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Research Article

Some Antiperiodic Boundary Value Problem for Nonlinear Fractional Impulsive Differential Equations

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This paper is concerned with the sufficient conditions for the existence of solutions for a class of generalized antiperiodic boundary value problem for nonlinear fractional impulsive differential equations involving the Riemann-Liouville fractional derivative. Firstly, we introduce the fractional calculus and give the generalized R-L fractional integral formula of R-L fractional derivative involving impulsive. Secondly, the sufficient condition for the existence and uniqueness of solutions is presented. Finally, we give some examples to illustrate our main results.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration; it is also as old as ordinary differential calculus. For the last decades, fractional differential equations have been receiving intensive attention because they provide an excellent tool for the description of memory and hereditary properties of various materials and processes, such as physics, mechanics, chemistry, and engineering; for more details, one can see Kilbas et al. [1] and Podlubny [2] and the references therein.

There have been considerable developments in the theory of impulsive differential equations in the last few decades. Impulsive differential equations have become more important in some mathematical models of real phenomena, especially in control, biology, medicine, and information (see [3, 4]). So the study of fractional impulsive differential equations is a more meaningful work. Some significant developments in fractional impulsive differential equations with Caputo derivative have been presented [5–27]. Recently, Fečkan et al. defined the solutions for fractional impulsive differential equations with Caputo derivative (for more details, see [17]). They considered the Cauchy problems for the following impulsive fractional differential equations:

$$\begin{gathered} {}^{c}D_{t}^{q}u(t) = f(t,u(t)), \quad t \in J', \\ u(t_k^+) = u(t_k^-) + y_k, \quad k = 1,\dots,m, \\ u(0) = u_0, \end{gathered} \tag{1}$$

where y_k $(k = 1,\dots,m)$, u_0 are constants. ${}^{c}D_{t}^{q}$ $(0 < q < 1)$ denotes Caputo's fractional derivative. Some sufficient conditions for existence of the solutions have been established by applying Schaefer's fixed point theorem, Banach fixed point theorem, and the theorem of nonlinear alternative of Leray-Schauder type.

But as far as we know, there are few papers that consider the fractional impulsive differential equations with Riemann-Liouville derivative (only see [15, 24]).

Motivated by [15, 17, 24] and some related literature, we study the existence and uniqueness of solutions for the generalized antiperiodic boundary value problem for fractional differential equations with impulsive effects

$$\begin{aligned} &{}^{L}D_{0^+}^{\alpha}u(t) \\ &\quad = f(t,u(t)), \quad t \in J,\ t \neq t_k,\ k = 1,\dots,m, \\ &\Delta I_{0^+}^{1-\alpha}u(t)\big|_{t=t_k} = y_k, \quad k = 1,\dots,m, \\ &I_{0^+}^{1-\alpha}u(t)\Big|_{t=0} = -I_{0^+}^{1-\alpha}u(t)\Big|_{t=T}, \end{aligned} \tag{2}$$

where ${}^{L}D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative, $f \in C(J \times R, R)$, $J = (0, T]$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $y_k \in R$ are given constants and

$$\Delta I_{0^+}^{1-\alpha} u(t)\Big|_{t=t_k} = I_{0^+}^{1-\alpha} u(t)\Big|_{t=t_k^+} - I_{0^+}^{1-\alpha} u(t)\Big|_{t=t_k^-}, \quad (3)$$

and $I_{0^+}^{1-\alpha} u(t)|_{t=t_k^-}$ and $I_{0^+}^{1-\alpha} u(t)|_{t=t_k^+}$ denote the left and the right limit of $I_{0^+}^{1-\alpha} u(t)$ at $t = t_k$, respectively.

For clarity and brevity, we restrict that the impulsive functions are constants y_k, $k = 1, \ldots, m$. Indeed, we can also define the impulsive functions as $J_k(u(t_k))$ $(J_k \in C(R, R))$.

Remark 1. For $\alpha = 1$, (2) reduces to the first order nonlinear impulsive differential equation with antiperiodic boundary value problem.

To the best of the authors' knowledge, no one has studied the existence of solutions for (2). The purpose of this paper is to study the existence and uniqueness of solution of the generalized antiperiodic boundary value problem for nonlinear fractional impulsive differential equation involving Riemann-Liouville fractional derivative by using some fixed point theorems.

2. Preliminaries and Lemmas

In this section, we introduce notations, definitions, and preliminaries that will be used in this paper. In order to define the solution of (2), we will consider the following spaces.

$PC(J, R) = \{x : J \to R : x(t) \in C(t_k, t_{k+1}]$, $k = 0, \ldots, m$; there exist $x(t_k^+)$ and $x(t_k^-)$ with $x(t_k^-) = x(t_k)$, $k = 1, \ldots, m\}$.

$PC^n(J, R) = \{x \in PC^{n-1}(J, R) : x^{(n)}(t) \in C(t_k, t_{k+1}]$, $k = 0, \ldots, m$; there exist $x^{(n)}(t_k^+) x^{(n)}(t_k^-)$ with $x^{(n)}(t_k^-) = x^{(n)}(t_k)$, $k = 1, \ldots, m\}$.

$PC_\gamma(J, R) = \{x : (t - t_k)^\gamma x|_{[t_k, t_{k+1}]} \in C[t_k, t_{k+1}]$, $k = 0, \ldots, m\}$, where $0 \le \gamma < 1$.

It is easy to check that the space $PC_\gamma(J, R)$ is a Banach space with norm

$$\|x\|_{PC_\gamma} = \sup_{t \in (t_k, t_{k+1}]} (t - t_k)^\gamma |x(t)|, \quad k = 0, \ldots, m. \quad (4)$$

Let us recall the following known definitions. For more details see [1].

Definition 2. Let $\Omega = [a, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis R. The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by

$$I_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s)\, ds, \quad t > 0. \quad (5)$$

Definition 3. The Riemann-Liouville derivative of order $\alpha > 0$, $n = [\alpha] + 1$ can be written as

$$\begin{aligned} {}^{L}D_{0^+}^{\alpha} y(t) &= \left(\frac{d}{dt}\right)^n (I_{0^+}^{n-\alpha} y(t)) \\ &= \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n - \alpha - 1} y(s)\, ds, \quad t > 0. \end{aligned} \quad (6)$$

Lemma 4 (see Lemma 2.5 in [1]). *Let $\alpha > 0$, $n = [\alpha] + 1$ and let $f_{n-\alpha}(t) = I_{0^+}^{n-\alpha} f(t)$ be the fractional integral of order $n - \alpha$. If $f(t) \in L_1(a, b)$ and $f_{n-\alpha}(t) \in C^n[a, b]$, we have the following equality:*

$$I_{0^+}^{\alpha}\, {}^{L}D_{0^+}^{\alpha} f(t) = f(t) - \sum_{i=1}^{n} \frac{f_{n-\alpha}^{(n-i)}(0)}{\Gamma(\alpha - i + 1)} t^{\alpha - i}. \quad (7)$$

Lemma 5. *If $a_i, b_j \neq 0$ $(i = 1, 2, \ldots, k,\ j = 0, 1, \ldots, k-1)$, then*

$$\sum_{i=1}^{k} \sum_{j=0}^{i-1} a_i b_j = \sum_{j=0}^{k-1} \sum_{i=j+1}^{k} a_i b_j, \quad (8)$$

where $k \in N_+$.

Proof. If $k = 1$, then we obtain

$$a_1 b_0 = a_1 b_0. \quad (9)$$

Suppose $k = n$; the result holds; that is,

$$\sum_{i=1}^{n} \sum_{j=0}^{i-1} a_i b_j = \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} a_i b_j. \quad (10)$$

When $k = n + 1$, we obtain that

$$\begin{aligned} \sum_{i=1}^{n+1} \sum_{j=0}^{i-1} a_i b_j &= \sum_{i=1}^{n} \sum_{j=0}^{i-1} a_i b_j + a_{n+1} \sum_{j=0}^{n} b_j \\ &= \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} a_i b_j + a_{n+1} \sum_{j=0}^{n} b_j \\ &= \sum_{j=0}^{n-1} \sum_{i=j+1}^{n+1} a_i b_j - a_{n+1} \sum_{j=0}^{n-1} b_j + a_{n+1} \sum_{j=0}^{n} b_j \\ &= \sum_{j=0}^{n-1} \sum_{i=j+1}^{n+1} a_i b_j + a_{n+1} b_n = \sum_{j=0}^{n} \sum_{i=j+1}^{n+1} a_i b_j. \end{aligned} \quad (11)$$

The proof is completed. □

Lemma 6. *Let $\alpha > 0$, $n - 1 < \alpha < n$, $f_{n-\alpha}(t) = I_{0^+}^{n-\alpha} f(t)$. If $f(t) \in L_1(0, T)$ and $f_{n-\alpha}(t) \in PC^n(J, R)$, then for $t \in [0, t_1]$, one has*

$$I_{0^+}^{\alpha}\, {}^{L}D_{0^+}^{\alpha} f(t) = f(t) - \sum_{i=1}^{n} \frac{f_{n-\alpha}^{(n-i)}(0)}{\Gamma(\alpha - i + 1)} t^{\alpha - i}; \quad (12)$$

for $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$, one has

$$\begin{aligned} I_{0^+}^{\alpha}\, {}^{L}D_{0^+}^{\alpha} f(t) = f(t) &- \sum_{i=1}^{n} \frac{f_{n-\alpha}^{(n-i)}(0)}{\Gamma(\alpha - i + 1)} t^{\alpha - i} \\ &- \sum_{j=1}^{k} \sum_{i=1}^{n} \frac{\Delta f_{n-\alpha}^{(n-i)}(t_j)}{\Gamma(\alpha - i + 1)} (t - t_j)^{\alpha - i}, \end{aligned} \quad (13)$$

where

$$\Delta f_{n-\alpha}^{(n-i)}(t_j) = f_{n-\alpha}^{(n-i)}(t_j^+) - f_{n-\alpha}^{(n-i)}(t_j^-). \quad (14)$$

Proof. Firstly, according to the fractional integral definitions, we get

$$I_{0^+}^{\alpha}\,{}^{L}D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,{}^{L}D_{0^+}^{\alpha} f(s)\,ds = \frac{d}{dt}\left\{\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha}\,{}^{L}D_{0^+}^{\alpha} f(s)\,ds\right\}. \tag{15}$$

If $t\in[0,t_1]$, by Lemma 4, the result is easily to get.

If $t\in(t_1,t_2]$, integrating by parts repeatedly, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha}\,{}^{L}D_{0^+}^{\alpha} f(s)\,ds \\
&= \frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha}\frac{d^n}{ds^n}\left\{I_{0^+}^{n-\alpha} f(s)\right\} ds \\
&= \frac{1}{\Gamma(\alpha+1)\Gamma(n-\alpha)} \times \int_0^t (t-s)^{\alpha}\frac{d^n}{ds^n}\int_0^s (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau\,ds \\
&= \frac{1}{\Gamma(\alpha+1)\Gamma(n-\alpha)} \times \int_0^{t_1} (t-s)^{\alpha}\frac{d^n}{ds^n}\int_0^s (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau\,ds \\
&\quad + \frac{1}{\Gamma(\alpha+1)\Gamma(n-\alpha)} \times \int_{t_1}^t (t-s)^{\alpha}\frac{d^n}{ds^n}\int_0^{t_1} (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau\,ds \\
&\quad + \frac{1}{\Gamma(\alpha+1)\Gamma(n-\alpha)} \times \int_{t_1}^t (t-s)^{\alpha}\frac{d^n}{ds^n}\int_{t_1}^s (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau\,ds \\
&= \frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \times \int_0^{t_1} (t-s)^{\alpha-n}\int_0^s (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau\,ds \\
&\quad + \frac{1}{\Gamma(\alpha-i+2)\Gamma(n-\alpha)} \times \sum_{i=1}^{n}\left[(t-s)^{\alpha-i+1}\frac{d^{n-i}}{ds^{n-i}}\int_0^s (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau\right]\Bigg|_{s=0}^{s=t_1} \\
&\quad + \frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \times \int_{t_1}^t (t-s)^{\alpha-n}\int_0^{t_1} (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau\,ds \\
&\quad + \frac{1}{\Gamma(\alpha-i+2)\Gamma(n-\alpha)} \times \sum_{i=1}^{n}\left[(t-s)^{\alpha-i+1}\frac{d^{n-i}}{ds^{n-i}}\int_0^{t_1} (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau\right]\Bigg|_{s=t_1}^{s=t} \\
&\quad + \frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \times \int_{t_1}^t (t-s)^{\alpha-n}\int_{t_1}^s (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau\,ds \\
&\quad + \frac{1}{\Gamma(\alpha-i+2)\Gamma(n-\alpha)} \times \sum_{i=1}^{n}\left[(t-s)^{\alpha-i+1}\frac{d^{n-i}}{ds^{n-i}} \times \int_{t_1}^s (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau\right]\Bigg|_{s=t_1}^{s=t} \\
&= \frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \times \int_0^{t_1} f(\tau)\,d\tau\int_{\tau}^{t_1} (t-s)^{\alpha-n}(s-\tau)^{n-\alpha-1}ds \\
&\quad + \frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \times \int_0^{t_1} f(\tau)\,d\tau\int_{t_1}^{t} (t-s)^{\alpha-n}(s-\tau)^{n-\alpha-1}ds \\
&\quad + \frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \times \int_{t_1}^{t} f(\tau)\,d\tau\int_{\tau}^{t} (t-s)^{\alpha-n}(s-\tau)^{n-\alpha-1}ds \\
&\quad - \sum_{i=1}^{n}\frac{f_{n-\alpha}^{(n-i)}(0)}{\Gamma(\alpha-i+2)}t^{\alpha-i+1} - \sum_{i=1}^{n}\frac{\Delta f_{n-\alpha}^{(n-i)}(t_1)}{\Gamma(\alpha-i+2)}(t-t_1)^{\alpha-i+1} \\
&= \int_0^t f(\tau)\,d\tau - \sum_{i=1}^{n}\frac{f_{n-\alpha}^{(n-i)}(0)}{\Gamma(\alpha-i+2)}t^{\alpha-i+1} - \sum_{i=1}^{n}\frac{\Delta f_{n-\alpha}^{(n-i)}(t_1)}{\Gamma(\alpha-i+2)}(t-t_1)^{\alpha-i+1},
\end{aligned} \tag{16}$$

where the integral

$$\begin{aligned}
&\int_{\tau}^{t} (t-s)^{\alpha-n}(s-\tau)^{n-\alpha-1}ds \\
&= \int_0^1 (1-z)^{\alpha-n}z^{n-\alpha-1}dz = B(\alpha-n+1,n-\alpha) \\
&= \Gamma(\alpha-n+1)\Gamma(n-\alpha),
\end{aligned} \tag{17}$$

where using the substitution $s=\tau+z(t-\tau)$.

So, if $t \in (t_1, t_2]$, by (15), we have

$$I_{0^+}^{\alpha}\, {}^{L}D_{0^+}^{\alpha} f(t) = f(t) - \sum_{i=1}^{n} \frac{f_{n-\alpha}^{(n-i)}(0)}{\Gamma(\alpha-i+1)} t^{\alpha-i} - \sum_{i=1}^{n} \frac{\Delta f_{n-\alpha}^{(n-i)}(t_1)}{\Gamma(\alpha-i+1)} (t-t_1)^{\alpha-i}. \tag{18}$$

If $t \in (t_k, t_{k+1}]$, $k = 2, \ldots, m$, integrating by parts and using Lemma 5 and (17) repeatedly, we get

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}\, {}^{L}D_{0^+}^{\alpha} f(s)\, ds \\
&= \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha} \frac{d^n}{ds^n} \{I_{0^+}^{n-\alpha} f(s)\}\, ds \\
&= \frac{1}{\Gamma(\alpha+1)\Gamma(n-\alpha)} \times \int_0^t (t-s)^{\alpha} \frac{d^n}{ds^n} \int_0^s (s-\tau)^{n-\alpha-1} f(\tau)\, d\tau\, ds \\
&= \frac{1}{\Gamma(\alpha+1)\Gamma(n-\alpha)} \times \int_0^{t_1} (t-s)^{\alpha} \frac{d^n}{ds^n} \int_0^s (s-\tau)^{n-\alpha-1} f(\tau)\, d\tau\, ds \\
&\quad + \frac{1}{\Gamma(\alpha+1)\Gamma(n-\alpha)} \times \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} (t-s)^{\alpha} \frac{d^n}{ds^n} \times \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (s-\tau)^{n-\alpha-1} f(\tau)\, d\tau\, ds \\
&\quad + \frac{1}{\Gamma(\alpha+1)\Gamma(n-\alpha)} \times \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} (t-s)^{\alpha} \frac{d^n}{ds^n} \int_{t_i}^{s} (s-\tau)^{n-\alpha-1} f(\tau)\, d\tau\, ds \\
&\quad + \frac{1}{\Gamma(\alpha+1)\Gamma(n-\alpha)} \times \int_{t_k}^{t} (t-s)^{\alpha} \frac{d^n}{ds^n} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (s-\tau)^{n-\alpha-1} f(\tau)\, d\tau\, ds \\
&\quad + \frac{1}{\Gamma(\alpha+1)\Gamma(n-\alpha)} \times \int_{t_k}^{t} (t-s)^{\alpha} \frac{d^n}{ds^n} \int_{t_k}^{s} (s-\tau)^{n-\alpha-1} f(\tau)\, d\tau\, ds \\
&= \frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \times \int_0^{t_1} (t-s)^{\alpha-n} \int_0^s (s-\tau)^{n-\alpha-1} f(\tau)\, d\tau\, ds \\
&\quad + \frac{1}{\Gamma(n-\alpha)} \times \sum_{p=1}^{n} \left[(t-s)^{\alpha-p+1} \frac{d^{n-p}}{ds^{n-p}} \times \int_0^s (s-\tau)^{n-\alpha-1} f(\tau)\, d\tau \right] \Bigg|_{s=0}^{s=t_1} \times \frac{1}{\Gamma(\alpha-p+2)} \\
&\quad + \frac{1}{\Gamma(\alpha+1)\Gamma(n-\alpha)} \times \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} (t-s)^{\alpha} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (s-\tau)^{n-\alpha-1} f(\tau)\, d\tau\, ds \\
&\quad + \frac{1}{\Gamma(n-\alpha)} \times \sum_{i=1}^{k-1} \left\{ \sum_{p=1}^{n} \left[(t-s)^{\alpha-p+1} \frac{d^{n-p}}{ds^{n-p}} \times \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (s-\tau)^{n-\alpha-1} \times f(\tau)\, d\tau \right) \right] \Bigg|_{s=t_i}^{s=t_{i+1}} \right\} \times \frac{1}{\Gamma(\alpha-p+2)} \\
&\quad + \frac{1}{\Gamma(\alpha+1)\Gamma(n-\alpha)} \times \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} (t-s)^{\alpha} \int_{t_i}^{s} (s-\tau)^{n-\alpha-1} f(\tau)\, d\tau\, ds \\
&\quad + \frac{1}{\Gamma(n-\alpha)} \times \sum_{i=1}^{k-1} \left\{ \sum_{p=1}^{n} \left[(t-s)^{\alpha-p+1} \frac{d^{n-p}}{ds^{n-p}} \times \int_{t_i}^{s} (s-\tau)^{n-\alpha-1} f(\tau)\, d\tau \right] \Bigg|_{s=t_i}^{s=t_{i+1}} \right\} \times \frac{1}{\Gamma(\alpha-p+2)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha+1)\Gamma(n-\alpha)} \\
&\times\int_{t_k}^{t}(t-s)^{\alpha}\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(s-\tau)^{n-\alpha-1}f(\tau)\,d\tau\,ds \\
&+\frac{1}{\Gamma(n-\alpha)} \\
&\times\sum_{p=1}^{n}\left[(t-s)^{\alpha-p+1}\frac{d^{n-p}}{ds^{n-p}}\right. \\
&\qquad\left.\times\left(\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(s-\tau)^{n-\alpha-1}f(\tau)\,d\tau\right)\right]\Bigg|_{s=t_k}^{s=t} \\
&\qquad\times\frac{1}{\Gamma(\alpha-p+2)} \\
&+\frac{1}{\Gamma(\alpha+1)\Gamma(n-\alpha)} \\
&\times\int_{t_k}^{t}(t-s)^{\alpha}\int_{t_k}^{s}(s-\tau)^{n-\alpha-1}f(\tau)\,d\tau\,ds \\
&+\frac{1}{\Gamma(n-\alpha)} \\
&\times\sum_{p=1}^{n}\left[(t-s)^{\alpha-p+1}\frac{d^{n-p}}{ds^{n-p}}\right. \\
&\qquad\left.\times\int_{t_k}^{s}(s-\tau)^{n-\alpha-1}f(\tau)d\tau\right]\Bigg|_{s=t_k}^{s=t} \\
&\qquad\times\frac{1}{\Gamma(\alpha-p+2)} \\
=&\frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \\
&\times\int_{0}^{t_1}f(\tau)\,d\tau\int_{\tau}^{t_1}(t-s)^{\alpha-n}(s-\tau)^{n-\alpha-1}ds \\
&+\frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \\
&\times\sum_{i=1}^{k-1}\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}f(\tau)\,d\tau\int_{t_i}^{t_{i+1}}(t-s)^{\alpha-n}(s-\tau)^{n-\alpha-1}ds \\
&+\frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \\
&\times\sum_{i=1}^{k-1}\int_{t_i}^{t_{i+1}}f(\tau)\,d\tau\int_{\tau}^{t_{i+1}}(t-s)^{\alpha-n}(s-\tau)^{n-\alpha-1}ds \\
&+\frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \\
&\times\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}f(\tau)\,d\tau\int_{t_k}^{t}(t-s)^{\alpha-n}(s-\tau)^{n-\alpha-1}ds \\
&+\frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \\
&\times\int_{t_k}^{t}f(\tau)\,d\tau\int_{\tau}^{t}(t-s)^{\alpha-n}(s-\tau)^{n-\alpha-1}ds \\
&+\frac{1}{\Gamma(n-\alpha)} \\
&\times\sum_{p=1}^{n}\left[(t-s)^{\alpha-p+1}\frac{d^{n-p}}{ds^{n-p}}\right. \\
&\qquad\left.\times\int_{0}^{s}(s-\tau)^{n-\alpha-1}f(\tau)\,d\tau\right]\Bigg|_{s=0}^{s=t_1} \\
&\qquad\times\frac{1}{\Gamma(\alpha-p+2)} \\
&+\frac{1}{\Gamma(n-\alpha)} \\
&\times\sum_{i=1}^{k-1}\sum_{j=0}^{i-1}\left\{\sum_{p=1}^{n}\left[(t-s)^{\alpha-p+1}\frac{d^{n-p}}{ds^{n-p}}\right.\right. \\
&\qquad\left.\left.\times\int_{t_j}^{t_{j+1}}(s-\tau)^{n-\alpha-1}f(\tau)\,d\tau\right]\Bigg|_{s=t_i}^{s=t_{i+1}}\right\} \\
&\qquad\times\frac{1}{\Gamma(\alpha-p+2)} \\
&+\frac{1}{\Gamma(n-\alpha)} \\
&\times\sum_{i=1}^{k-1}\left\{\sum_{p=1}^{n}\left[(t-s)^{\alpha-p+1}\frac{d^{n-p}}{ds^{n-p}}\right.\right. \\
&\qquad\left.\left.\times\int_{t_i}^{s}(s-\tau)^{n-\alpha-1}f(\tau)\,d\tau\right]\Bigg|_{s=t_i}^{s=t_{i+1}}\right\} \\
&\qquad\times\frac{1}{\Gamma(\alpha-p+2)} \\
&+\frac{1}{\Gamma(n-\alpha)} \\
&\times\sum_{j=0}^{k-1}\sum_{p=1}^{n}\left[(t-s)^{\alpha-p+1}\frac{d^{n-p}}{ds^{n-p}}\right. \\
&\qquad\left.\times\int_{t_j}^{t_{j+1}}(s-\tau)^{n-\alpha-1}f(\tau)\,d\tau\right]\Bigg|_{s=t_k}^{s=t} \\
&\qquad\times\frac{1}{\Gamma(\alpha-p+2)} \\
&+\frac{1}{\Gamma(n-\alpha)} \\
&\times\sum_{p=1}^{n}\left[(t-s)^{\alpha-p+1}\frac{d^{n-p}}{ds^{n-p}}\right. \\
&\qquad\left.\times\int_{t_k}^{s}(s-\tau)^{n-\alpha-1}f(\tau)\,d\tau\right]\Bigg|_{s=t_k}^{s=t}
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{1}{\Gamma(\alpha-p+2)} \\
&= \frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \\
&\quad\times \int_0^{t_1} f(\tau)\,d\tau \int_\tau^{t_1} (t-s)^{\alpha-n}(s-\tau)^{n-\alpha-1} ds \\
&\quad+ \frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \\
&\quad\times \sum_{j=0}^{k-2}\sum_{i=j+1}^{k-1} \int_{t_j}^{t_{j+1}} f(\tau)\,d\tau \int_{t_i}^{t_{i+1}} (t-s)^{\alpha-n}(s-\tau)^{n-\alpha-1} ds \\
&\quad+ \frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \\
&\quad\times \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} f(\tau)\,d\tau \int_\tau^{t_{i+1}} (t-s)^{\alpha-n}(s-\tau)^{n-\alpha-1} ds \\
&\quad+ \frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \\
&\quad\times \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} f(\tau)\,d\tau \int_{t_k}^{t} (t-s)^{\alpha-n}(s-\tau)^{n-\alpha-1} ds \\
&\quad+ \frac{1}{\Gamma(\alpha-n+1)\Gamma(n-\alpha)} \\
&\quad\times \int_{t_k}^{t} f(\tau)\,d\tau \int_\tau^{t} (t-s)^{\alpha-n}(s-\tau)^{n-\alpha-1} ds \\
&\quad+ \frac{1}{\Gamma(n-\alpha)} \\
&\quad\times \sum_{p=1}^{n} \left[(t-s)^{\alpha-p+1} \frac{d^{n-p}}{ds^{n-p}} \right. \\
&\qquad\qquad \left. \times \int_0^{s} (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau \right]\Bigg|_{s=0}^{s=t_1} \frac{1}{\Gamma(\alpha-p+2)} \\
&\quad+ \frac{1}{\Gamma(n-\alpha)} \\
&\quad\times \sum_{i=1}^{k-1}\sum_{j=0}^{i-1} \left\{ \sum_{p=1}^{n} \left[(t-s)^{\alpha-p+1} \frac{d^{n-p}}{ds^{n-p}} \right.\right. \\
&\qquad\qquad \left.\left. \times \int_{t_j}^{t_{j+1}} (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau \right]\Bigg|_{s=t_i}^{s=t_{i+1}} \right\} \\
&\qquad \times \frac{1}{\Gamma(\alpha-p+2)} \\
&\quad+ \frac{1}{\Gamma(n-\alpha)} \\
&\quad\times \sum_{i=1}^{k-1} \left\{ \sum_{p=1}^{n} \left[(t-s)^{\alpha-p+1} \frac{d^{n-p}}{ds^{n-p}} \right.\right. \\
&\qquad\qquad \left.\left. \times \int_{t_i}^{s} (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau \right]\Bigg|_{s=t_i}^{s=t_{i+1}} \right\} \\
&\qquad \times \frac{1}{\Gamma(\alpha-p+2)} \\
&\quad+ \frac{1}{\Gamma(n-\alpha)} \\
&\quad\times \sum_{j=0}^{k-1}\sum_{p=1}^{n} \left[(t-s)^{\alpha-p+1} \frac{d^{n-p}}{ds^{n-p}} \right. \\
&\qquad\qquad \left. \times \int_{t_j}^{t_{j+1}} (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau \right]\Bigg|_{s=t_k}^{s=t} \\
&\qquad \times \frac{1}{\Gamma(\alpha-p+2)} \\
&\quad+ \frac{1}{\Gamma(n-\alpha)} \\
&\quad\times \sum_{p=1}^{n} \left[(t-s)^{\alpha-p+1} \frac{d^{n-p}}{ds^{n-p}} \right. \\
&\qquad\qquad \left. \times \int_{t_k}^{s} (s-\tau)^{n-\alpha-1} f(\tau)\,d\tau \right]\Bigg|_{s=t_k}^{s=t} \\
&\qquad \times \frac{1}{\Gamma(\alpha-p+2)} \\
&= \int_0^t f(\tau)\,d\tau - \sum_{p=1}^{n} \frac{f_{n-\alpha}^{(n-p)}(0)}{\Gamma(\alpha-p+2)} t^{\alpha-p+1} \\
&\quad - \sum_{i=1}^{k}\sum_{p=1}^{n} \frac{\Delta f_{n-\alpha}^{(n-p)}(t_i)}{\Gamma(\alpha-p+2)} (t-t_i)^{\alpha-p+1}.
\end{aligned} \tag{19}
$$

By (15), if $t \in (t_k, t_{k+1}]$, $k = 2, \ldots, m$, we have

$$
\begin{aligned}
I_{0^+}^{\alpha}\, {}^{L}D_{0^+}^{\alpha} f(t) = f(t) &- \sum_{i=1}^{n} \frac{f_{n-\alpha}^{(n-i)}(0)}{\Gamma(\alpha-i+1)} t^{\alpha-i} \\
&- \sum_{j=1}^{k}\sum_{i=1}^{n} \frac{\Delta f_{n-\alpha}^{(n-i)}(t_j)}{\Gamma(\alpha-i+1)} (t-t_j)^{\alpha-i}.
\end{aligned} \tag{20}
$$

The proof is completed. □

Remark 7. In Lemma 6, if the assumption $f_{n-\alpha}(t) \in PC^n(J,R)$ is replaced by $f_{n-\alpha}(t) \in C^n(J,R)$, we will get the same result of Lemma 4.

Lemma 8. *The impulsive antiperiodic boundary value problem*

$$
\begin{gathered}
{}^{L}D_{0^+}^{\alpha} u(t) = \rho(t), \\
0 < \alpha \le 1,\ t \in J,\ t \ne t_k,\ k = 1, \ldots, m, \\
\Delta I_{0^+}^{1-\alpha} u(t)\big|_{t=t_k} = y_k, \quad k = 1, \ldots, m, \\
I_{0^+}^{1-\alpha} u(t)\big|_{t=0} = -I_{0^+}^{1-\alpha} u(t)\big|_{t=T},
\end{gathered} \tag{21}
$$

where $\rho(t) \in C(J,R)$, has a unique solution $u(t) \in PC_{1-\alpha}(J,R)$ given by

$$u(t) = \begin{cases} \dfrac{a}{\Gamma(\alpha)} t^{\alpha-1} + \dfrac{1}{\Gamma(\alpha)} \times \displaystyle\int_0^t (t-s)^{\alpha-1} \rho(s)\, ds, & t \in (0,t_1], \\ \dfrac{a}{\Gamma(\alpha)} t^{\alpha-1} + \displaystyle\sum_{0<t_i<t} \dfrac{y_i}{\Gamma(\alpha)} (t-t_i)^{\alpha-1} + \dfrac{1}{\Gamma(\alpha)} \times \displaystyle\int_0^t (t-s)^{\alpha-1} \rho(s)\, ds, & t \in (t_k, t_{k+1}],\ k = 1,2,\ldots,m, \end{cases} \tag{22}$$

where

$$a = -\frac{1}{2}\left(\sum_{i=1}^m y_i + \int_0^T \rho(s)\, ds\right). \tag{23}$$

Proof. Let $u(t)$ be a solution of (21). By Lemma 6 $(n = 1)$, we have

$$u(t) = \begin{cases} \dfrac{a}{\Gamma(\alpha)} t^{\alpha-1} + \dfrac{1}{\Gamma(\alpha)} \times \displaystyle\int_0^t (t-s)^{\alpha-1} \rho(s)\, ds, & t \in (0,t_1], \\ \dfrac{a}{\Gamma(\alpha)} t^{\alpha-1} + \displaystyle\sum_{0<t_i<t} \dfrac{y_i}{\Gamma(\alpha)} (t-t_i)^{\alpha-1} + \dfrac{1}{\Gamma(\alpha)} \times \displaystyle\int_0^t (t-s)^{\alpha-1} \rho(s)\, ds, & t \in (t_k, t_{k+1}],\ k = 1,2,\ldots,m, \end{cases} \tag{24}$$

where $I_{0^+}^{1-\alpha} u(t)|_{t=0} = a$.

According to the following properties:

$$I_{0^+}^{1-\alpha} t^{\alpha-1} = \Gamma(\alpha),$$

$$\begin{aligned} I_{0^+}^{1-\alpha} \sum_{0<t_i<t} y_i (t-t_i)^{\alpha-1} &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \sum_{0<t_i<s} y_i (s-t_i)^{\alpha-1} ds \\ &= \frac{1}{\Gamma(1-\alpha)} \int_{t_i}^t (t-s)^{-\alpha} \sum_{0<t_i<s} y_i (s-t_i)^{\alpha-1} ds \\ &= \Gamma(\alpha) \sum_{0<t_i<t} y_i, \end{aligned} \tag{25}$$

we obtain

$$I_{0^+}^{1-\alpha} u(t) = \begin{cases} a + \displaystyle\int_0^t \rho(s)\, ds, & t \in (0,t_1], \\ a + \displaystyle\sum_{0<t_i<t} y_i + \displaystyle\int_0^t \rho(s)\, ds, & t \in (t_k, t_{k+1}],\ k = 1,2,\ldots,m. \end{cases} \tag{26}$$

Then by the antiperiodic boundary value condition, we have

$$a = -\frac{1}{2}\left(\sum_{i=1}^m y_i + \int_0^T \rho(s)\, ds\right). \tag{27}$$

Conversely, assuming that $u(t)$ is a solution of the impulsive fractional integral equation (22), we can obtain the impulsive fractional differential equation (21).

This completes the proof. □

3. Main Results

This section deals with the existence and uniqueness of solutions for the problem (2).

Firstly, for $t \in (t_k, t_{k+1}]$, $k = 0,1,\ldots,m$, we define an operator T as

$$\begin{aligned} (Tu)(t) = &-\frac{t^{\alpha-1}}{2\Gamma(\alpha)}\left(\sum_{i=1}^m y_i + \int_0^T f(s,u(s))\, ds\right) \\ &+ \sum_{0<t_i<t} \frac{y_i}{\Gamma(\alpha)} (t-t_i)^{\alpha-1} \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s))\, ds. \end{aligned} \tag{28}$$

Theorem 9. *If the following condition is satisfied:*

(H1): *there exist constants $\ell, N > 0$ such that*

$$\begin{gathered} |f(t,u)| \le \ell + N\|u\|^\theta, \quad \forall t \in J, \\ u \in R, \quad 0 \le \theta < 1, \end{gathered} \tag{29}$$

then the fractional impulsive differential equation (2) *has at least one solution.*

Proof. Assume (H1) hold; let

$$\lambda = \max\left\{\frac{9}{2\Gamma(\alpha)}\sum_{i=1}^m |y_i|, \left[3\left(\frac{1}{2\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)}\right)NT\right]^{1/(1-\theta)}, 3\left(\frac{1}{2\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)}\right)\ell T\right\} \tag{30}$$

and define $A^{\lambda}_{1-\alpha} = \{u \in PC_{1-\alpha}(J,R) : \|u\| \le \lambda\}$.

When $t \in (t_k, t_{k+1}]$, $k = 0,1,\ldots,m$, for $\forall u \in A^{\lambda}_{1-\alpha}$, by (H1), we have

$$\begin{aligned}
&(t-t_k)^{1-\alpha}|(Tu)(t)| \\
&\le \frac{(t-t_k)^{1-\alpha}t^{\alpha-1}}{2\Gamma(\alpha)} \\
&\quad\times\sum_{i=1}^{m}|y_i| + \frac{(t-t_k)^{1-\alpha}t^{\alpha-1}}{2\Gamma(\alpha)}\int_0^T|f(s,u(s))|\,ds \\
&\quad+\frac{(t-t_k)^{1-\alpha}}{\Gamma(\alpha)}\sum_{0<t_i<t}|y_i|(t-t_i)^{\alpha-1} \\
&\quad+\frac{(t-t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,u(s))|\,ds \\
&\le \frac{3}{2\Gamma(\alpha)}\sum_{i=1}^{m}|y_i| + \frac{T}{2\Gamma(\alpha)}\left(\ell+N\lambda^{\theta}\right) \\
&\qquad+\frac{T}{\Gamma(\alpha+1)}\left(\ell+N\lambda^{\theta}\right) \\
&= \frac{3}{2\Gamma(\alpha)}\sum_{i=1}^{m}|y_i| + \left(\frac{1}{2\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\right)NT\lambda^{\theta} \\
&\quad+\left(\frac{1}{2\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\right)\ell T \le \lambda,
\end{aligned} \tag{31}$$

which implies that $T : A^{\lambda}_{1-\alpha} \to A^{\lambda}_{1-\alpha}$.

In view of the continuity of f, we get that the operator T is continuous easily. □

Next, we will prove that T is a completely continuous operator.

For $(t_k, t_{k+1}]$, $k = 0,\ldots,m$, if $t_k < \tau_1 < \tau_2 \le t_{k+1}$, $u \in A^{\lambda}_{1-\alpha}$, when $\tau_1 \to \tau_2$, by (H1), we have

$$\begin{aligned}
&\Big|(\tau_1-t_k)^{1-\alpha}(Tu)(\tau_1) \\
&\quad-(\tau_2-t_k)^{1-\alpha}(Tu)(\tau_2)\Big| \\
&\le \frac{\left|(\tau_1-t_k)^{1-\alpha}\tau_1^{\alpha-1}-(\tau_2-t_k)^{1-\alpha}\tau_2^{\alpha-1}\right|}{2\Gamma(\alpha)} \\
&\quad\times\sum_{i=1}^{m}|y_i| \\
&\quad+\frac{\left|(\tau_1-t_k)^{1-\alpha}\tau_1^{\alpha-1}-(\tau_2-t_k)^{1-\alpha}\tau_2^{\alpha-1}\right|}{2\Gamma(\alpha)} \\
&\quad\times\int_0^T|f(s,u(s))|\,ds \\
&\quad+\left|\frac{(\tau_1-t_k)^{1-\alpha}}{\Gamma(\alpha)}\sum_{0<t_i<t}y_i(\tau_1-t_i)^{\alpha-1}\right. \\
&\qquad\left.-\frac{(\tau_2-t_k)^{1-\alpha}}{\Gamma(\alpha)}\sum_{0<t_i<t}y_i(\tau_2-t_i)^{\alpha-1}\right| \\
&\quad+\frac{\left|(\tau_1-t_k)^{1-\alpha}-(\tau_2-t_k)^{1-\alpha}\right|}{\Gamma(\alpha)} \\
&\quad\times\int_0^t(t-s)^{\alpha-1}|f(s,u(s))|\,ds \\
&\le \frac{\left|(1-t_k/\tau_1)^{1-\alpha}-(1-t_k/\tau_2)^{1-\alpha}\right|}{2\Gamma(\alpha)}\sum_{i=1}^{m}|y_i| \\
&\quad+\frac{\left|(1-t_k/\tau_1)^{1-\alpha}-(1-t_k/\tau_2)^{1-\alpha}\right|}{2\Gamma(\alpha)} \\
&\quad\times\left(\ell+N\lambda^{\theta}\right)T+\frac{1}{\Gamma(\alpha)} \\
&\quad\times\left|\sum_{i=1}^{k-1}y_i(\tau_1-t_i)^{\alpha-1}\right. \\
&\qquad\left.-\sum_{i=1}^{k-1}y_i(\tau_2-t_i)^{\alpha-1}\right| \\
&\quad+\frac{\left|(\tau_1-t_k)^{1-\alpha}-(\tau_2-t_k)^{1-\alpha}\right|}{\Gamma(\alpha+1)} \\
&\quad\times\left(\ell+N\lambda^{\theta}\right)T^{\alpha}\longrightarrow 0.
\end{aligned} \tag{32}$$

According to the Ascoli-Arzela theorem, we can obtain $T : A^{\lambda}_{1-\alpha} \to A^{\lambda}_{1-\alpha}$ which is a completely continuous operator. Therefore, by Schauder's fixed point theorem, the operator T has at least one fixed point, which implies that fractional impulsive differential equation (2) has at least one solution $u(t)$.

Theorem 10. *Assume that*

(H2): *there exists constant $L \ge 0$ such that*

$$|f(t,u_1)-f(t,u_2)| \le L\|u_1-u_2\|, \quad \forall u_1,u_2 \in PC_{\gamma}(J,R). \tag{33}$$

Then problem (2) *has a unique solution if*

$$\left(\frac{1}{2\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\right)LT < 1. \tag{34}$$

Proof. We define $\sup_{t\in J}|f(t,0)| = M$ and choose

$$r = \frac{(3/2\Gamma(\alpha))\sum_{i=1}^{m}|y_i| + (1/2\Gamma(\alpha)+1/\Gamma(\alpha+1))MT}{1-(1/2\Gamma(\alpha)+1/\Gamma(\alpha+1))LT}. \tag{35}$$

Firstly, we prove that $Tu \in B^{r}_{1-\alpha}$, where $B^{r}_{1-\alpha} = \{u \in PC_{1-\alpha}(J,R) : \|u\| \le r\}$.

For $\forall u \in B^{r}_{1-\alpha}$, by (H2), we have

$$\begin{aligned}
&(t-t_k)^{1-\alpha}|(Tu)(t)| \\
&\le \frac{(t-t_k)^{1-\alpha}t^{\alpha-1}}{2\Gamma(\alpha)}\sum_{i=1}^{m}|y_i|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(t-t_k)^{1-\alpha}t^{\alpha-1}}{2\Gamma(\alpha)}\int_0^T|f(s,u(s))|\,ds\\
&+\frac{(t-t_k)^{1-\alpha}}{\Gamma(\alpha)}\\
&\times\Bigg(\sum_{0<t_i<t}|y_i|(t-t_i)^{\alpha-1}\\
&\qquad+\int_0^t(t-s)^{\alpha-1}|f(s,u(s))|\,ds\Bigg)\\
&\le\frac{3}{2\Gamma(\alpha)}\sum_{i=1}^m|y_i|+\frac{1}{2\Gamma(\alpha)}\\
&\quad\times\int_0^T(|f(s,u(s))-f(s,0)|+|f(s,0)|)\,ds\\
&\quad+\frac{(t-t_k)^{1-\alpha}}{\Gamma(\alpha)}\\
&\quad\times\int_0^t(t-s)^{\alpha-1}\\
&\qquad\quad\times(|f(s,u(s))-f(s,0)|+|f(s,0)|)\,ds\\
&\le\frac{3}{2\Gamma(\alpha)}\sum_{i=1}^m|y_i|+\frac{LTr}{2\Gamma(\alpha)}\\
&\quad+\frac{MT}{2\Gamma(\alpha)}+\frac{LTr}{\Gamma(\alpha+1)}+\frac{MT}{\Gamma(\alpha+1)}\\
&=\frac{3}{2\Gamma(\alpha)}\sum_{i=1}^m|y_i|+\left(\frac{1}{2\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\right)LTr\\
&\quad+\left(\frac{1}{2\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\right)MT=r.
\end{aligned}
\tag{36}
$$

Next, for $\forall u,v\in B^r_{1-\alpha}$, by (H2), we get

$$
\begin{aligned}
&(t-t_k)^{1-\alpha}|(Tu)(t)-(Tv)(t)|\\
&\quad\le\frac{(t-t_k)^{1-\alpha}t^{\alpha-1}}{2\Gamma(\alpha)}\\
&\qquad\times\int_0^T|f(s,u(s))-f(s,v(s))|\,ds\\
&\qquad+\frac{(t-t_k)^{1-\alpha}}{\Gamma(\alpha)}\\
&\qquad\times\int_0^t(t-s)^{\alpha-1}(|f(s,u(s))-f(s,v(s))|)\,ds\\
&\quad\le\left(\frac{LT}{2\Gamma(\alpha)}+\frac{LT}{\Gamma(\alpha+1)}\right)\|u-v\|_{B^r_{1-\alpha}}.
\end{aligned}
\tag{37}
$$

According to inequality (34), we obtain that the operator T is a contractive mapping on $B^r_{1-\alpha}$. Hence, by Banach fixed point theorem, problem (2) has a unique solution.

The proof is completed. □

4. Examples

Example 1. Choose $\alpha=1/2$, $t_1=1/2$, and $T=1$, and consider the following fractional impulsive generalized antiperiodic boundary value problem:

$$
\begin{gathered}
{}^LD^{1/2}_{0^+}u(t)=f(t,u(t)),\quad t\neq\frac{1}{2},\ t\in[0,1],\\
\Delta u\left(\frac{1}{2}\right)=b,\quad b\in R,\\
I^{1/2}_{0^+}u(t)\big|_{t=0}=-I^{1/2}_{0^+}u(t)\big|_{t=1},
\end{gathered}
\tag{38}
$$

where

$$f(t,u(t))=\sin t+e^{-t}\sqrt{tu(t)}.\tag{39}$$

Let $\ell=N=1$, $\theta=1/2$; clearly, assumption (H1) is satisfied. By Theorem 9, the fractional impulsive generalized antiperiodic boundary value problem (38) has at least one solution.

Example 2. Choose $\alpha=2/3$, $t_1=1/2$, and $T=1$; consider the following fractional impulsive generalized antiperiodic boundary value problem:

$$
\begin{gathered}
{}^LD^{2/3}_{0^+}u(t)=f(t,u(t)),\quad t\neq\frac{1}{2},\ t\in[0,1],\\
\Delta u\left(\frac{1}{2}\right)=b,\quad b\in R,\\
I^{2/3}_{0^+}u(t)\big|_{t=0}=-I^{2/3}_{0^+}u(t)\big|_{t=1},
\end{gathered}
\tag{40}
$$

where

$$f(t,u(t))=t^3+\frac{\Gamma(2/3)\sin t}{3e^{t^{1/2}}}t^{1/3}u(t).\tag{41}$$

Letting $L=\Gamma(2/3)/3$, condition (H2) of Theorem 10 can be verified, so Example 2 has at least one solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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