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*Research Article*

# Random Attractors for Stochastic Ginzburg-Landau Equation on Unbounded Domains

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We prove the existence of a pullback attractor in $\mathbb{L}^2(\mathbb{R}^n)$ for the stochastic Ginzburg-Landau equation with additive noise on the entire $n$-dimensional space $\mathbb{R}^n$. We show that the stochastic Ginzburg-Landau equation with additive noise can be recast as a random dynamical system. We demonstrate that the system possesses a unique $\mathscr{D}$-random attractor, for which the asymptotic compactness is established by the method of uniform estimates on the tails of its solutions.

## 1. Introduction

In this paper, we study the following stochastic Ginzburg-Landau equation with additive noise defined in the entire space $\mathbb{R}^n$:

$$\begin{aligned} \mathrm{d}u &= (\lambda + i\mu)\,\Delta u dt - (\kappa + i\beta)\,|u|^2 u dt \\ &\quad - \gamma u dt + \sum_{j=1}^{m} \varphi_j \mathrm{d}\omega_j(t), \end{aligned} \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{2}$$

where $\lambda$, $\mu$, $\kappa$, $\beta$, $\gamma$ are real coefficients, with $\lambda > 0$, $\kappa > 0$, $\gamma > 0$, and $\varphi_j \in H^2(\mathbb{R}^n) \cap W^{2,4}(\mathbb{R}^n)$, $j = 1,\ldots,m$, being time independent defined on $\mathbb{R}^n$ and $\{\omega_j\}_{j=1}^m$ being independent two-sided real-valued Wiener processes on a complete probability space $(\Omega, \mathscr{F}, P)$. Our aim is to study its long time behavior defined in the entire space $\mathbb{R}^n$.

Attractors are quite well investigated to describe the long time behavior of the deterministic equations (see, e.g., [1–7]). Recently, the concept of random attractors, which is in fact compact invariant set, was introduced to stochastic dynamical systems from the theory of attractors for deterministic equations in [8–10]. The existence of such random attractors for the Ginzburg-Landau equation perturbed by additive white noise and multiplicative white noise on bounded domains has been investigated, respectively, in [11, 12].

However, for unbounded domains, we cannot guarantee the compactness of solutions by the standard method since the Sobolev embeddings are no longer compact. Hence, to prove the existence of an attractor, we have to first overcome this difficulty. For deterministic equations, this difficulty has been overcome by employing the energy equation approach, introduced in [13, 14], and then used by others to prove the asymptotic compactness of deterministic equations in unbounded domains (see, e.g., [15–22]). In this paper, we prove the existence of a random attractor for the stochastic Ginzburg-Landau equation (1), defined on the unbounded domain $\mathbb{R}^n$ with the help of tail estimates method, which was firstly established in [23] to the case of stochastic dissipative PDEs.

For the mathematical setting, we introduce complex Sobolev spaces. In general, we denote by $\mathbb{X}, \mathbb{Y}, \ldots$ the complexified space of a function space $X, Y, \ldots$. For example, $\mathbb{L}^2(\mathbb{R}^n)$ is the complexified space of $L^2(\mathbb{R}^n)$. Denote by $(\cdot,\cdot)$ and $\|\cdot\|_{L^2}$ the scalar product and the norm in either $L^2(\mathbb{R}^n)$ or $\mathbb{L}^2(\mathbb{R}^n)$. So, if $u \in \mathbb{L}^2(\mathbb{R}^n)$, then $u = \{u_1, u_2\}$, $u_j \in L^2(\mathbb{R}^n)$, $j = 1, 2$, and

$$\|u\|_{L^2} = \left\{ \|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2 \right\}^{1/2}. \tag{3}$$

If $u = u_1 + iu_2$, $v = v_1 + iv_2$ are in $\mathbb{L}^2(\mathbb{R}^n)$,

$$(u, v) = \{(u_1, v_1) + (u_2, v_2)\} + i\{(u_2, v_1) - (u_1, v_2)\}. \quad (4)$$

We use letter $c > 0$ to denote any positive constant which may change its value from line to line or even in the same line when necessary.

The whole paper is organized as follows. In Section 2, we first recall some definitions and propositions on random attractors for random dynamical systems (RDS). And then, by Ornstein-Uhlenbeck process, we obtain the continuous RDS $\phi$ associated with the stochastic Ginzburg-Landau equation (1). In Section 3, we concentrate to get the uniform estimate on the far-field values of the solution as $t \longrightarrow \infty$ and thus to further establish the asymptotic compactness of the solution operator $\phi$. Then, we can exhibit our main result in the following theorem.

**Theorem 1.** *The random dynamical system $\phi$ of stochastic Ginzburg-Landau equation with additive noise has a unique $\mathscr{D}$-random attractor in $\mathbb{L}^2(\mathbb{R}^n)$ provided that $\sqrt{3}\kappa \geq |\beta|$.*

## 2. RDS Associated with the Stochastic Ginzburg-Landau Equation on $\mathbb{R}^n$

*2.1. Preliminaries on RDS.* We first recall some definitions. For more details, one can refer to [8, 10, 24–26].

*Definition 2.* Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\{\theta_t : \Omega \longrightarrow \Omega, t \in \mathbb{R}\}$ a family of measures preserving transformation such that $(t, \omega) \longrightarrow \theta_t\omega$ is measurable, $\theta_0 = \text{id}$, and $\theta_{s+t} = \theta_t \circ \theta_s$, for all $s, t \in \mathbb{R}$, and then the flow $\theta_t$ together with the corresponding probability space $(\Omega, \mathscr{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system.

For Wiener process $\omega_j$ in (1), we consider the probability space $(\Omega, \mathscr{F}, P)$, where

$$\Omega = \{\omega = (\omega_1, \omega_2, \ldots, \omega_m) \in C(\mathbb{R}, \mathbb{R}^m), \omega(0) = 0\}, \quad (5)$$

$\mathscr{F}$ is the Borel $\sigma$-algebra induced by the compact-open topology of $\Omega$, and $P$ is the corresponding Wiener measure on $(\Omega, \mathscr{F})$. The time shift is simply defined by

$$\theta_t\omega(s) = \omega(s + t) - \omega(s), \quad t, s \in \mathbb{R}, \ \omega \in \Omega. \quad (6)$$

Then $(\Omega, \mathscr{F}, P, (\theta_t)_{t\in R})$ is a metric dynamical system.

*Definition 3.* A continuous random dynamical system (RDS) on $X$ over a metric dynamical system $(\Omega, \mathscr{F}, P, (\theta_t)_{t\in R})$ is a mapping:

$$\phi : \mathbb{R}^+ \times \Omega \times X \longrightarrow X, \qquad (t, \omega, x) \longrightarrow \phi(t, \omega, x), \quad (7)$$

which is $(\mathscr{B}(\mathbb{R}^+) \times \mathscr{F} \times \mathscr{B}(X), \mathscr{B}(X))$-measurable such that, for $P$-a.e. $\omega \in \Omega$,

(i) $\phi(0, \omega, \cdot)$ is the identity on $X$;

(ii) $\phi(t + s, \omega, \cdot) = \phi(t, \theta_s\omega, \cdot) \circ \phi(s, \omega, \cdot)$ for all $t, s \in \mathbb{R}^+$;

(iii) $\phi(t, \omega, \cdot) : X \longrightarrow X$ is continuous for all $t \in \mathbb{R}^+$.

Hereafter, we always assume that $\phi$ is a continuous RDS on $X$ over $(\Omega, \mathscr{F}, P, (\theta_t)_{t\in R})$.

*Definition 4.* A random variable $R : \Omega \longrightarrow (0, \infty)$ is called tempered with respect to the dynamical system $\theta$ if, for the associated stationary stochastic process $t \longrightarrow R(\theta_t\cdot)$, the invariant set for which

$$\lim_{t\longrightarrow\pm\infty} \frac{1}{t} \log R(\theta_t\omega) = 0 \quad (8)$$

($t \longrightarrow -\infty$ applies only to two-sided time) has full $P$-measure.

*Definition 5.* A random bounded set $\{B(\omega)\}_{\omega\in\Omega}$ of $X$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if, for $P$-a.e. $\omega \in \Omega$,

$$\lim_{t\longrightarrow\infty} e^{-\varepsilon t} d(B(\theta_{-t}\omega)) = 0 \quad \forall\varepsilon > 0, \quad (9)$$

where $d(B) = \sup_{x\in B}\|x\|_X$.

*Definition 6.* Let $\mathscr{D}$ be a collection of random subsets of $X$ and $\{K(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$. Then $\{K(\omega)\}_{\omega\in\Omega}$ is called a random absorbing set for $\phi$ in $\mathscr{D}$ if, for every $B \in \mathscr{D}$ and $P$-a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega) \quad \forall t \geq t_B(\omega). \quad (10)$$

*Definition 7.* Let $\mathscr{D}$ be a collection of random subsets of $X$. Then $\phi$ is said to be $\mathscr{D}$-pullback asymptotically compact in $X$ if, for $P$-a.e. $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^\infty$ has a convergent subsequence in $X$ whenever $t_n \longrightarrow \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$.

*Definition 8.* Let $\mathscr{D}$ be a collection of random subsets of $X$. Then a random set $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ of $X$ is called a $\mathscr{D}$-random attractor (or $\mathscr{D}$-pullback attractor) for $\phi$ if the following conditions are satisfied: for $P$-a.e. $\omega \in \Omega$,

(i) $\mathscr{A}(\omega)$ is compact, and $\omega \longrightarrow d(x, \mathscr{A}(\omega))$ is measurable for every $x \in X$;

(ii) $\mathscr{A}(\omega)$ is invariant; that is,

$$\phi(t, \omega, \mathscr{A}(\omega)) = \mathscr{A}(\theta_t\omega) \quad \forall t \geq 0; \quad (11)$$

(iii) $\mathscr{A}(\omega)$ attracts every set in $\mathscr{D}$; that is, for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$,

$$\lim_{t\longrightarrow\infty} d(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathscr{A}(\omega)) = 0, \quad (12)$$

where $d$ is the Hausdorff semimetric given by $d(Y, Z) = \sup_{y\in Y}\inf_{z\in Z}\|y - z\|_X$ for any $Y \subseteq X$ and $Z \subseteq X$.

**Proposition 9** (see [10, 25]). *Let $\mathscr{D}$ be an inclusion-closed collection of random subsets of $X$ and $\phi$ a continuous RDS on $X$ over $(\Omega, \mathscr{F}, P, (\theta_t)_{t\in R})$. Suppose that $\{K(\omega)\}_{\omega\in\Omega}$ is a closed random absorbing set for $\phi$ in $\mathscr{D}$ and $\phi$ is $\mathscr{D}$-pullback asymptotically compact in $X$. Then $\phi$ has a unique $\mathscr{D}$-random attractor $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ which is given by*

$$\mathscr{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}. \quad (13)$$

*Remark 10.* A collection $\mathscr{D}$ of random subsets is called inclusion closed if, whenever $\{E(\omega)\}_{\omega\in\Omega}$ is an arbitrary random set and $\{F(\omega)\}_{\omega\in\Omega}$ is in $\mathscr{D}$ with $E(\omega) \subset F(\omega)$, for all $\omega \in \Omega$, $\{E(\omega)\}_{\omega\in\Omega}$ must belong to $\mathscr{D}$.

*2.2. RDS Associated with the Stochastic Ginzburg-Landau Equation on $\mathbb{R}^n$.* Denote $z(t) = z(\theta_t\omega) = \sum_{j=1}^{m} \varphi_j z_j(\theta_t\omega_j)$, where

$$z_j(t) = z_j(\theta_t\omega_j) = \int_{-\infty}^{t} e^{\gamma(s-t)} \mathrm{d}\omega_j(s), \quad t \in \mathbb{R}, \tag{14}$$

satisfies the one-dimensional Ornstein-Uhlenbeck equation

$$\mathrm{d}z_j = -\gamma z_j dt + \mathrm{d}\omega_j(t). \tag{15}$$

Since the random variable $|z_j(\omega_j)|$ is tempered and $|z_j(\theta_t\omega_j)|$ is $P$-a.e. continuous, there exists a tempered function $r(\omega) > 0$ such that

$$\sum_{j=1}^{m} \left( \left|z_j(\omega_j)\right|^2 + \left|z_j(\omega_j)\right|^4 \right) \le r(\omega), \tag{16}$$

where $r(\omega)$ satisfies, for $P$-a.e. $\omega \in \Omega$,

$$r(\theta_t\omega) \le e^{(\gamma/2)|t|} r(\omega), \quad t \in \mathbb{R}, \tag{17}$$

thanks to Proposition 4.3.3 in [24]. From (16) to (17), we get, for $P$-a.e. $\omega \in \Omega$,

$$\sum_{j=1}^{m} \left( \left|z_j(\theta_t\omega_j)\right|^2 + \left|z_j(\theta_t\omega_j)\right|^4 \right) \le e^{(\gamma/2)|t|} r(\omega), \quad t \in \mathbb{R}. \tag{18}$$

Introduce the transformation

$$v(t) = u(t) - z(\theta_t\omega), \tag{19}$$

where $u$ is the solution of (1)-(2); then $v$ should satisfy

$$\begin{aligned} \frac{\partial v}{\partial t} &= (\lambda + i\mu)\Delta v - (\kappa + i\beta)|v+z|^2(v+z) \\ &\quad - \gamma v + (\lambda + i\mu)\Delta z. \end{aligned} \tag{20}$$

Similar to the procedure in [23], we can obtain that (20) has a unique solution $v(t, \omega, v_0)$ with $v(0, \omega, v_0) = v_0$, which is continuous with respect to $v_0$ in $\mathbb{L}^2(\mathbb{R}^n)$. Let $u(t, \omega, u_0) = v(t, \omega, u_0 - z(\omega)) + z(\theta_t\omega)$, and then $u$ is the solution of (1)-(2). Define $\phi : \mathbb{R}^+ \times \Omega \times \mathbb{L}^2(\mathbb{R}^n) \longrightarrow \mathbb{L}^2(\mathbb{R}^n)$ by

$$\phi(t, \omega, u_0) = u(t, \omega, u_0) = v(t, \omega, u_0 - z(\omega)) + z(\theta_t\omega), \tag{21}$$

for all $(t, \omega, u_0) \in \mathbb{R}^+ \times \Omega \times \mathbb{L}^2(\mathbb{R}^n)$. Then, we can claim that $\phi$ is a continuous random dynamical system associated with the stochastic Ginzburg-Landau equation on $\mathbb{R}^n$.

## 3. Existence of Random Attractors

In the following paper, we always assume that $\mathscr{D}$ is the collection of all tempered subsets of $\mathbb{L}^2(\mathbb{R}^n)$ with respect to $(\Omega, \mathscr{F}, P, (\theta_t)_{t\in R})$. And then we are devoted to prove that $\phi$ has a random absorbing set in $\mathscr{D}$, and it is also $\mathscr{D}$-pullback asymptotically compact.

**Proposition 11.** *There exists $\{K(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ such that $\{K(\omega)\}_{\omega\in\Omega}$ is a random absorbing set for $\phi$ in $\mathscr{D}$. Precisely, for any $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ and $P$-a.e. $\omega \in \Omega$, there is $t_B(\omega) > 0$ such that*

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega) \quad \forall t \ge t_B(\omega). \tag{22}$$

*Proof.* By multiplying (20) by $\overline{v}$, integrating over $\mathbb{R}^n$, and taking the real part, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|^2 &= \operatorname{Re}(\lambda + i\mu)(\Delta v, \overline{v}) \\ &\quad - \operatorname{Re}(\kappa + i\beta)\left(|v+z|^2(v+z), \overline{v}\right) \\ &\quad - \gamma\|v\|^2 + \operatorname{Re}(\lambda + i\mu)(\Delta z(\theta_t\omega), \overline{v}). \end{aligned} \tag{23}$$

Here

$$\begin{aligned} &\operatorname{Re}(\lambda + i\mu)(\Delta v, \overline{v}) = -\lambda\|\nabla v\|^2, \\ &-\operatorname{Re}(\kappa + i\beta)\left(|v+z|^2(v+z), \overline{v}\right) \\ &\quad = -\operatorname{Re}(\kappa + i\beta)\left(|v+z|^2(v+z), \overline{v+z}\right) \\ &\qquad + \operatorname{Re}(\kappa + i\beta)\left(|v+z|^2(v+z), \overline{z}\right) \\ &\quad = -\kappa\|u\|_4^4 + \int_{\mathbb{R}^n} |\kappa + i\beta| \cdot |u|^3 |z| \,\mathrm{d}x \\ &\quad \le -\kappa\|u\|_4^4 + \frac{1}{2}\kappa\|u\|_4^4 + \frac{27(\kappa^2+\beta^2)^2}{32\kappa^3}\|z\|_4^4 \\ &\quad \le -\frac{1}{2}\kappa\|u\|_4^4 + \frac{27(\kappa^2+\beta^2)^2}{32\kappa^3}\|z\|_4^4, \\ &\operatorname{Re}(\lambda + i\mu)(\Delta z(\theta_t\omega), \overline{v}) \\ &\quad \le \int_{\mathbb{R}^n} |\lambda + i\mu| \cdot |\nabla z(\theta_t\omega)| |\nabla v| \,\mathrm{d}x \\ &\quad \le \frac{\lambda}{2}\|\nabla v\|^2 + \frac{\lambda^2+\mu^2}{2\lambda}\|\nabla z\|^2. \end{aligned} \tag{24}$$

From (23) to (24),

$$\begin{aligned} &\frac{d}{dt}\|v\|^2 + \lambda\|\nabla v\|^2 + 2\gamma\|v\|^2 + \kappa\|u\|_4^4 \\ &\quad \le \frac{27(\kappa^2+\beta^2)^2}{16\kappa^3}\|z\|_4^4 + \frac{\lambda^2+\mu^2}{\lambda}\|\nabla z\|^2. \end{aligned} \tag{25}$$

We can see that the right-hand side of (25) can be bounded by

$$c \cdot \sum_{j=1}^{m} \left( \left| z_j \left( \theta_t \omega_j \right) \right|^2 + \left| z_j \left( \theta_t \omega_j \right) \right|^4 \right) \triangleq h \left( \theta_t \omega \right), \tag{26}$$

since $z(\theta_t \omega) = \sum_{j=1}^{m} \varphi_j z_j(\theta_t \omega_j)$, where $\varphi_j \in H^2(\mathbb{R}^n) \cap W^{2,4}(\mathbb{R}^n)$.

So, for $\forall t \geq 0$,

$$\frac{d}{dt} \|v\|^2 + \gamma \|v\|^2 \leq \frac{d}{dt} \|v\|^2 + 2\gamma \|v\|^2 \leq h \left( \theta_t \omega \right), \tag{27}$$

which leads to

$$\begin{aligned} \left\| v \left( t, \omega, v_0 \left( \omega \right) \right) \right\|^2 &\leq e^{-\gamma t} \left\| v_0(\omega) \right\|^2 \\ &\quad + \int_0^t e^{\gamma (s-t)} h \left( \theta_s \omega \right) \mathrm{d}s, \quad \forall t \geq 0, \end{aligned} \tag{28}$$

by multiplying (27) by $e^{\gamma t}$ and integrating from 0 to $t$.

By replacing $\omega$ by $\theta_{-t}\omega$, we derive from (18) and (28) that, for all $t \geq 0$,

$$\begin{aligned} &\left\| v \left( t, \theta_{-t}\omega, v_0 \left( \theta_{-t}\omega \right) \right) \right\|^2 \\ &\quad \leq e^{-\gamma t} \left\| v_0(\theta_{-t}\omega) \right\|^2 + \int_0^t e^{\gamma (s-t)} h \left( \theta_{s-t} \omega \right) \mathrm{d}s \\ &\quad \leq e^{-\gamma t} \left\| v_0(\theta_{-t}\omega) \right\|^2 + \int_{-t}^0 e^{\gamma \tau} h \left( \theta_\tau \omega \right) \mathrm{d}\tau, \\ &\quad \leq e^{-\gamma t} \left\| v_0(\theta_{-t}\omega) \right\|^2 + c \cdot \int_{-t}^0 e^{\gamma \tau} e^{-(\gamma/2)\tau} r \left( \omega \right) \mathrm{d}\tau, \\ &\quad \leq e^{-\gamma t} \left\| v_0(\theta_{-t}\omega) \right\|^2 + c \cdot \frac{1}{\gamma} r \left( \omega \right). \end{aligned} \tag{29}$$

By replacing $\omega$ by $\theta_{-t}\omega$ in (21), one has $\phi(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega)) = v(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega) - z(\theta_{-t}\omega)) + z(\omega)$. Thereafter,

$$\begin{aligned} &\left\| \phi \left( t, \theta_{-t}\omega, u_0 \left( \theta_{-t}\omega \right) \right) \right\|^2 \\ &\quad = \left\| v \left( t, \theta_{-t}\omega, u_0 \left( \theta_{-t}\omega \right) - z \left( \theta_{-t}\omega \right) \right) + z(\omega) \right\|^2 \\ &\quad \leq 2 \left\| v \left( t, \theta_{-t}\omega, u_0 \left( \theta_{-t}\omega \right) - z \left( \theta_{-t}\omega \right) \right) \right\|^2 + 2 \|z(\omega)\|^2 \\ &\quad \leq 2e^{-\gamma t} \left\| u_0(\theta_{-t}\omega) - z(\theta_{-t}\omega) \right\|^2 \\ &\qquad + \frac{2c}{\gamma} r \left( \omega \right) + 2 \|z(\omega)\|^2 \\ &\quad \leq 4e^{-\gamma t} \left( \left\| u_0(\theta_{-t}\omega) \right\|^2 + \left\| z(\theta_{-t}\omega) \right\|^2 \right) \\ &\qquad + \frac{2c}{\gamma} r \left( \omega \right) + 2 \|z(\omega)\|^2. \end{aligned} \tag{30}$$

Recall that both the random variable $\|z(\omega)\|^2$ and the random bounded set $\{B(\omega)\}_{\omega \in \Omega} \in \mathscr{D}$ are tempered. Then, for any $u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, there exists $t_B(\omega) > 0$ such that, for all $t > t_B(\omega)$,

$$\begin{aligned} &4e^{-\gamma t} \left( \left\| u_0(\theta_{-t}\omega) \right\|^2 + \left\| z(\theta_{-t}\omega) \right\|^2 \right) \\ &\quad = 4 \left[ \left( e^{-(\gamma/2)t} \left\| u_0 \left( \theta_{-t}\omega \right) \right\| \right)^2 + \left( e^{-(\gamma/2)t} \left\| z \left( \theta_{-t}\omega \right) \right\| \right)^2 \right] \\ &\quad \leq \frac{2c}{\gamma} r \left( \omega \right). \end{aligned} \tag{31}$$

So far, for all $t > t_B(\omega)$,

$$\left\| \phi \left( t, \theta_{-t}\omega, u_0 \left( \theta_{-t}\omega \right) \right) \right\|^2 \leq \frac{4c}{\gamma} r \left( \omega \right) + 2 \|z(\omega)\|^2. \tag{32}$$

Select

$$K \left( \omega \right) = \left\{ u \in \mathbb{L}^2 \left( \mathbb{R}^n \right) : \|u\|^2 \leq \frac{4c}{\gamma} r \left( \omega \right) + 2 \|z \left( \omega \right)\|^2 \right\}; \tag{33}$$

then $\{K(\omega)\}_{\omega \in \Omega} \in \mathscr{D}$ is a random absorbing set for $\phi$ in $\mathscr{D}$.

The proof is completed. □

**Lemma 12.** *Let $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathscr{D}$ and $u_0(\omega) \in B(\omega)$, and then, for any $T_1 \geq 0$ and P-a.e. $\omega \in \Omega$, that is, the two inequalities of* (34) *hold true for the solution $u(t, \omega, u_0(\omega))$ of* (1)-(2) *and $v(t, \omega, v_0(\omega))$ of* (20) *with $v_0(\omega) = u_0(\omega) - z(\omega)$, $t \geq T_1$, such that*

$$\begin{aligned} &\int_{T_1}^t e^{\gamma (s-t)} \left\| u \left( s, \theta_{-t}\omega, u_0 \left( \theta_{-t}\omega \right) \right) \right\|_4^4 ds \\ &\quad \leq \frac{1}{\kappa} e^{-\gamma t} \left\| v_0 \left( \theta_{-t}\omega \right) \right\|^2 + \frac{2c}{\gamma \kappa} \cdot r \left( \omega \right), \\ &\int_{T_1}^t e^{\gamma (s-t)} \left\| \nabla v \left( s, \theta_{-t}\omega, v_0 \left( \theta_{-t}\omega \right) \right) \right\|^2 ds \\ &\quad \leq \frac{1}{\lambda} e^{-\gamma t} \left\| v_0 \left( \theta_{-t}\omega \right) \right\|^2 + \frac{2c}{\gamma \lambda} \cdot r \left( \omega \right). \end{aligned} \tag{34}$$

*Proof.* Fix $T_1 \geq 0$, and then replace $t$ by $T_1$ and $\omega$ by $\theta_{-t}\omega$ in (28); we then obtain

$$\begin{aligned} &\left\| v \left( T_1, \theta_{-t}\omega, v_0 \left( \theta_{-t}\omega \right) \right) \right\|^2 \\ &\quad \leq e^{-\gamma T_1} \left\| v_0 \left( \theta_{-t}\omega \right) \right\|^2 + \int_0^{T_1} e^{\gamma (s-T_1)} h \left( \theta_{s-t}\omega \right) \mathrm{d}s. \end{aligned} \tag{35}$$

With (18) and (26) in mind, by multiplying $e^{\gamma(T_1 - t)}$ at both sides of the above equation, one can easily get

$$\begin{aligned} &e^{\gamma (T_1 - t)} \left\| v \left( T_1, \theta_{-t}\omega, v_0 \left( \theta_{-t}\omega \right) \right) \right\|^2 \\ &\quad \leq e^{-\gamma t} \left\| v_0 \left( \theta_{-t}\omega \right) \right\|^2 + \int_0^{T_1} e^{\gamma (s-t)} h \left( \theta_{s-t}\omega \right) \mathrm{d}s \\ &\quad \leq e^{-\gamma t} \left\| v_0 \left( \theta_{-t}\omega \right) \right\|^2 + \int_{-t}^{T_1 - t} e^{\gamma \tau} h \left( \theta_\tau \omega \right) \mathrm{d}\tau \\ &\quad \leq e^{-\gamma t} \left\| v_0 \left( \theta_{-t}\omega \right) \right\|^2 + c \cdot r \left( \omega \right) \int_{-t}^{T_1 - t} e^{(\gamma/2)\tau} \mathrm{d}\tau \\ &\quad \leq e^{-\gamma t} \left\| v_0 \left( \theta_{-t}\omega \right) \right\|^2 + \frac{c}{\gamma} \cdot r \left( \omega \right) e^{(\gamma/2)(T_1 - t)}. \end{aligned} \tag{36}$$

From (25) to (26),

$$\begin{aligned}\frac{d}{dt}\|v\|^2 + \lambda\|\nabla v\|^2 + \gamma\|v\|^2 + \kappa\|u\|_4^4 &\\ \leq \frac{d}{dt}\|v\|^2 + \lambda\|\nabla v\|^2 + 2\gamma\|v\|^2 &\\ + \kappa\|u\|_4^4 \leq h(\theta_t\omega). &\end{aligned} \tag{37}$$

Multiply (37) by $e^{\gamma(s-t)}$ and then integrate from $T_1$ to $t$; we then obtain

$$\begin{aligned}&\|v(t,\omega,v_0(\omega))\|^2 + \lambda\cdot\int_{T_1}^t e^{\gamma(s-t)}\|\nabla v(s,\omega,v_0(\omega))\|^2 ds\\ &\quad + \kappa\cdot\int_{T_1}^t e^{\gamma(s-t)}\|u(s,\omega,u_0(\omega))\|_4^4 ds\\ &\leq e^{\gamma(T_1-t)}\|v(T_1,\omega,v_0(\omega))\|^2\\ &\quad + \int_{T_1}^t e^{\gamma(s-t)} h(\theta_s\omega)\,ds.\end{aligned} \tag{38}$$

Keep the last two terms on the left-hand side of (38), and replace $\omega$ by $\theta_{-t}\omega$; we then have

$$\begin{aligned}&\lambda\cdot\int_{T_1}^t e^{\gamma(s-t)}\|\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2 ds\\ &\quad + \kappa\cdot\int_{T_1}^t e^{\gamma(s-t)}\|u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\|_4^4 ds\\ &\leq e^{\gamma(T_1-t)}\|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\\ &\quad + \int_{T_1}^t e^{\gamma(s-t)} h(\theta_{s-t}\omega)\,ds\\ &\leq e^{\gamma(T_1-t)}\|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\\ &\quad + \int_{T_1-t}^0 e^{\gamma\tau} h(\theta_\tau\omega)\,d\tau.\end{aligned} \tag{39}$$

However the second term on the right-hand side can be bounded by

$$c\cdot r(\omega)\int_{T_1-t}^0 e^{(\gamma/2)\tau}d\tau \leq \frac{c}{\gamma}\cdot r(\omega), \tag{40}$$

due to (18) and (26). Together with (36), there is

$$\begin{aligned}&\lambda\cdot\int_{T_1}^t e^{\gamma(s-t)}\|\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2 ds\\ &\quad + \kappa\cdot\int_{T_1}^t e^{\gamma(s-t)}\|u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\|_4^4 ds\\ &\leq e^{-\gamma t}\|v_0(\theta_{-t}\omega)\|^2 + \frac{c}{\gamma}\cdot r(\omega)\, e^{(\gamma/2)(T_1-t)} + \frac{c}{\gamma}\cdot r(\omega)\\ &\leq e^{-\gamma t}\|v_0(\theta_{-t}\omega)\|^2 + \frac{2c}{\gamma}\cdot r(\omega) \quad \forall t \geq T_1.\end{aligned} \tag{41}$$

The proof is completed. □

**Corollary 13.** *Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ and $u_0(\omega) \in B(\omega)$, and then, for P-a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that the solutions $u(t,\omega,u_0(\omega))$ of* (1)-(2) *and $v(t,\omega,v_0(\omega))$ of* (20)*, with $v_0(\omega) = u_0(\omega) - z(\omega)$, satisfy the following uniform estimates, for all $t \geq t_B(\omega)$:*

$$\begin{aligned}&\int_t^{t+1} \|u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_4^4 ds \leq \frac{4c}{\gamma}\cdot e^{\gamma}\cdot r(\omega),\\ &\int_t^{t+1} \|\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 ds \leq \frac{4c}{\gamma}\cdot e^{\gamma}\cdot r(\omega).\end{aligned} \tag{42}$$

*Proof.* Replace $t$ by $(t+1)$ and then replace $T_1$ by $t$ in (34); we then deduce

$$\begin{aligned}&e^{-\gamma}\int_t^{t+1} \|u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_4^4 ds\\ &\leq \int_t^{t+1} e^{\gamma(s-t-1)}\|u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_4^4 ds\\ &\leq \frac{1}{\kappa}e^{-\gamma(t+1)}\|v_0(\theta_{-t-1}\omega)\|^2 + \frac{2c}{\gamma\kappa}\cdot r(\omega)\\ &\leq \frac{2}{\kappa}e^{-\gamma(t+1)}\left(\|u_0(\theta_{-t-1}\omega)\|^2 + \|z(\theta_{-t-1}\omega)\|^2\right)\\ &\quad + \frac{2c}{\gamma}\cdot r(\omega).\end{aligned} \tag{43}$$

As both random variables $u_0(\omega) \in B(\omega)$ and $z(\omega)$ are tempered, there exists $t_B(\omega) > 0$, such that, for all $t \geq t_B(\omega)$,

$$\frac{2}{\kappa}e^{-\gamma(t+1)}\left(\|u_0(\theta_{-t-1}\omega)\|^2 + \|z(\theta_{-t-1}\omega)\|^2\right) \leq \frac{2c}{\gamma}\cdot r(\omega), \tag{44}$$

which, together with (43), claims that, for all $t \geq t_B(\omega)$,

$$\int_t^{t+1} \|u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_4^4 ds \leq \frac{4c}{\gamma}\cdot e^{\gamma}\cdot r(\omega). \tag{45}$$

With the same procedure as the above, we can also verify that, for all $t \geq t_B(\omega)$,

$$\int_t^{t+1} \|\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 ds \leq \frac{4c}{\gamma}\cdot e^{\gamma}\cdot r(\omega). \tag{46}$$

The proof is completed. □

**Corollary 14.** *Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ and $u_0(\omega) \in B(\omega)$, and then, for P-a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that the solution $u(t,\omega,u_0(\omega))$ of* (1)-(2) *satisfies*

$$\begin{aligned}&\int_t^{t+1} \|\nabla u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|^2 ds\\ &\quad \leq \frac{4c}{\gamma}\cdot e^{\gamma}\cdot r(\omega), \quad \forall t \geq t_B(\omega).\end{aligned} \tag{47}$$

*Proof.* Let $t_B(\omega) > 0$ just be the one in Corollary 13, and take $t \geq t_B(\omega)$ and $s \in (t, t+1)$. Note that by (21) one has

$$\begin{aligned}
&\|\nabla u(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2 \\
&\quad = \|\nabla v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) + \nabla z(\theta_{s-t-1}\omega)\|^2 \\
&\quad \leq 2\|\nabla v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2 \\
&\qquad + 2\|\nabla z(\theta_{s-t-1}\omega)\|^2.
\end{aligned} \tag{48}$$

Owing to (18), one has

$$\begin{aligned}
2\|\nabla z(\theta_{s-t-1}\omega)\|^2 &\leq c \cdot \sum_{j=1}^{m} \left|z_j(\theta_{s-t-1}\omega)\right|^2 \\
&\leq ce^{(\gamma/2)(t+1-s)} r(\omega) \leq ce^{\gamma/2} r(\omega).
\end{aligned} \tag{49}$$

Together with Corollary 13, we derive

$$\begin{aligned}
&\int_t^{t+1} \|\nabla u(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2 ds \\
&\quad \leq 2\int_t^{t+1} \|\nabla v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2 ds \\
&\qquad + 2\int_t^{t+1} \|\nabla z(\theta_{s-t-1}\omega)\|^2 ds \\
&\quad \leq \frac{4c}{\gamma} \cdot e^{\gamma} \cdot r(\omega) + ce^{\gamma/2} r(\omega) \\
&\quad \leq \frac{4c}{\gamma} \cdot e^{\gamma} \cdot r(\omega),
\end{aligned} \tag{50}$$

by integrating (48) with respect to $s$ over $(t, t+1)$.

The proof is completed. □

**Lemma 15.** *Suppose $\sqrt{3}\kappa \geq |\beta|$, and let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ and $u_0(\omega) \in B(\omega)$; then, for P-a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that, for all $t \geq t_B(\omega)$,*

$$\|\nabla u(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^2 \leq \frac{4c}{\gamma} \cdot e^{\gamma} \cdot r(\omega). \tag{51}$$

*Proof.* By multiplying (20) by $\Delta\overline{v}$, integrating over $\mathbb{R}^n$, and then taking the real part, we get

$$\begin{aligned}
&\frac{1}{2} \cdot \frac{d}{dt}\|\nabla v\|^2 + \lambda\|\Delta v\|^2 + \gamma\|\nabla v\|^2 \\
&\quad = \operatorname{Re}\left((\kappa + i\beta)\left(|v+z|^2(v+z), \Delta\overline{v}\right)\right) \\
&\qquad - \operatorname{Re}\left((\lambda + i\mu)\left(\Delta z(\theta_t\omega), \Delta\overline{v}\right)\right).
\end{aligned} \tag{52}$$

Since

$$\left(|v+z|^2(v+z), \Delta\overline{v}\right) = \left(|u|^2 u, \Delta\overline{u}\right) - \left(|u|^2 u, \Delta\overline{z}(\theta_t\omega)\right), \tag{53}$$

while

$$\left(|u|^2 u, \Delta\overline{u}\right) = -\int_{\mathbb{R}^n} \left(|u|^2|\nabla u|^2 + u\nabla\overline{u}\nabla|u|^2\right) dx, \tag{54}$$

we have

$$\begin{aligned}
&\operatorname{Re}\left((\kappa + i\beta)\left(|u|^2 u, \Delta\overline{u}\right)\right) \\
&\quad = -\kappa \int_{\mathbb{R}^n} |u|^2|\nabla u|^2 dx - \kappa \int_{\mathbb{R}^n} \operatorname{Re}\left(u\nabla\overline{u}\nabla|u|^2\right) dx \\
&\qquad + \beta \int_{\mathbb{R}^n} \operatorname{Im}\left(u\nabla\overline{u}\nabla|u|^2\right) dx \\
&\quad = -\kappa \int_{\mathbb{R}^n} |u|^2|\nabla u|^2 dx - \frac{\kappa}{2} \int_{\mathbb{R}^n} \left(\nabla|u|^2\right)^2 dx \\
&\qquad - \frac{\beta}{2} \int_{\mathbb{R}^n} i\left(u\nabla\overline{u} - \overline{u}\nabla u\right)\nabla|u|^2 dx \\
&\quad = -\frac{1}{4} \int_{\mathbb{R}^n} \left(3\kappa\left(\nabla|u|^2\right)^2 + 2\beta i\left(u\nabla\overline{u} - \overline{u}\nabla u\right)\nabla|u|^2\right. \\
&\qquad \left. + \kappa|u\nabla\overline{u} - \overline{u}\nabla u|^2\right) dx \\
&\quad \leq 0,
\end{aligned} \tag{55}$$

provided that $\sqrt{3}\kappa \geq |\beta|$.

Therefore, for the first term at the right-hand side of (52), we have

$$\begin{aligned}
&\operatorname{Re}\left((\kappa + i\beta)\left(|v+z|^2(v+z), \Delta\overline{v}\right)\right) \\
&\quad = \operatorname{Re}\left((\kappa + i\beta)\left(|u|^2 u, \Delta\overline{u}\right)\right) \\
&\qquad - \operatorname{Re}\left((\kappa + i\beta)\left(|u|^2 u, \Delta\overline{z}(\theta_t\omega)\right)\right) \\
&\quad \leq -\operatorname{Re}\left((\kappa + i\beta)\left(|u|^2 u, \Delta\overline{z}(\theta_t\omega)\right)\right) \\
&\quad \leq |\kappa + i\beta| \cdot \int_{\mathbb{R}^n} |u|^3 \cdot \left|\Delta z(\theta_t\omega)\right| dx \\
&\quad \leq \frac{3}{4}\|u\|_4^4 + \frac{1}{4}\left(\kappa^2 + \beta^2\right)^2 \cdot \|\Delta z(\theta_t\omega)\|_4^4.
\end{aligned} \tag{56}$$

On the other hand, the second term at the right-hand side of (52) can be bounded by

$$\begin{aligned}
&|\lambda + i\mu| \cdot \int_{\mathbb{R}^n} \left|\Delta z(\theta_t\omega)\right| \cdot |\Delta v| dx \\
&\quad \leq \lambda\|\Delta v\|^2 + \frac{\lambda^2 + \mu^2}{4\lambda}\|\Delta z(\theta_t\omega)\|^2.
\end{aligned} \tag{57}$$

By (52), (56)-(57), we can see that

$$\begin{aligned}
&\frac{d}{dt}\|\nabla v\|^2 + 2\gamma\|\nabla v\|^2 \\
&\quad \leq \frac{3}{2}\|u\|_4^4 + \frac{1}{2}\left(\kappa^2 + \beta^2\right)^2 \\
&\qquad \cdot \|\Delta z(\theta_t\omega)\|_4^4 + \frac{\lambda^2 + \mu^2}{2\lambda}\|\Delta z(\theta_t\omega)\|^2.
\end{aligned} \tag{58}$$

That is,

$$\frac{d}{dt}\|\nabla v\|^2 \leq \frac{3}{2}\|u\|_4^4 + g(\theta_t\omega), \tag{59}$$

where

$$g(\theta_t\omega) \triangleq \frac{1}{2}\left(\kappa^2+\beta^2\right)^2 \cdot \|\Delta z(\theta_t\omega)\|_4^4 + \frac{\lambda^2+\mu^2}{2\lambda}\|\Delta z(\theta_t\omega)\|^2. \tag{60}$$

Since $z(\theta_t\omega) = \sum_{j=1}^m \varphi_j z_j(\theta_t\omega_j)$, where $\varphi_j \in H^2(\mathbb{R}^n) \cap W^{2,4}(\mathbb{R}^n)$, there exists a constant $c > 0$ such that

$$\begin{aligned} g(\theta_t\omega) &\le c \cdot \sum_{j=1}^{m}\left(\left|z_j(\theta_t\omega_j)\right|^2 + \left|z_j(\theta_t\omega_j)\right|^4\right) \\ &\le c \cdot e^{(\gamma/2)|t|} r(\omega), \quad \forall t \in \mathbb{R}. \end{aligned} \tag{61}$$

Let $t > t_B(\omega)$, $s \in (t, t+1)$, where $t_B(\omega)$ is the positive time taken in Corollary 13. By integrating (59) from $s$ to $t+1$, we obtain

$$\begin{aligned} \|\nabla v(t+1,\omega,v_0(\omega))\|^2 \le{}& \|\nabla v(s,\omega,v_0(\omega))\|^2 \\ &+ \frac{3}{2}\int_s^{t+1} \|u(\tau,\omega,u_0(\omega))\|_4^4 d\tau \\ &+ \int_s^{t+1} g(\theta_\tau\omega)\, d\tau. \end{aligned} \tag{62}$$

Integrate the above equation with respect to $s$ over $(t, t+1)$ to have

$$\begin{aligned} \|\nabla v(t+1,\omega,v_0(\omega))\|^2 \le{}& \int_t^{t+1} \|\nabla v(s,\omega,v_0(\omega))\|^2 ds \\ &+ \frac{3}{2}\int_t^{t+1} \|u(\tau,\omega,u_0(\omega))\|_4^4 d\tau \\ &+ \int_t^{t+1} g(\theta_\tau\omega)\, d\tau. \end{aligned} \tag{63}$$

By replacing $\omega$ by $\theta_{-t-1}\omega$, we derive

$$\begin{aligned} &\|\nabla v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 \\ &\quad\le \int_t^{t+1} \|\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 ds \\ &\qquad+ \frac{3}{2}\int_t^{t+1} \|u(\tau,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_4^4 d\tau \\ &\qquad+ \int_t^{t+1} g(\theta_{\tau-t-1}\omega)\, d\tau. \end{aligned} \tag{64}$$

Thanks to Corollary 13, it follows from (61) and (64) that, for all $t > t_B(\omega)$,

$$\begin{aligned} &\|\nabla v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 \\ &\quad\le \frac{4c}{\gamma}\cdot e^{\gamma}\cdot r(\omega) + \frac{3}{2}\cdot\frac{4c}{\gamma}\cdot e^{\gamma}\cdot r(\omega) + \int_{-1}^{0} g(\theta_\tau\omega)\, d\tau \\ &\quad\le \frac{4c}{\gamma}\cdot e^{\gamma}\cdot r(\omega) + c\cdot r(\omega)\int_{-1}^{0} e^{-(\gamma/2)\tau} d\tau \\ &\quad\le \frac{4c}{\gamma}\cdot e^{\gamma}\cdot r(\omega). \end{aligned} \tag{65}$$

Then, together with (16), we obtain that, for all $t > t_B(\omega)$,

$$\begin{aligned} &\|\nabla u(t+1,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|^2 \\ &\quad= \|\nabla v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)) + \nabla z(\omega)\|^2 \\ &\quad\le 2\|\nabla v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 \\ &\qquad+ 2\|\nabla z(\omega)\|^2 \\ &\quad\le \frac{4c}{\gamma}\cdot e^{\gamma}\cdot r(\omega). \end{aligned} \tag{66}$$

The proof is completed. □

**Lemma 16.** *Suppose $\sqrt{3}\kappa \ge |\beta|$, and let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ and $u_0(\omega) \in B(\omega)$; then, for every $\epsilon > 0$ and P-a.e. $\omega \in \Omega$, there exist $T^* = T(B,\omega,\epsilon) > 0$ and $R^* = R^*(\omega,\epsilon)$ such that the solution $v(t,\omega,v_0(\omega))$ of (20) with $v_0(\omega) = u_0(\omega) - z(\omega)$ satisfies, for all $t \ge T^*$,*

$$\int_{|x|\ge R^*} \left|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))(x)\right|^2 dx \le \epsilon. \tag{67}$$

*Proof.* Let $\rho$ be a smooth function defined on $\mathbb{R}^+$ such that $0 \le \rho(s) \le 1$ for all $s \in \mathbb{R}^+$, and

$$\rho(s) = \begin{cases} 0 & \text{for } 0 \le s \le 1, \\ 1 & \text{for } s \ge 2. \end{cases} \tag{68}$$

Then there exists a constant $c > 0$ such that $|\rho'(s)| \le c$, for all $s \in \mathbb{R}^+$. Multiply (20) by $\rho(|x|^2/l^2)\bar{v}$, integrate over $\mathbb{R}^n$, and then take the real part to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho\left(\frac{|x^2|}{l^2}\right)|v|^2 dx \\ &\quad= \operatorname{Re}\left((\lambda+i\mu)\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)\Delta v\bar{v}dx\right) \\ &\qquad- \operatorname{Re}\left((\kappa+i\beta)\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v+z|^2(v+z)\bar{v}dx\right) \\ &\qquad- \gamma\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v|^2 dx \\ &\qquad+ \operatorname{Re}\left((\lambda+i\mu)\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)\Delta z\bar{v}dx\right). \end{aligned} \tag{69}$$

We now concentrate to estimate the terms in (69). Firstly,

$$\begin{aligned}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)\Delta v\bar{v}\mathrm{d}x &= -\int_{\mathbb{R}^n}|\nabla v|^2\rho\left(\frac{|x^2|}{l^2}\right)\mathrm{d}x-\int_{\mathbb{R}^n}\bar{v}\rho'\left(\frac{|x^2|}{l^2}\right)\frac{2x}{l^2}\nabla v\mathrm{d}x\\ &= -\int_{\mathbb{R}^n}|\nabla v|^2\rho\left(\frac{|x^2|}{l^2}\right)\mathrm{d}x\\ &\quad -\int_{l\le|x|\le\sqrt{2}l}\bar{v}\rho'\left(\frac{|x^2|}{l^2}\right)\frac{2x}{l^2}\nabla v\mathrm{d}x.\end{aligned}\tag{70}$$

Since

$$\begin{aligned}\left|\int_{l\le|x|\le\sqrt{2}l}\bar{v}\rho'\left(\frac{|x^2|}{l^2}\right)\frac{2x}{l^2}\nabla v\mathrm{d}x\right| &\le\frac{2\sqrt{2}}{l}\int_{l\le|x|\le\sqrt{2}l}|v|\cdot\left|\rho'\left(\frac{|x^2|}{l^2}\right)\right|\cdot|\nabla v|\,\mathrm{d}x\\ &\le\frac{c}{l}\int_{\mathbb{R}^n}|v|\cdot|\nabla v|\,\mathrm{d}x\\ &\le\frac{c}{l}\left(\|v\|^2+\|\nabla v\|^2\right),\end{aligned}\tag{71}$$

then we find that

$$\begin{aligned}\operatorname{Re}\left((\lambda+i\mu)\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)\Delta v\bar{v}\mathrm{d}x\right) &\le-\lambda\cdot\int_{\mathbb{R}^n}|\nabla v|^2\rho\left(\frac{|x^2|}{l^2}\right)\mathrm{d}x\\ &\quad+\frac{c\cdot|\lambda+i\mu|}{l}\left(\|v\|^2+\|\nabla v\|^2\right)\\ &\le-\lambda\cdot\int_{\mathbb{R}^n}|\nabla v|^2\rho\left(\frac{|x^2|}{l^2}\right)\mathrm{d}x\\ &\quad+\frac{c}{l}\left(\|v\|^2+\|\nabla v\|^2\right).\end{aligned}\tag{72}$$

Secondly,

$$\begin{aligned}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v+z|^2(v+z)\bar{v}\mathrm{d}x &=\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|u|^4\mathrm{d}x\\ &\quad-\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|u|^2\cdot u\cdot\bar{z}(\theta_t\omega)\,\mathrm{d}x.\end{aligned}\tag{73}$$

Due to

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|u|^2\cdot u\cdot\bar{z}(\theta_t\omega)\,\mathrm{d}x\right| &\le\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|u|^3\cdot|z(\theta_t\omega)|\,\mathrm{d}x\\ &\le\frac{\kappa}{2|\kappa+i\beta|}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|u|^4\mathrm{d}x\\ &\quad+\frac{c}{|\kappa+i\beta|}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|z(\theta_t\omega)|^4\mathrm{d}x,\end{aligned}\tag{74}$$

we have

$$\begin{aligned}-\operatorname{Re}\left((\kappa+i\beta)\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v+z|^2(v+z)\bar{v}\mathrm{d}x\right) &=-\kappa\cdot\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|u|^4\mathrm{d}x\\ &\quad+\operatorname{Re}\left((\kappa+i\beta)\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|u|^2\cdot u\cdot\bar{z}(\theta_t\omega)\,\mathrm{d}x\right)\\ &\le-\kappa\cdot\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|u|^4\mathrm{d}x+\frac{\kappa}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|u|^4\mathrm{d}x\\ &\quad+c\cdot\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|z(\theta_t\omega)|^4\mathrm{d}x\\ &\le-\frac{\kappa}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|u|^4\mathrm{d}x\\ &\quad+c\cdot\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|z(\theta_t\omega)|^4\mathrm{d}x.\end{aligned}\tag{75}$$

Thirdly,

$$\begin{aligned}\operatorname{Re}\left((\lambda+i\mu)\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)\Delta z\bar{v}\mathrm{d}x\right) &\le\frac{\gamma}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v|^2\mathrm{d}x\\ &\quad+\frac{\lambda^2+\mu^2}{2\gamma}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|\Delta z|^2\mathrm{d}x.\end{aligned}\tag{76}$$

Finally, from (69) to (76),

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v|^2\mathrm{d}x+\frac{\gamma}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v|^2\mathrm{d}x\\ &\quad+\frac{\kappa}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|u|^4\mathrm{d}x+\lambda\cdot\int_{\mathbb{R}^n}|\nabla v|^2\rho\left(\frac{|x^2|}{l^2}\right)\mathrm{d}x\\ &\le\frac{c}{l}\left(\|v\|^2+\|\nabla v\|^2\right)+c\cdot\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|z(\theta_t\omega)|^4\mathrm{d}x\\ &\quad+\frac{\lambda^2+\mu^2}{2\gamma}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|\Delta z|^2\mathrm{d}x,\end{aligned}\tag{77}$$

which implies

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v|^2 dx+\gamma\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v|^2 dx \le \frac{c}{l}\left(\|v\|^2+\|\nabla v\|^2\right)+c\cdot\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|z(\theta_t\omega)|^4 dx + \frac{\lambda^2+\mu^2}{\gamma}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|\Delta z|^2 dx. \quad (78)$$

Proposition 11 together with Lemma 15 shows that there is $T_1 = t_B(\omega)$ such that, for all $t \ge T_1$,

$$\|v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2_{H^1(\mathbb{R}^n)} \le \frac{4c}{\gamma}\cdot e^{\gamma}\cdot r(\omega). \quad (79)$$

Now, multiply (78) with $e^{\gamma(s-t)}$, and then integrate over $(T_1, t)$ with respect to $s$ so that, for all $t \ge T_1$,

$$\begin{aligned}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v(t,\omega,v_0(\omega))|^2 dx &\le e^{\gamma(T_1-t)}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v(T_1,\omega,v_0(\omega))|^2 dx \\ &+\frac{c}{l}\int_{T_1}^t e^{\gamma(s-t)}\left(\|v(s,\omega,v_0(\omega))\|^2 +\|\nabla v(s,\omega,v_0(\omega))\|^2\right)ds \\ &+c\cdot\int_{T_1}^t e^{\gamma(s-t)}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|z(\theta_s\omega)|^4 dx\, ds \\ &+\frac{\lambda^2+\mu^2}{\gamma}\int_{T_1}^t e^{\gamma(s-t)}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|\Delta z(\theta_s\omega)|^2 dx\, ds.\end{aligned} \quad (80)$$

By replacing $\omega$ by $\theta_{-t}\omega$ in (80), we obtain that, for all $t \ge T_1$,

$$\begin{aligned}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2 dx &\le e^{\gamma(T_1-t)}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2 dx \\ &+\frac{c}{l}\int_{T_1}^t e^{\gamma(s-t)}\|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2 ds \\ &+\frac{c}{l}\int_{T_1}^t e^{\gamma(s-t)}\|\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2 ds \\ &+c\cdot\int_{T_1}^t e^{\gamma(s-t)}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|z(\theta_{s-t}\omega)|^4 dx\, ds \\ &+\frac{\lambda^2+\mu^2}{\gamma}\int_{T_1}^t e^{\gamma(s-t)}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|\Delta z(\theta_{s-t}\omega)|^2 dx\, ds.\end{aligned} \quad (81)$$

We now estimate the terms in (81) as follows.

Firstly, from (28), one deduces

$$\|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2 \le e^{-\gamma T_1}\|v_0(\theta_{-t}\omega)\|^2 + \int_0^{T_1} e^{\gamma(\tau-T_1)}h(\theta_{\tau-t}\omega)\,d\tau. \quad (82)$$

Thus,

$$\begin{aligned}e^{\gamma(T_1-t)}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2 dx &\le e^{\gamma(T_1-t)}\left(e^{-\gamma T_1}\|v_0(\theta_{-t}\omega)\|^2 + \int_0^{T_1} e^{\gamma(\tau-T_1)}h(\theta_{\tau-t}\omega)\,d\tau\right) \\ &\le e^{-\gamma t}\|v_0(\theta_{-t}\omega)\|^2 + \int_{-t}^{T_1-t} e^{\gamma s}h(\theta_s\omega)\,ds \\ &\le e^{-\gamma t}\|v_0(\theta_{-t}\omega)\|^2 + \frac{c}{2\gamma}\cdot r(\omega)\,e^{(\gamma/2)(T_1-t)},\end{aligned} \quad (83)$$

due to (18) and (26). Thus, for any given $\epsilon > 0$, there is $T_2(B,\omega,\epsilon) > T_1$ such that, for all $t \ge T_2$,

$$e^{\gamma(T_1-t)}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2 dx \le \epsilon. \quad (84)$$

For the second, replace $T_1$ by $s$ in (82); then we can find that the second term at the right-hand side of (81) satisfies

$$\begin{aligned}\frac{c}{l}\int_{T_1}^t e^{\gamma(s-t)}\|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2 ds &\le \frac{c}{l}\int_{T_1}^t e^{-\gamma t}\|v_0(\theta_{-t}\omega)\|^2 ds + \frac{c}{l}\int_{T_1}^t\int_0^s e^{\gamma(\tau-t)}h(\theta_{\tau-t}\omega)\,d\tau\, ds \\ &\le \frac{c}{l}e^{-\gamma t}(t-T_1)\|v_0(\theta_{-t}\omega)\|^2 + \frac{c}{l}\int_{T_1}^t\int_{-t}^{s-t} e^{\gamma\tau}h(\theta_\tau\omega)\,d\tau\, ds \\ &\le \frac{c}{l}e^{-\gamma t}(t-T_1)\|v_0(\theta_{-t}\omega)\|^2 + \frac{cr(\omega)}{l}\int_{T_1}^t\int_{-t}^{s-t} e^{(\gamma/2)\tau}d\tau\, ds \\ &\le \frac{c}{l}e^{-\gamma t}(t-T_1)\|v_0(\theta_{-t}\omega)\|^2 + \frac{cr(\omega)}{\gamma^2 l},\end{aligned} \quad (85)$$

which implies that there exist $T_3(B,\omega,\epsilon) > T_1$ and $R_1(\omega,\epsilon) > 0$ such that, for all $t \ge T_3$ and $l \ge R_1$,

$$\frac{c}{l}\int_{T_1}^t e^{\gamma(s-t)}\|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2 ds \le \epsilon. \quad (86)$$

For the third, from Lemma 12, we know that there is $T_4(B,\omega) > T_1$ such that, for all $t \geq T_4$, the third term at the right-hand side of (81) satisfies

$$\frac{c}{l}\int_{T_1}^{t} e^{\gamma(s-t)}\|\nabla v(s,\theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 ds \leq \frac{2c}{l\gamma} r(\omega). \quad (87)$$

Therefore, there is $R_2(\omega,\epsilon) > 0$ such that, for all $t \geq T_4$ and $l \geq R_2$,

$$\frac{c}{l}\int_{T_1}^{t} e^{\gamma(s-t)}\|\nabla v(s,\theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 ds \leq \epsilon. \quad (88)$$

Finally, note that the last two terms in (81) can be bounded by

$$c\cdot\int_{T_1}^{t} e^{\gamma(s-t)} \times \int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)\left(|\Delta z(\theta_{s-t}\omega)|^2 + |z(\theta_{s-t}\omega)|^4\right) dx\, ds \quad (89)$$

and $z(\theta_t\omega) = \sum_{j=1}^{m}\varphi_j z_j(\theta_t\omega_j)$, where $\varphi_j \in H^2(\mathbb{R}^n)\cap W^{2,4}(\mathbb{R}^n)$, and we can find $R_3(\omega,\epsilon) > 0$ such that, for all $l \geq R_3$ and $j = 1,2,\ldots,m$,

$$\begin{aligned}&\int_{|x|\geq l}\left(|\varphi_j(x)|^2 + |\varphi_j(x)|^4 + |\Delta\varphi_j(x)|^2\right) dx \\ &\quad\leq \min\left\{\frac{\gamma\epsilon}{m^4 cr(\omega)}, \frac{\epsilon}{2mr(\omega)}\right\}.\end{aligned} \quad (90)$$

Accordingly, we have the following estimates for the last two terms in (81):

$$\begin{aligned}&c\cdot\int_{T_1}^{t} e^{\gamma(s-t)}\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|z(\theta_{s-t}\omega)|^4 dx\, ds \\ &\quad + \frac{\lambda^2+\mu^2}{\gamma}\int_{T_1}^{t} e^{\gamma(s-t)} \times \int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|\Delta z(\theta_{s-t}\omega)|^2 dx\, ds \\ &\leq c\cdot\int_{T_1}^{t} e^{\gamma(s-t)} \times \int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right) \times \left(|\Delta z(\theta_{s-t}\omega)|^2 + |z(\theta_{s-t}\omega)|^4\right) dx\, ds \\ &\leq c\cdot\int_{T_1}^{t} e^{\gamma(s-t)} \times \int_{|x|\geq l}\left(|\Delta z(\theta_{s-t}\omega)|^2 + |z(\theta_{s-t}\omega)|^4\right) dx\, ds \\ &\leq cm^4\cdot\int_{T_1}^{t} e^{\gamma(s-t)} \times \sum_{j=1}^{m}\int_{|x|\geq l}\left(|\Delta\varphi_j|^2|z_j(\theta_{s-t}\omega_j)|^2 + |\varphi_j|^4|z_j(\theta_{s-t}\omega_j)|^4\right) dx\, ds \\ &\leq \frac{\gamma\epsilon}{r(\omega)}\int_{T_1}^{t} e^{\gamma(s-t)} \times \sum_{j=1}^{m}\left(|z_j(\theta_{s-t}\omega_j)|^2 + |z_j(\theta_{s-t}\omega_j)|^4\right) ds \\ &\leq \frac{\gamma\epsilon}{r(\omega)}\int_{T_1}^{t} e^{\gamma(s-t)} h(\theta_{s-t}\omega)\, ds \\ &\leq \frac{\gamma\epsilon}{r(\omega)}\int_{T_1-t}^{0} e^{\gamma\tau} h(\theta_\tau\omega)\, d\tau \\ &\leq \frac{\gamma\epsilon}{r(\omega)}\int_{T_1-t}^{0} e^{(\gamma/2)\tau} d\tau \leq \epsilon.\end{aligned} \quad (91)$$

Let $T^* = T(B,\omega,\epsilon) = \max\{T_1,T_2,T_3,T_4\}$ and $R^* = R(\omega,\epsilon) = \max\{R_1,R_2,R_3\}$. Then from (81), (84), (86), (88), and (91), we know that, for all $t \geq T^*$ and $l \geq R^*$,

$$\int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2 dx \leq 4\epsilon. \quad (92)$$

That is, for any $t \geq T^*$ and $l \geq R^*$,

$$\begin{aligned}&\int_{|x|\geq l}|v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2 dx \\ &\quad\leq \int_{\mathbb{R}^n}\rho\left(\frac{|x^2|}{l^2}\right)|v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2 dx \leq 4\epsilon.\end{aligned} \quad (93)$$

The proof is completed. □

**Lemma 17.** *Suppose $\sqrt{3}\kappa \geq |\beta|$, and let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ and $u_0(\omega) \in B(\omega)$; then, for every $\epsilon > 0$ and P-a.e. $\omega \in \Omega$, there exist $T^* = T(B,\omega,\epsilon) > 0$ and $R^* = R^*(\omega,\epsilon)$ such that the solution $u(t,\omega,u_0(\omega))$ of (1)-(2) satisfies, for all $t \geq T^*$,*

$$\int_{|x|\geq R^*}|u(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega))(x)|^2 dx \leq \epsilon. \quad (94)$$

*Proof.* Let $T^*$ and $R^*$ be the constants in Lemma 16. Then due to (16) and (90) we know that, for all $t \geq T^*$ and $l \geq R^*$,

$$\begin{aligned}\int_{|x|\geq R^*}|z(\omega)|^2 dx &= \int_{|x|\geq R^*}\left|\sum_{j=1}^{m}\varphi_j z_j(\omega_j)\right|^2 dx \\ &\leq m\int_{|x|\geq R^*}\sum_{j=1}^{m}|\varphi_j|^2|z_j(\omega_j)|^2 dx \\ &\leq \frac{\epsilon}{2r(\omega)}\sum_{j=1}^{m}|z_j(\omega_j)|^2 \leq \frac{\epsilon}{2}.\end{aligned} \quad (95)$$

Thus, together with Lemma 16, we derive, for all $t \geq T^*$ and $l \geq R^*$,

$$\begin{aligned}
&\int_{|x|\geq R^*} \left|u\left(t,\theta_{-t}\omega,u_0\left(\theta_{-t}\omega\right)\right)(x)\right|^2 \mathrm{d}x \\
&\quad = \int_{|x|\geq R^*} \left|v\left(t,\theta_{-t}\omega,v_0\left(\theta_{-t}\omega\right)\right)(x)+z(\omega)\right|^2 \mathrm{d}x \\
&\quad \leq 2\int_{|x|\geq R^*} \left|v\left(t,\theta_{-t}\omega,v_0\left(\theta_{-t}\omega\right)\right)(x)\right|^2 \\
&\qquad + 2\int_{|x|\geq R^*} |z(\omega)|^2 \mathrm{d}x \\
&\quad \leq 3\epsilon.
\end{aligned} \tag{96}$$

The proof is completed. □

Up to now, we are ready to give the $\mathscr{D}$-pullback asymptotic compactness of $\phi$, based on the former uniform estimates referring to the tails of solutions.

**Proposition 18.** *Suppose that $\sqrt{3}\kappa \geq |\beta|$, and then the random dynamical system $\phi$ is $\mathscr{D}$-pullback asymptotically compact in $\mathbb{L}^2(\mathbb{R}^n)$. That is to say, for $P$-a.e. $\omega \in \Omega$, the sequence $\{\phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))\}_{n=1}^{\infty}$ has a convergent subsequence in $\mathbb{L}^2(\mathbb{R}^n)$ for $t_n \longrightarrow \infty$, $B=\{B(\omega)\}_{\omega\in\Omega}\in\mathscr{D}$, and $u_{0,n}(\theta_{-t_n}\omega)\in B(\theta_{-t_n}\omega)$.*

*Proof.* Let $t_n \longrightarrow \infty$, $B=\{B(\omega)\}_{\omega\in\Omega}\in\mathscr{D}$, and $u_{0,n}(\theta_{-t_n}\omega)\in B(\theta_{-t_n}\omega)$. By Proposition 11, we know that, for $P$-a.e. $\omega\in\Omega$,

$$\left\{\phi\left(t_n,\theta_{-t_n}\omega,u_{0,n}\left(\theta_{-t_n}\omega\right)\right)\right\}_{n=1}^{\infty} \quad \text{is bounded in } \mathbb{L}^2\left(\mathbb{R}^n\right). \tag{97}$$

So, there is $\xi\in\mathbb{L}^2(\mathbb{R}^n)$ such that, up to a subsequence,

$$\phi\left(t_n,\theta_{-t_n}\omega,u_{0,n}\left(\theta_{-t_n}\omega\right)\right) \longrightarrow \xi \quad \text{weakly in } \mathbb{L}^2\left(\mathbb{R}^n\right). \tag{98}$$

It only remains to prove that the weak convergence of (98) is indeed strong convergence. Let $\epsilon>0$ be small enough. Since $\xi\in\mathbb{L}^2(\mathbb{R}^n)$, there exists $R_1=R(\epsilon)>0$, such that

$$\int_{|x|\geq R_1} |\xi(x)|^2 \mathrm{d}x \leq \epsilon. \tag{99}$$

From Lemma 17, there are $T_1(B,\omega,\epsilon)$ and $R_2(\omega,\epsilon)>R_1(\epsilon)>0$, for $P$-a.e. $\omega\in\Omega$, such that, for all $t\geq T_1$,

$$\int_{|x|\geq R_2} \left|\phi\left(t,\theta_{-t}\omega,u_0\left(\theta_{-t}\omega\right)\right)\right|^2 \mathrm{d}x \leq \epsilon. \tag{100}$$

Since $t_n \longrightarrow \infty$, let $N_1=N_1(B,\omega,\epsilon)$ be large enough such that $t_n\geq T_1$ for every $n\geq N_1$. Hence, it follows from (100) that, for all $n\geq N_1$,

$$\int_{|x|\geq R_2} \left|\phi\left(t_n,\theta_{-t_n}\omega,u_{0,n}\left(\theta_{-t_n}\omega\right)\right)\right|^2 \mathrm{d}x \leq \epsilon. \tag{101}$$

On the other hand, from Proposition 11 and Lemma 15, there is $T_2=T_2(B,\omega)$ such that, for all $t\geq T_2$,

$$\left\|\phi\left(t,\theta_{-t}\omega,u_0\left(\theta_{-t}\omega\right)\right)\right\|_{H^1(\mathbb{R}^n)}^2 \leq \frac{4c}{\gamma}\cdot e^{\gamma}\cdot r(\omega). \tag{102}$$

Let $N_2=N_2(B,\omega)>N_1$ such that $t_n\geq T_2$ for $n\geq N_2$. Thus, from (102), we know that, for all $n\geq N_2$,

$$\left\|\phi\left(t_n,\theta_{-t_n}\omega,u_{0,n}\left(\theta_{-t_n}\omega\right)\right)\right\|_{H^1(\mathbb{R}^n)}^2 \leq \frac{4c}{\gamma}\cdot e^{\gamma}\cdot r(\omega). \tag{103}$$

Denote $Q_{R_2}$ for the set $\{x\in\mathbb{R}^n : |x|\leq R_2\}$. Due to the compactness of embedding $H^1(Q_{R_2})\hookrightarrow\mathbb{L}^2(Q_{R_2})$, we deduce from (103) that, up to a subsequence,

$$\phi\left(t_n,\theta_{-t_n}\omega,u_{0,n}\left(\theta_{-t_n}\omega\right)\right) \longrightarrow \xi \quad \text{strongly in } \mathbb{L}^2\left(Q_{R_2}\right), \tag{104}$$

which tells us that, for the given $\epsilon>0$, there exists $N_3=N_3(B,\omega,\epsilon)>N_2$ such that, for all $n\geq N_3$,

$$\left\|\phi\left(t_n,\theta_{-t_n}\omega,u_{0,n}\left(\theta_{-t_n}\omega\right)\right)-\xi\right\|_{\mathbb{L}^2(Q_{R_2})}^2 \leq \epsilon. \tag{105}$$

By (99), (101), and (105), we conclude that, for all $n\geq N_3$,

$$\begin{aligned}
&\left\|\phi\left(t_n,\theta_{-t_n}\omega,u_{0,n}\left(\theta_{-t_n}\omega\right)\right)-\xi\right\|_{\mathbb{L}^2(\mathbb{R}^n)}^2 \\
&\quad \leq \int_{|x|\geq R_2} \left|\phi\left(t_n,\theta_{-t_n}\omega,u_{0,n}\left(\theta_{-t_n}\omega\right)\right)-\xi\right|^2 \mathrm{d}x \\
&\qquad + \int_{|x|\leq R_2} \left|\phi\left(t_n,\theta_{-t_n}\omega,u_{0,n}\left(\theta_{-t_n}\omega\right)\right)-\xi\right|^2 \mathrm{d}x \leq 5\epsilon.
\end{aligned} \tag{106}$$

Therefore, up to a subsequence,

$$\phi\left(t_n,\theta_{-t_n}\omega,u_{0,n}\left(\theta_{-t_n}\omega\right)\right) \longrightarrow \xi \quad \text{strongly in } \mathbb{L}^2\left(\mathbb{R}^n\right) \tag{107}$$

is verified. □

Up to now, we have proved that $\phi$ has a closed random absorbing set $\{K(\omega)\}_{\omega\in\Omega}$ in $\mathscr{D}$ by Proposition 11 and is $\mathscr{D}$-pullback asymptotically compact in $\mathbb{L}^2(\mathbb{R}^n)$, which is present in Proposition 18. So, the existence of unique $\mathscr{D}$-random attractor for $\phi$ stated in Theorem 1 immediately follows from Proposition 9.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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