(S_k - S_k^*)(I_k - I_k^*) \right. \\
 &\quad \left. + \frac{\sigma_{k1}^2 S_k^2 + \sigma_{k2}^2 I_k^2}{2} \right] \\
 &\leq \sum_{k=1}^n m_k \left[ -\left( d - \frac{(2d + \epsilon)^2}{2(d + \epsilon)} \right) (S_k - S_k^*)^2 \right. \\
 &\quad \left. - \frac{(d + \epsilon)}{2} (I_k - I_k^*)^2 + \frac{\sigma_{k1}^2 S_k^2 + \sigma_{k2}^2 I_k^2}{2} \right] \\
 &= \sum_{k=1}^n m_k \left[ \frac{(d + \epsilon)^2 + d^2}{2(d + \epsilon)} (S_k - S_k^*)^2 - \frac{(d + \epsilon)}{2} (I_k - I_k^*)^2 \right. \\
 &\quad \left. + \frac{\sigma_{k1}^2 S_k^2 + \sigma_{k2}^2 I_k^2}{2} \right].
 \end{aligned}$$

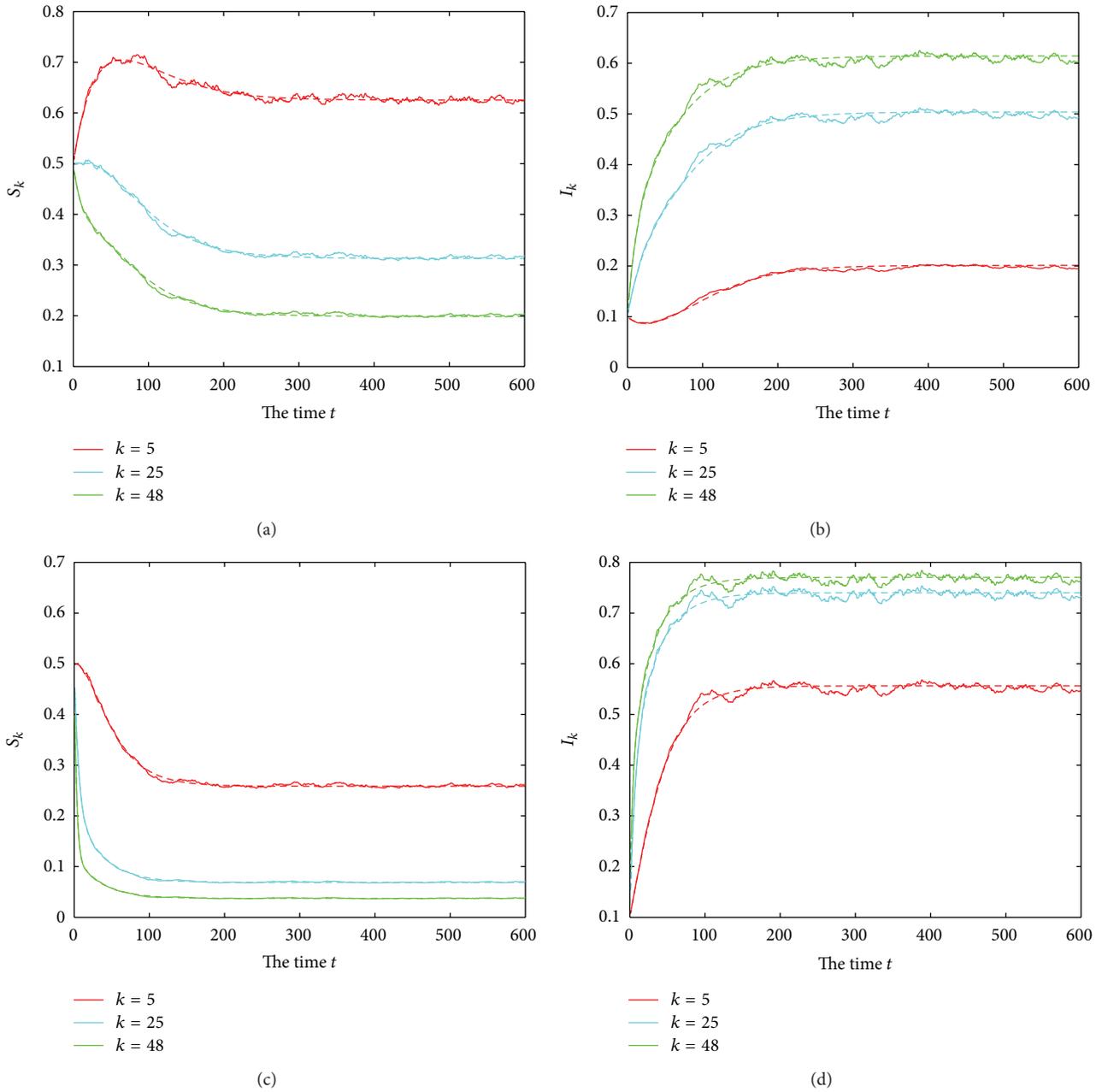


FIGURE 3:  $\lambda = 0.08$ ,  $\sigma_{k1} = 0.01$ ,  $\sigma_{k2} = 0.01$ ,  $R_0 \geq 1$ . (a), (b)  $P(k) = m^k \exp(-m)/k!$ ,  $m = 6$ ,  $R_0 = 1.5054 \geq 1$ ; (c), (d)  $P(k) = 2m^2 k^{-3}$ ,  $m = 3$ ,  $R_0 = 2.9027 \geq 1$ .

$$\begin{aligned}
 LV_4 &= \sum_{k=1}^n p_k (S_k - S_k^*) \left( b_k - \sum_{j=1}^n \beta_{kj} S_k I_j - d S_k \right) + \sum_{k=1}^n p_k \frac{\sigma_{k1}^2 S_k^2}{2} \\
 &+ \sum_{k=1}^n p_k \frac{\sigma_{k1}^2 S_k^2}{2} \\
 &= \sum_{k=1}^n p_k (S_k - S_k^*) \\
 &\quad \times \left( \sum_{j=1}^n \beta_{kj} (S_k^* I_j^* - S_k I_j) - d (S_k - S_k^*) \right) \\
 &= - \sum_{k=1}^n p_k \left[ \sum_{j=1}^n \beta_{kj} S_k^* (S_k - S_k^*) (I_j - I_j^*) \right. \\
 &\quad \left. + \sum_{j=1}^n \beta_{kj} I_j (S_k - S_k^*)^2 + d (S_k - S_k^*)^2 - \frac{\sigma_{k1}^2 S_k^2}{2} \right]. \tag{40}
 \end{aligned}$$

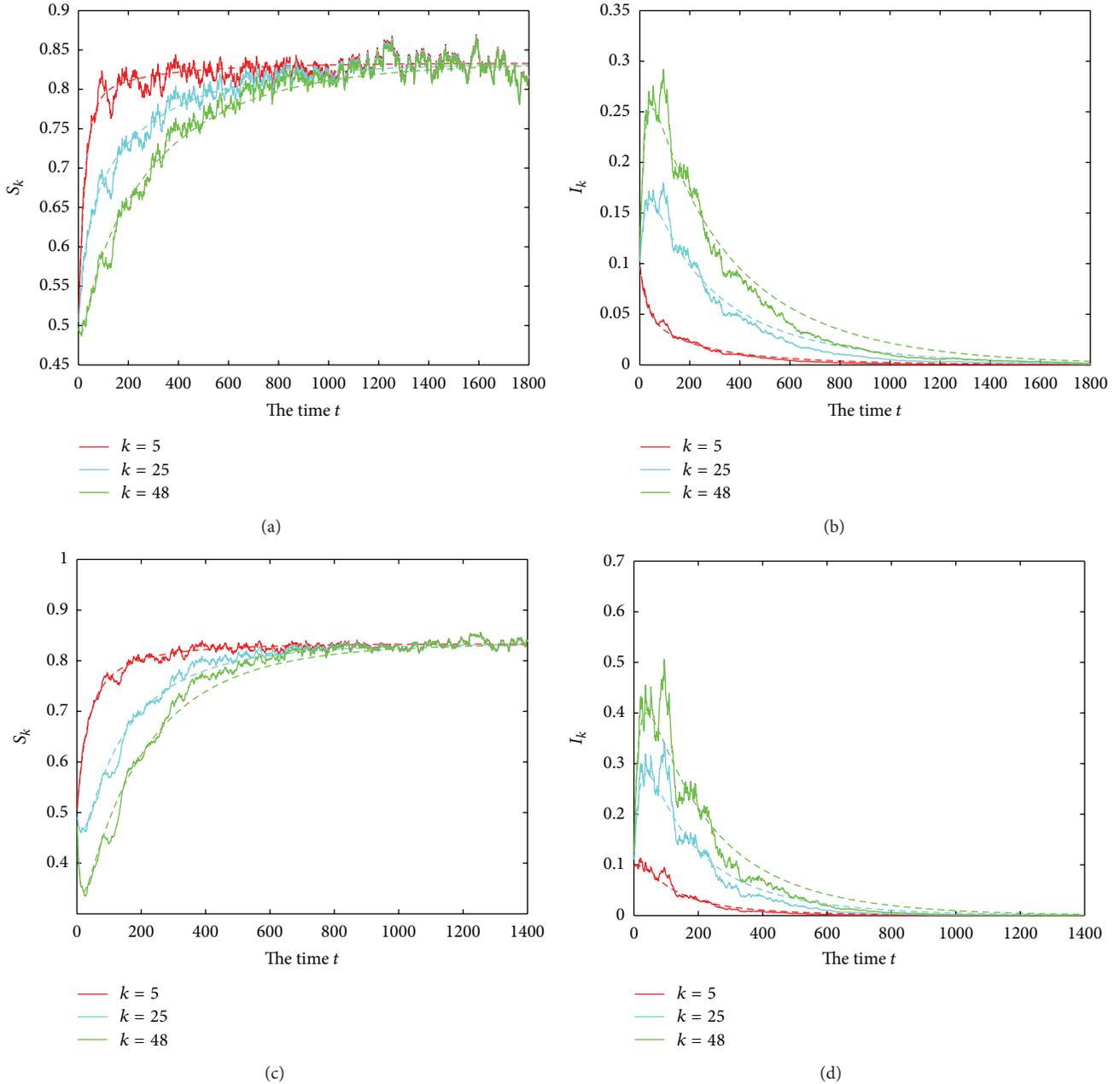


FIGURE 4: (a), (b)  $\lambda = 0.05, \sigma_{k1} = 0.015, \sigma_{k2} = 0.05, P(k) = m^k \exp(-m)/k!, m = 6, R_0 = 0.949 \leq 1$ . (c), (d)  $\lambda = 0.025, \sigma_{k1} = 0.01, \sigma_{k2} = 0.1, P(k) = 2m^2 k^{-3}, m = 3, R_0 = 0.9071 \leq 1$ .

By property (2) of Lemma A.2 (see [20]), we know

$$\sum_{k=1}^n \bar{c}_k \left( \sum_{j=1}^n \bar{\beta}_{kj} \frac{I_j}{I_j^*} - \sum_{j=1}^n \bar{\beta}_{kj} \frac{I_k}{I_k^*} \right) = 0, \tag{41}$$

$$\sum_{k=1}^n \bar{c}_k \left( \sum_{j=1}^n \bar{\beta}_{kj} \ln \frac{I_j}{I_j^*} - \sum_{j=1}^n \bar{\beta}_{kj} \ln \frac{I_k}{I_k^*} \right) = 0.$$

Besides, note that  $x - 1 - \ln x \geq 0$  for  $x > 0$ ; then

$$\frac{S_k^*}{S_k} \geq 1 + \ln \frac{S_k^*}{S_k}, \quad \frac{S_k I_j I_k^*}{I_k S_k^* I_j^*} \geq 1 + \ln \frac{S_k I_j I_k^*}{I_k S_k^* I_j^*}. \tag{42}$$

According to (41) and (42), we get

$$\sum_{k=1}^n \bar{c}_k \sum_{j=1}^n \bar{\beta}_{kj} \left( 2 - \frac{S_k^*}{S_k} - \frac{S_k I_j I_k^*}{I_k S_k^* I_j^*} \right)$$

$$\leq \sum_{k=1}^n \bar{c}_k \sum_{j=1}^n \bar{\beta}_{kj} \left[ 2 - \left( 1 + \ln \frac{S_k^*}{S_k} \right) - \left( 1 + \ln \frac{S_k I_j I_k^*}{I_k S_k^* I_j^*} \right) \right]$$

$$= \sum_{k=1}^n \bar{c}_k \sum_{j=1}^n \bar{\beta}_{kj} \left( \ln \frac{I_k}{I_k^*} - \ln \frac{I_j}{I_j^*} \right) = 0, \tag{43}$$

$$\begin{aligned}
 \sum_{k=1}^n \bar{c}_k \sum_{j=1}^n \bar{\beta}_{kj} \frac{S_k I_j I_k^*}{I_k S_k^* I_j^*} &\geq \sum_{k=1}^n \bar{c}_k \sum_{j=1}^n \bar{\beta}_{kj} \left( 1 + \ln \frac{S_k I_j I_k^*}{I_k S_k^* I_j^*} \right) \\
 &= \sum_{k=1}^n \bar{c}_k \sum_{j=1}^n \bar{\beta}_{kj} \left( 1 - \ln \frac{S_k^*}{S_k} \right) \\
 &\quad + \sum_{k=1}^n \bar{c}_k \sum_{j=1}^n \bar{\beta}_{kj} \left( \ln \frac{I_j}{I_j^*} - \ln \frac{I_k}{I_k^*} \right) \\
 &\geq \sum_{k=1}^n \bar{c}_k \sum_{j=1}^n \bar{\beta}_{kj} \left( 2 - \frac{S_k^*}{S_k} \right).
 \end{aligned} \tag{44}$$

Substituting (41) and (43) into (38), we get

$$\begin{aligned}
 LV_1 &\leq \sum_{k=1}^n \bar{c}_k \left[ -d S_k^* \left( \frac{S_k^*}{S_k} + \frac{S_k}{S_k^*} - 2 \right) + \frac{1}{2} (S_k^* \sigma_{k1}^2 + I_k^* \sigma_{k2}^2) \right] \\
 &= - \sum_{k=1}^n \bar{c}_k d \frac{(S_k - S_k^*)^2}{S_k} + \sum_{k=1}^n \frac{\bar{c}_k}{2} (S_k^* \sigma_{k1}^2 + I_k^* \sigma_{k2}^2).
 \end{aligned} \tag{45}$$

Substituting (44) into (39), we get

$$\begin{aligned}
 LV_2 &\leq \sum_{k=1}^n \bar{c}_k \left[ \sum_{j=1}^n \beta_{kj} S_k I_j - \sum_{j=1}^n \bar{\beta}_{kj} \frac{I_k}{I_k^*} - \sum_{j=1}^n \bar{\beta}_{kj} \left( 2 - \frac{S_k^*}{S_k} \right) \right. \\
 &\quad \left. + \sum_{j=1}^n \bar{\beta}_{kj} + \frac{I_k^* \sigma_{k2}^2}{2} \right] \\
 &= \sum_{k=1}^n \bar{c}_k \left[ \sum_{j=1}^n \beta_{kj} (S_k - S_k^*) (I_j - I_j^*) \right. \\
 &\quad \left. - \sum_{j=1}^n \bar{\beta}_{kj} \left( \frac{I_k}{I_k^*} - \frac{I_j}{I_j^*} \right) \right. \\
 &\quad \left. - \sum_{j=1}^n \bar{\beta}_{kj} \left( 2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \right) + \frac{I_k^* \sigma_{k2}^2}{2} \right] \\
 &= \sum_{k=1}^n \bar{c}_k \left[ \sum_{j=1}^n \beta_{kj} (S_k - S_k^*) (I_j - I_j^*) \right. \\
 &\quad \left. + \sum_{j=1}^n \beta_{kj} I_j^* \frac{(S_k - S_k^*)^2}{S_k} + \frac{I_k^* \sigma_{k2}^2}{2} \right].
 \end{aligned} \tag{46}$$

Therefore,

$$\begin{aligned}
 LV &= aLV_1 + LV_2 + LV_3 + LV_4 \\
 &\leq - \sum_{k=1}^n \left[ p_k d - m_k \frac{(d + \epsilon)^2 + d^2}{2(d + \epsilon)} \right] (S_k - S_k^*)^2
 \end{aligned}$$

$$\begin{aligned}
 &- \frac{1}{2} \sum_{k=1}^n m_k (d + \epsilon) (I_k - I_k^*)^2 \\
 &- \sum_{k=1}^n \bar{c}_k \left( ad - \sum_{j=1}^n \beta_{kj} I_j^* \right) \frac{(S_k - S_k^*)^2}{S_k} \\
 &+ \sum_{k=1}^n \sum_{j=1}^n \beta_{kj} (\bar{c}_k - p_k S_k^*) (S_k - S_k^*) (I_j - I_j^*) \\
 &- \sum_{k=1}^n \sum_{j=1}^n p_k \beta_{kj} I_j (S_k - S_k^*)^2 \\
 &+ \sum_{k=1}^n \frac{a \bar{c}_k S_k^* + (m_k + p_k) S_k^2}{2} \sigma_{k1}^2 \\
 &+ \sum_{k=1}^n \frac{(a + 1) \bar{c}_k I_k^* + m_k I_k^2}{2} \sigma_{k2}^2 \\
 &\leq - \sum_{k=1}^n \left[ p_k d - m_k \frac{(d + \epsilon)^2 + d^2}{2(d + \epsilon)} \right] (S_k - S_k^*)^2 \\
 &- \frac{1}{2} \sum_{k=1}^n m_k (d + \epsilon) (I_k - I_k^*)^2 \\
 &- \sum_{k=1}^n \bar{c}_k \left( ad - \sum_{j=1}^n \beta_{kj} I_j^* \right) \frac{(S_k - S_k^*)^2}{S_k} \\
 &+ \sum_{k=1}^n \sum_{j=1}^n \beta_{kj} (\bar{c}_k - p_k S_k^*) (S_k - S_k^*) (I_j - I_j^*) \\
 &+ \sum_{k=1}^n \frac{a \bar{c}_k S_k^* + (m_k + p_k) S_k^2}{2} \sigma_{k1}^2 \\
 &+ \sum_{k=1}^n \frac{(a + 1) \bar{c}_k I_k^* + m_k I_k^2}{2} \sigma_{k2}^2.
 \end{aligned} \tag{47}$$

Choose  $a = \max\{(\sum_{j=1}^n \beta_{kj} I_j^*)/d, k = 1, 2, \dots, n\}$ ,  $m_k = (d + \epsilon)p_k d / ((d + \epsilon)^2 + d^2)$ ,  $p_k = \bar{c}_k / S_k^*$ ,  $k = 1, 2, \dots, n$ ; then

$$\begin{aligned}
 LV &\leq - \sum_{k=1}^n \frac{p_k d}{2} (S_k - S_k^*)^2 - \frac{1}{2} \sum_{k=1}^n m_k (d + \epsilon) (I_k - I_k^*)^2 \\
 &+ \frac{1}{2} \sum_{k=1}^n \left[ \left( \frac{a \bar{c}_k b}{d} + \frac{(m_k + p_k) b^2}{d^2} \right) \sigma_{k1}^2 \right. \\
 &\quad \left. + \left( \frac{(a + 1) \bar{c}_k b}{d} + \frac{m_k b^2}{d^2} \right) \sigma_{k2}^2 \right] \\
 &:= F(t).
 \end{aligned} \tag{48}$$

Therefore,

$$\begin{aligned}
 dV \leq F(t) dt &+ \sum_{k=1}^n [[a\bar{c}_k \sigma_{k1} (S_k - S_k^*) + (m_k + p_k) \sigma_{k1} (S_k - S_k^*) S_k] \\
 &\times dB_{k1}(t) \\
 &+ [\bar{c}_k (a + 1) \sigma_{k2} (I_k - I_k^*) + m_k \sigma_{k2} (I_k - I_k^*) I_k] \\
 &\times dB_{k2}(t)].
 \end{aligned} \tag{49}$$

Integrating both sides of (49) from 0 to  $t$  yields

$$\begin{aligned}
 V(t) - V(0) &\leq \int_0^t F(s) ds \\
 &+ \int_0^t \sum_{k=1}^n [a\bar{c}_k \sigma_{k1} (S_k - S_k^*) + (m_k + p_k) \sigma_{k1} (S_k - S_k^*) S_k] \\
 &\quad \times dB_{k1}(s) \\
 &+ \int_0^t \sum_{k=1}^n [\bar{c}_k (a + 1) \sigma_{k2} (I_k - I_k^*) + m_k \sigma_{k2} (I_k - I_k^*) I_k] \\
 &\quad \times dB_{k2}(s).
 \end{aligned} \tag{50}$$

Let  $M_1(t) := \int_0^t \sum_{k=1}^n [a\bar{c}_k \sigma_{k1} (S_k - S_k^*) + (m_k + p_k) \sigma_{k1} (S_k - S_k^*) S_k] dB_{k1}(s)$ ,  $M_2(t) := \int_0^t \sum_{k=1}^n [\bar{c}_k (a + 1) \sigma_{k2} (I_k - I_k^*) + m_k \sigma_{k2} (I_k - I_k^*) I_k] dB_{k2}(s)$ , which are local continuous martingale, and  $M_1(0) = M_2(0) = 0$ . Moreover

$$\begin{aligned}
 \limsup_{t \rightarrow \infty} \frac{\langle M_1, M_1 \rangle_t}{t} &\leq 8 \sum_{k=1}^n \sigma_{k1}^2 \left[ a^2 \bar{c}_k^2 + (m_k + p_k)^2 \frac{b_k^2}{d^2} \right] \frac{b_k^2}{d^2} \\
 &< \infty, \\
 \limsup_{t \rightarrow \infty} \frac{\langle M_2, M_2 \rangle_t}{t} &\leq 8 \sum_{k=1}^n \sigma_{k2}^2 \left[ \bar{c}_k^2 (a + 1)^2 + m_k^2 \frac{b_k^2}{d^2} \right] \frac{b_k^2}{d^2} < \infty.
 \end{aligned} \tag{51}$$

By Lemma A.4 (see [20]), we obtain

$$\lim_{t \rightarrow \infty} \frac{M_1(t)}{t} = 0, \quad \lim_{t \rightarrow \infty} \frac{M_2(t)}{t} = 0 \quad \text{a.s.}, \tag{52}$$

which together with (50) implies

$$\liminf_{t \rightarrow \infty} \frac{\int_0^t F(s) ds}{t} \geq 0 \quad \text{a.s.} \tag{53}$$

Consequently,

$$\begin{aligned}
 \limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{k=1}^n \int_0^t [p_k d(S_k - S_k^*)^2 + m_k (d + \epsilon) (I_k - I_k^*)^2] ds \\
 \leq \sum_{k=1}^n \left[ \left( \frac{a\bar{c}_k b_k}{d} + \frac{(m_k + p_k) b_k^2}{d^2} \right) \sigma_{k1}^2 \right. \\
 \left. + \left( \frac{(a + 1) \bar{c}_k b_k}{d} + \frac{m_k b_k^2}{d^2} \right) \sigma_{k2}^2 \right], \quad \text{a.s.}
 \end{aligned} \tag{54}$$

Thus Theorem 5 is proved.  $\square$

*Remark 6.* Theorem 5 shows that the solution of system (5) fluctuates around the certain level which is relevant to  $E^*$  of system (3) and  $\sigma_{k1}^2, \sigma_{k2}^2, k = 1, 2, \dots, n$ . The distance between the solution  $X(t) = (S_1(t), I_1(t), \dots, S_n(t), I_n(t))$  and  $E^*$  of system (3) has the following form:

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|X(s) - E^*\|^2 ds \leq C \|\sigma\|^2, \tag{55}$$

where  $C$  is a positive constant and  $\|\sigma\|^2 = \sum_{k=1}^n (\sigma_{k1}^2 + \sigma_{k2}^2)$ . Although the solution of system (5) does not have stability as the deterministic system, we can draw a conclusion that system (5) is persistent on the basis of the result of Theorem 5, which also accounts for the fact that the disease is prevalent.

### 5. Simulations and Conclusions

*5.1. Numerical Simulations.* In order to confirm the results above, we numerically simulate the solution of system (5) with  $n = 50, b_k = 0.25, d = 0.3, \epsilon = 0.01$ , and initial value  $S_k(0) = 0.5, I_k(0) = 0.1, k = 1, 2, \dots, 50$ . Using Milstein's Higher Order Method [31], we get the discretization equation:

$$\begin{aligned}
 S_{k,i+1} &= S_{k,i} + \Delta t \left( b_k - \sum_{j=1}^2 \beta_{kj} S_{k,i} I_{j,i} - d S_{k,i} \right) \\
 &\quad + \sigma_{k1} S_{k,i} \sqrt{\Delta t} \xi_{k1,i} + \frac{\sigma_{k1}^2}{2} S_{k,i} \Delta t (\xi_{k1,i}^2 - 1), \\
 I_{k,i+1} &= I_{k,i} + \Delta t \left( \sum_{j=1}^2 \beta_{kj} S_{k,i} I_{j,i} - (d + \epsilon) I_{k,i} \right) \\
 &\quad + \sigma_{k2} I_{k,i} \sqrt{\Delta t} \xi_{k2,i} + \frac{\sigma_{k2}^2}{2} I_{k,i} \Delta t (\xi_{k2,i}^2 - 1),
 \end{aligned} \tag{56}$$

where  $k = 1, 2, \dots, n$  and  $\xi_{k1,i}, \xi_{k2,i}, i = 1, 2, \dots, N$ , are the independent Gaussian random variables  $N(0, 1)$ .

From Theorem 3 and Remark 4, it is shown that the expectations of  $S_k(t), I_k(t), k = 1, 2, \dots, n$ , are converging under some conditions, and the solution of system (5) will oscillate around the disease-free equilibrium of system (3). In Figure 1, we choose parameters  $\lambda = 0.025, \sigma_{k1} = 0.005$ , and  $\sigma_{k2} = 0.03$ , such that  $R_0 \leq 1$ , and in Figures 1(a) and 1(b) we choose  $P(k) = m^k \exp(-m)/k!$ , and in Figures 1(c) and 1(d)

we choose  $P(k) = 2m^2k^{-3}$ . From Figure 1, we can see that the disease-free equilibrium  $E_0$  of system (3) (imaginary lines) is globally asymptotically stable and the curves of system (5) (real lines) always fluctuate around the curves of system (3) (imaginary lines). From Figure 2, we can see that, due to the difference of the degree distribution, the critical value of spread is different.

In Figure 3, parameters  $\lambda = 0.08$ ,  $\sigma_{k1} = 0.01$ , and  $\sigma_{k2} = 0.01$  and others are the same as the previous. From Figure 3, we can see that the position of the equilibrium state is different due to the difference of the degree distribution. From Figures 1(d), 2(b), 4(b), and 4(d), we found that the solution of stochastic system converging to the disease-free equilibrium is faster than that of the deterministic system with the increase of noise intensity.

**5.2. Conclusions.** The numerical simulations illustrate the mathematical theorems well. Due to the existence of the noise, the solution of the stochastic system goes around the solution of the deterministic system. With intensities decreasing, the turbulence intensity is weaker. From numerical simulations, we have a new discovery. When  $R_0 \leq 1$ , with the increase of noise intensity, the solution of stochastic system converging to the disease-free equilibrium is faster than the deterministic system. This is because of the fact that, when  $R_0 \leq 1$ , the disease will die out after some time. However in the real world many stochastic factors contributed to the extinction of the disease.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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