

Research Article

A Note on the Minimal Period Problem for Second Order Hamiltonian Systems

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We study periodic solutions of second order Hamiltonian systems with even potential. By making use of generalized Nehari manifold, some sufficient conditions are obtained to guarantee the multiplicity and minimality of periodic solutions for second order Hamiltonian systems. Our results generalize the outcome in the literature.

1. Introduction

Denote by $\mathbb{N}, \mathbb{Z}, \mathbb{R}^*, \mathbb{R}$ the sets of all natural numbers, integers, nonnegative real numbers, and real numbers, respectively. For $N \in \mathbb{N}$, denote by $\mathbb{R}^N$ the N -dimensional Euclidean space with the usual inner product $(\cdot, \cdot)$ and norm $|\cdot|$.

Consider the second order Hamiltonian systems

$$\ddot{x} + Ax + V'(x) = 0, \quad \forall x \in \mathbb{R}^N. \quad (1)$$

Assume that

- (A1) A is a symmetric, negative semidefinite matrix;
- (V1) $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and $V(x) \geq 0$ for all $x \in \mathbb{R}^N$;
- (V2) $V(x) = o(|x|^2)$ as $x \rightarrow 0$;
- (V3) $|x|^{-2}V(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$;
- (V4) for any $x \in \mathbb{R}^N$ with $|x| = 1$, $\lambda \rightarrow (1/\lambda)(V'(\lambda x), x)$ is strictly increasing on $(0, \infty)$;
- (V5) there exist $s > 2$ and $a_1, a_2 > 0$ such that $|V(x)| \leq a_1 + a_2|x|^s$;
- (V6) V is even; that is, $V(-x) = V(x)$ for all $x \in \mathbb{R}^N$.

When $A = 0$, in his pioneering work [1] of 1978, Rabinowitz established the existence of periodic solutions

of (1) when V satisfies (V1), (V2), and the well-known AR-condition:

$$\begin{aligned} & (V7) \\ & \exists \mu > 2, \quad r_0 > 0 \text{ such that } 0 < \mu V(x) \leq (x, V'(x)), \\ & \quad \quad \quad \forall |x| \geq r_0. \end{aligned} \quad (2)$$

He conjectured that (1) possesses a nonconstant solution with any prescribed minimal period under the same assumptions. Since then, many authors devoted themselves to the study of periodic solutions with prescribed minimal period of (1).

When $A = 0$, in 1993, Long (cf. [2]) proved the existence of T -periodic solutions with minimal period T or $T/3$ of (1) under the assumptions that $V \in C^2(\mathbb{R}^N, \mathbb{R}^*)$ satisfies (V2), (V6), and (V7). Releasing assumption (V6), Long showed that (1) possesses periodic solutions with minimal period T/k for some integer k satisfying $1 \leq k \leq n+2$ (cf. [3]) or $1 \leq k \leq n+1$ (cf. [4]). In 2001, Fei et al. (cf. [5]) proved that (1) has a nonconstant periodic solution with any prescribed minimal period when $V \in C^2(\mathbb{R}^N, \mathbb{R}^*)$ satisfies (V2), (V6), and (V7), and $V''(x)$ is positive semidefinite.

When A is positive semidefinite, $V \in C^2(\mathbb{R}^N, \mathbb{R}^*)$ satisfies (V2) and (V7); in 1997, Fei and Wang (cf. [6]) showed that (1) possesses nonconstant T -periodic solutions with minimal period T/k for some odd interger k satisfying $1 \leq k \leq$

$2(i_T(A) + \nu_T(A)) + 3$, where $i_T(A) = \sum_{m=0}^{\infty} M^-(T_m(A))$, $\nu_T(A) = \sum_{m=0}^{\infty} M^0(T_m(A))$, and $T_m(A) = (2\pi m/T)^2 I - A$. If V is even, replacing (V7) by (V3), and the following condition

$$\begin{aligned} \exists \beta \in [0, 2], \\ d \geq 0 \text{ such that } \mu V(x) - (V'(x), x) \leq d|x|^\beta, \quad \forall |x| \geq r_0, \end{aligned} \quad (3)$$

Fei and Wang (cf. [7]) estimated that the minimal period is not smaller than $T/(i_T(A) + \nu_T(A) + 2)$.

The above results on minimal period problem were obtained by making use of index theory. Another method being used to study such a problem is Nehari manifold. As is well known, the Nehari manifold, introduced by Nehari (cf. [8, 9]), has been used widely to study the existence of ground state solutions of partial differential equations (cf. [10–14]) and that of (1) (cf. [13, 15, 16]). A ground state solution of a system is a solution which possesses the minimal energy of all solutions of the system. However, such kind of solutions may not have prescribed minimal periods. In 1981, Ambrosetti and Mancini made use of Nehari manifold to study the existence of periodic solutions with any prescribed minimal period of first order Hamiltonian systems with convex potential (cf. [17]). In 2010, Xiao (cf. [18]) proved that (1) possesses a periodic solution with any prescribed minimal period and when $A = 0$, V satisfies (V6) and the following assumption:

(V8)

$$\begin{aligned} V \in C^2(\mathbb{R}^N, \mathbb{R}), \quad 0 < \nu(V'(x), x) \leq (V''(x)x, x), \\ \forall x \in \mathbb{R}^N \setminus \{0\}, \end{aligned} \quad (4)$$

where $\nu > 1$. For more results on this direction, we refer to [19, 20].

Motivated by [13, 17, 18], in this paper, we consider the multiplicity and minimality of periodic solutions of (1) under the assumption (A1), (V1)–(V6). Our main result reads as follows.

Theorem 1. *Assume that (A1), (V1)–(V6) hold. Then, for any $T > 0$, (1) possesses a periodic solution with minimal period T .*

Corollary 2. *If $A = 0$ and V satisfies (V6) and (V8), then, for any $T > 0$, (1) possesses a periodic solution with minimal period T .*

Theorem 3. *Assume that (A1), (V1)–(V6) hold. Then, for any $T > 0$, (1) possesses infinitely many pairs of T -periodic solutions.*

The rest of this paper is divided into two parts. In Section 2, we establish the variational functional and state some useful lemmas. In Section 3, we introduce Nehari Manifold and prove our main results.

2. Preliminary

Denote by $S_T = \mathbb{R}/(T\mathbb{Z})$. $H = W^{1,2}(S_T, \mathbb{R}^N)$ is a Hilbert space equipping with the usual norm and inner product

$$\begin{aligned} \|x\|_H^2 &= \int_0^T (|\dot{x}|^2 + |x|^2) dt, \\ \langle x, y \rangle_H &= \int_0^T [(\dot{x}, \dot{y}) + (x, y)] dt. \end{aligned} \quad (5)$$

The variational functional defined on H , corresponding to (1), is

$$\begin{aligned} \varphi(x) &= \int_0^T \left[\frac{1}{2} |\dot{x}(t)|^2 \right. \\ &\quad \left. - \frac{1}{2} (Ax(t), x(t)) - V(x(t)) \right] dt. \end{aligned} \quad (6)$$

If (A1), (V1)–(V5) hold, then φ is continuous differentiable on H and

$$\begin{aligned} \langle \varphi'(x), y \rangle_H &= \int_0^T [(\dot{x}(t), \dot{y}(t)) - (Ax(t), y(t)) \\ &\quad - (V'(x(t)), y(t))] dt, \\ &\quad \forall x, y \in H. \end{aligned} \quad (7)$$

Moreover, $\phi(x) = \int_0^T V(x(t))dt$ is weakly continuous and $\phi' : H \rightarrow H^*$ is compact.

Define a subspace of H as follows:

$$E = \{x \in H \mid x(-t) = -x(t)\}. \quad (8)$$

Then E is a closed subspace of H . If $x \in E$, it has a Fourier expansion

$$x(t) = \sum_{n=1}^{\infty} a_n \sin \frac{2\pi n t}{T}. \quad (9)$$

Obviously, $E \cap \mathbb{R}^N = \{0\}$. We can define an equivalent inner product on E

$$\langle x, y \rangle_1 = \int_0^T (\dot{x}(t), \dot{y}(t)) dt. \quad (10)$$

Let us state a useful lemma and omit the proof. One can find the details in [21].

Lemma 4. *If x is a critical point of φ on E , then x is a critical point of φ on H .*

Define an operator on E by extending bilinear form

$$\langle Lx, y \rangle_1 = \int_0^T [(\dot{x}(t), \dot{y}(t)) - (Ax(t), y(t))] dt. \quad (11)$$

It is easy to verify that L is a linear self-adjoint operator. Since A is a negative semidefinite matrix, then L has a sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots \quad (12)$$

with $\lambda_n \rightarrow +\infty$ as $n \rightarrow \infty$. Thus for any $x \in E \setminus \{0\}$, $\langle Lx, x \rangle_1 \geq \lambda_1 \|x\|_1^2$, where $\|\cdot\|_1$ denotes the norm induced by $\langle \cdot, \cdot \rangle_1$. Define another equivalent inner product on E

$$\langle x, y \rangle = \langle Lx, y \rangle_1. \quad (13)$$

Then, for all $x \in E$, there exist $C_1, C_2 > 0$ such that $C_1 \|x\| \leq \|x\|_1 \leq C_2 \|x\|$, where $\|\cdot\|$ denotes the norm induced by $\langle \cdot, \cdot \rangle$.

Functional (6) can be rewritten as

$$\varphi(x) = \frac{1}{2} \|x\|^2 - \int_0^T V(x(t)) dt. \quad (14)$$

In the end of this section, we introduce two useful lemmas.

Lemma 5 (see [22]). *If $x \in H$ and $\int_0^T x(t) dt = 0$, then*

$$\begin{aligned} \int_0^T |x(t)|^2 dt &\leq \frac{T^2}{4\pi^2} \int_0^T |\dot{x}(t)|^2 dt, \\ \|x\|_\infty^2 &\leq \frac{T}{12} \int_0^T |\dot{x}|^2 dt, \end{aligned} \quad (15)$$

where $\|\cdot\|_\infty = \max_{t \in [0, T]} |x(t)|$.

If $x \in E$, $\int_0^T x(t) dt = 0$. Thus elements of E satisfy the above two inequalities.

Let X be a Banach space such that the unit sphere S^1 in X is a submanifold of class (at least) C^1 and let $J \in C^1(S^1, \mathbb{R})$. A sequence $\{x_n\} \subset S^1$ is called (PS) sequence for J if it satisfies $\{J(x_n)\}$ is bounded and $J'(x_n) \rightarrow 0$ as $n \rightarrow \infty$. We say that J satisfies the (PS) condition if every (PS) sequence contains a convergent subsequence.

Lemma 6 (see [13]). *If X is infinite-dimensional and $J \in C^1(S^1, \mathbb{R})$ is bounded below and satisfies the (PS) condition on S^1 , then J has infinitely many pairs of critical points.*

3. Proofs of the Main Results

Denote by $h(x) = \langle \varphi'(x), x \rangle$. We define the Nehari manifold

$$\mathcal{M} = \{x \in E \setminus \{0\} \mid h(x) = 0\}. \quad (16)$$

Denote by $S = \{x \in E \mid \|x\| = 1\}$. Fixing $x \in S$, define $g_x(r) = \varphi(rx)$ for $r \in (0, \infty)$. On the one hand, if $x \in \mathcal{M}$, then $y = x/\|x\| \in S$ and

$$\begin{aligned} g'_y(\|x\|) &= \langle \varphi'(\|x\| y), y \rangle \\ &= \frac{\langle \varphi'(x), x \rangle}{\|x\|} = \frac{h(x)}{\|x\|} = 0. \end{aligned} \quad (17)$$

On the other hand, if $g'_x(r) = 0$, then

$$\begin{aligned} h(rx) &= \langle \varphi'(rx), rx \rangle = r \langle \varphi'(rx), x \rangle \\ &= rg'_x(r) = 0. \end{aligned} \quad (18)$$

Thus, for any $x \in S$, $g'_x(r) = 0$ if and only if $rx \in \mathcal{M}$. The following lemma shows that $\mathcal{M} \neq \emptyset$.

Lemma 7. *For any $x \in S$, there exists a unique r_0 dependent on x such that*

$$g_x(r_0) = \sup_{r \in \mathbb{R}^+} g_x(r). \quad (19)$$

Proof. By the definition of g_x , we have

$$g_x(r) = \varphi(rx) = \frac{1}{2} r^2 - \int_0^T V(rx(t)) dt. \quad (20)$$

By (V2), for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that $V(x) \leq \epsilon |x|^2$ for $|x| \leq \delta_\epsilon$. If $x \in S$, then Lemma 5 implies that $\|x\|_\infty \leq \sqrt{T/12} C_2$ and $|x(t)| \leq \sqrt{T/12} C_2$ for $t \in [0, T]$. Fixing $\epsilon_1 < 3/(T^2 C_2^2)$, there exists $r_1 = \delta_{\epsilon_1}/(\sqrt{T/12} C_2)$ such that for all $x \in S$ and $0 < r < r_1$

$$\begin{aligned} \varphi(rx) &= \frac{1}{2} r^2 - \int_0^T V(rx(t)) dt \\ &\geq \frac{1}{2} r^2 - \epsilon_1 r^2 \int_0^T |x|^2 dt > \frac{r^2}{4}. \end{aligned} \quad (21)$$

Let $\alpha_0 = 1/T \int_0^T |x(t)| dt$. Then $\alpha_0 > 0$. For any $0 < \alpha < \alpha_0$, denote

$$A_\alpha = \{t \in [0, T] \mid |x(t)| \geq \alpha\}. \quad (22)$$

Then $\text{meas}(A_\alpha) > 0$, where $\text{meas}(A_\alpha)$ denotes the measure of A_α . Thus (V1) and (V3) yield that

$$\begin{aligned} \frac{g_x(r)}{r^2} &= \frac{1}{2} - \int_0^T \frac{V(rx(t))}{r^2} dt \\ &\leq \frac{1}{2} - \int_{A_\alpha} \frac{V(rx(t))}{|rx(t)|^2} |x(t)|^2 dt \rightarrow -\infty, \end{aligned} \quad (23)$$

as $r \rightarrow \infty$.

Consequently, there exists $r_2 > 0$ dependent on x such that $g_x(r) < 0$ for $r \geq r_2$. By the mean value theorem, there exists $r_0 > 0$ dependent on x such that $g_x(r_0) = \sup_{r \in \mathbb{R}^+} g_x(r)$ and $g'_x(r_0) = 0$.

By the definition, g_x is continuous differentiable on $(0, +\infty)$. Differentiating g_x , we have

$$\begin{aligned} \frac{g'_x(r)}{r} &= 1 - \frac{1}{r} \int_0^T (V'(rx), x) dt \\ &= 1 - \int_0^T |x|^2 \left(\frac{1}{r|x|} V'(r|x|y), y \right) dt, \end{aligned} \quad (24)$$

where $y = x/|x|$. Because of (V4), $g'_x(r)/r$ is strictly decreasing on $(0, +\infty)$. Thus $g'_x(r)/r$ and hence $g'_x(r)$ has a unique zero point on $(0, +\infty)$. It follows that $g_x(r)$ has a unique critical point, which is a maximum. $\square$

Remark 8. Fixing $x \in E \setminus \{0\}$, then $x/\|x\| \in S$. Obviously, $\varphi(rx) = \varphi(r\|x\| \cdot x/\|x\|)$ and $\sup_{r \in \mathbb{R}^+} \varphi(rx) = \sup_{r \in \mathbb{R}^+} \varphi(r\|x\|)$. If we extend the definition of g_x to $E \setminus \{0\}$, then Lemma 7 implies that there exists a unique $r_x > 0$ such that $g_x(r_x) = \sup_{r \in \mathbb{R}^+} \varphi(rx)$. Also, $g'_x(r) > 0$ for $0 < r < r_x$ and $g'_x(r) < 0$ for $r > r_x$.

Now, we study the properties of φ restricted on $\mathcal{M}$.

Lemma 9. Consider the following:

$$c_0 = \inf_{x \in \mathcal{M}} \varphi(x) > 0.$$

Proof. For any $x \in \mathcal{M}$, both Lemma 7 and Remark 8 imply that $\varphi(x) = \sup_{r \in \mathbb{R}^+} \varphi(rx) > r_1^2/16$. Hence $c_0 = \inf_{x \in \mathcal{M}} \varphi(x) \geq r_1^2/16 > 0$. $\square$

Lemma 10. φ satisfies the (PS) condition on $\mathcal{M}$.

Proof. Assume that $\{x_n\} \subset \mathcal{M}$ is a (PS) sequence of φ . Then there exists $d \geq 0$ such that $\varphi(x_n) \leq d$ and $\varphi'(x_n) \rightarrow 0$ as $n \rightarrow \infty$. We show that $\{x_n\}$ is bounded. Suppose, to the opposite, that $\|x_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Set $y_n = x_n/\|x_n\|$. Then $\{y_n\} \subset S$. Passing to a subsequence, $y_n \rightarrow y_0$. If $y_0 = 0$, then $y_n(t) \Rightarrow 0$ uniformly for $t \in [0, T]$. Hence, for any $r > 0$, we have

$$\begin{aligned} d \geq \varphi(x_n) &= \varphi(\|x_n\| y_n) \geq \varphi(r y_n) \\ &= \frac{1}{2} r^2 - \int_0^T V(r y_n(t)) dt \rightarrow \frac{1}{2} r^2. \end{aligned} \quad (25)$$

This is a contradiction if $r > \sqrt{2d}$. So $y_0 \neq 0$. Arguing similarly to [23], there exist $\delta_1 > 0, \delta_2 > 0$ such that

$$\text{meas}(\{t \in [0, T] \mid |y_0(t)| \geq \delta_1\}) \geq \delta_2. \quad (26)$$

Denote $A = \{t \in [0, T] \mid |y_0(t)| \geq \delta_1\}$. But (V3) implies that

$$\begin{aligned} 0 \leq \frac{\varphi(x_n)}{\|x_n\|^2} &\leq \frac{1}{2} \\ &- \int_A \frac{V(\|x_n\| y_n)}{\|x_n\|^2 |y_n|^2} |y_n|^2 dt \rightarrow -\infty, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (27)$$

This is a contradiction. Hence $\{\|x_n\|\}$ is bounded. Since ϕ' is compact, passing to a subsequence, $\{\phi'(x_n)\}$ converges strongly. Since $x_n = \varphi'(x_n) + \phi'(x_n)$, then $\{x_n\}$ contains a convergent subsequence. $\square$

Define the maps as follows:

$$\begin{aligned} \widehat{m} : E \setminus \{0\} &\longrightarrow \mathcal{M}, \quad \widehat{m}(x) = r_x x, \\ m &= \widehat{m}|_S : S \longrightarrow \mathcal{M}, \quad m = m|_S, \\ \widehat{\Psi} : E \setminus \{0\} &\longrightarrow \mathbb{R}, \quad \widehat{\Psi}(x) = \varphi(\widehat{m}(x)), \\ \Psi : S &\longrightarrow \mathbb{R}, \quad \Psi = \widehat{\Psi}|_S, \end{aligned} \quad (28)$$

where r_x is defined on Remark 8.

Lemma 11. Assume that (V1)–(V4) hold. Then the following statements hold:

(B1) there exists a normalization function φ such that

$$u \mapsto \psi(u) := \int_0^{\|u\|} \varphi(t) dt \in C^1(E \setminus \{0\}, \mathbb{R}); \quad (29)$$

$J := \psi'$ is bounded on bounded sets and $J(w)w = 1$ for all $w \in S$;

(B2) for each $w \in E \setminus \{0\}$ there exists s_w such that if $\alpha_w(s) := \Phi(sw)$, then $\alpha'_w(s) > 0$ for $0 < s < s_w$ and $\alpha'_w(s) < 0$ for $s > s_w$;

(B3) there exists $\delta > 0$ such that $s_w \geq \delta$ for all $w \in S$ and for each compact subset $W \subset S$ there exists a constant C_W such that $s_w \leq C_S$ for all $w \in W$.

Proof. It follows from the discussion in [13] and Remark 8 that (B1), (B2) hold, and $r_x \geq r_1/2$ for all $x \in S$. Next, we show that the last part of (B3) holds. Arguing by contradiction, we suppose that there exists a compact subset $W \subset S$ and a sequence $\{r_{x_n} x_n\} \subset \mathcal{M}$, where $\{x_n\} \subset W$ and $\{r_{x_n}\} \subset \mathbb{R}^+ \setminus \{0\}$, such that $r_{x_n} \rightarrow \infty$ as $n \rightarrow \infty$. Since W is compact and $x_n \in W$, passing to a subsequence, $x_n \rightarrow x_0 \in S$. Since the embedding of H into $C(S_T, \mathbb{R}^N)$ is compact and $x_n \in W$, then $x_n(t) \Rightarrow x_0(t)$ uniformly for $t \in [0, T]$. Denote by $A_{x_0} = \{t \in [0, T] \mid |x_0(t)| \geq 1/T \int_0^T |x_0(t)| dt\}$. Since $r_{x_n} |x_n(t)| \rightarrow \infty$, if $t \in A_{x_0}$, then Fatou's lemma yields

$$\frac{\phi(r_{x_n} x_n)}{r_{x_n}^2} \geq \int_{A_{x_0}} \frac{V(r_{x_n} x_n(t))}{r_{x_n}^2 |x_n(t)|^2} |x_n(t)|^2 dt \rightarrow \infty, \quad (30)$$

as $n \rightarrow \infty$.

Thus $\varphi(r_{x_n} x_n) = 1/2 r_{x_n}^2 - \phi(r_{x_n} x_n) \rightarrow -\infty$, which contradicts with the fact that $\varphi(r_{x_n} x_n) > r_1^2/16 > 0$. Consequently, there exists $C_W > 0$ such that $r_x \leq C_W$ for all $x \in W$ and (B3) holds. $\square$

Lemma 12 (see [13]). The mapping $\widehat{m}$ is continuous and m is a homeomorphism between S and $\mathcal{M}$.

Lemma 13 (see [13]). Consider the following.

(1) $\widehat{\Psi}, \Psi \in C^1(S, \mathbb{R})$ and

$$\begin{aligned} \langle \widehat{\Psi}'(x), y \rangle &= \frac{\|\widehat{m}(x)\|}{\|x\|} \langle \Psi'(\widehat{m}(x)), y \rangle, \\ &\forall x, y \in E, \quad x \neq 0, \end{aligned} \quad (31)$$

$$\langle \Psi'(x), y \rangle = \|m(x)\| \langle \Psi'(m(x)), y \rangle,$$

$$\forall y \in T_x(S),$$

where $T_x(S) = \{y \in E \mid \langle \varphi(x), y \rangle = 0\}$ is the tangent space of S at x .

(2) If $\{x_n\}$ is a (PS) sequence for Ψ , then $\{m(x_n)\}$ is a (PS) sequence for φ . If $\{x_n\} \subset \mathcal{M}$ is a bounded (PS) sequence for φ , then $\{m^{-1}(x_n)\}$ is a (PS) sequence for Ψ .

(3) x is a critical point of Ψ if and only if $m(x)$ is a non-trivial critical point of φ . Moreover, the corresponding values of Ψ and φ coincide and $\inf_S \Psi = \inf_{\mathcal{M}} \varphi$.

(4) If φ is even, then so is Ψ .

Lemma 14. If φ satisfies the (PS) condition, so does Ψ .

Proof. Assume that $\{x_n\}$ is a (PS) sequence for Ψ . According to Lemma 13, $\{m(x_n)\} \subset \mathcal{M}$ is a (PS) sequence for φ . Since

φ satisfies the (PS) condition, passing to a subsequence, $m(x_n) \rightarrow y$. Thus $\{x_n\}$ contains a convergent subsequence, which converges to $m^{-1}(y)$. Hence Ψ satisfies the (PS) condition. $\square$

Proof of Theorem 1. According to Lemmas 9 and 13, $\inf_{x \in S} \Psi(x) = \inf_{x \in \mathcal{M}} \varphi(x) = c_0$. Let $\{y_n\}$ be a minimizing sequence for Ψ restricted to S . By Ekeland's variational principle, we may assume that $\Psi'(y_n) \rightarrow 0$ as $n \rightarrow \infty$. The (PS) condition implies that $\{y_n\}$ contains a converging subsequence, whose limit is denoted by y_0 . Thus y_0 is a critical point of Ψ . According to Lemma 13 again, $x_0 = m(y_0)$ is a critical point of φ , which is also a nonconstant T -periodic solution of (1).

Claim. x_0 has minimal period T .

Suppose, to the opposite, that x_0 has minimal period T/k , where $k \geq 2$ is an integer. Let $y_0(t) = x_0(t/k)$. Obviously, $y_0 \in E$. It follows from Lemma 7 that there exists $\bar{r} > 0$ such that $\bar{r}y_0 \in \mathcal{M}$. It follows that

$$\begin{aligned} \inf_{x \in \mathcal{M}} \varphi(x) &\leq \varphi(\bar{r}y_0) \\ &= \int_0^T \left[\frac{1}{2} |\bar{r}\dot{y}_0(t)|^2 - V(\bar{r}y_0(t)) \right] dt \\ &= \int_0^T \left[\frac{1}{2k^2} |\bar{r}\dot{x}_0\left(\frac{t}{k}\right)|^2 - V\left(\bar{r}x_0\left(\frac{t}{k}\right)\right) \right] dt \\ &= \int_0^T \left[\frac{1}{2k^2} |\bar{r}\dot{x}_0(\tau)|^2 - V(\bar{r}x_0(\tau)) \right] d\tau \\ &< \int_0^T \left[\frac{1}{2} |\bar{r}\dot{x}_0(\tau)|^2 - V(\bar{r}x_0(\tau)) \right] d\tau \\ &= \varphi(\bar{r}x_0) \leq \varphi(x_0) = \inf_{x \in \mathcal{M}} \varphi(x), \end{aligned} \quad (32)$$

which is a contradiction. Hence x_0 has minimal period T . $\square$

Proof of Corollary 2. For $x \in \mathbb{R}^N$ with $|x| = 1$, set $a(\lambda) = \lambda^{-1}(V'(\lambda x), x)$. Then (V8) implies that

$$\begin{aligned} a'(\lambda) &= \frac{1}{\lambda^3} \left[(V''(\lambda x) \lambda x, \lambda x) \right. \\ &\quad \left. - (V'(\lambda x), \lambda x) \right] > 0; \end{aligned} \quad (33)$$

that is, $a(\lambda)$ is increasing strictly on $(0, +\infty)$. Thus (V4) is satisfied. It is easy to check that V satisfies (V1)–(V3) when (V8) is available.

Without assumption (V5), φ may not be continuous differentiable on H . However, we can use the method introduced in [1] to handle such a situation. Let $K > 1$ and $\chi \in C^\infty(\mathbb{R}^+, \mathbb{R}^+)$ such that $\chi(s) = 1$ if $s \leq K$, $\chi(s) = 0$ if $s \geq K+1$, and $\chi'(s) < 0$ if $s \in (K, K+1)$. Set

$$V_K(x) = \chi(|x|) V(x) + [1 - \chi(|x|)] \rho(K) |x|^\lambda, \quad (34)$$

where $\rho(K) \geq \max_{K \leq |x| \leq K+1} (V(x)/|x|^\lambda)$. Since V satisfies (V8), it follows from Lemma 2.3 in [18] that

$$\begin{aligned} 0 < (\nu + 1) V(x) &\leq (V'(x), x), \\ \forall x \in \mathbb{R}^N \setminus \{0\}. \end{aligned} \quad (35)$$

Arguing similarly to Lemma 2.9 in [1], we have

$$0 < \lambda V_K(x) \leq (V'_K(x), x), \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \quad (36)$$

where $\lambda = \min\{\nu + 1, 4\} > 2$. Let $a_3 = \min_{|x|=1} V_K(x)$ and $a_4 = \max_{|x|=1} V_K(x)$. Since $K > 1$, then $a_3 = \min_{|x|=1} V(x) > 0$ and $a_4 = \max_{|x|=1} V(x) > 0$. Obviously, a_3, a_4 are independent of K . Arguing similarly to Lemma 4.1 of [22], we can prove the following inequality:

$$V_K(x) \geq a_3 |x|^\lambda - a_4, \quad \forall x \in \mathbb{R}^N. \quad (37)$$

Hence V_K satisfies (V1)–(V6).

Consider the disturbed second order Hamiltonian systems

$$\ddot{x} + V'_K(x) = 0, \quad (38)$$

whose variational function is

$$\varphi_K(x) = \int_0^T \left[\frac{1}{2} |\dot{x}(t)|^2 - V_K(x(t)) \right] dt. \quad (39)$$

Since V_K satisfies (V1)–(V6), applying Theorem 1, (38) possesses a periodic solution x_K with minimal period T .

Claim. There exists $K_0 > 0$ independent of K such that $\|x_K\|_{L^\infty} < K_0$.

Let $x_1(t) = b_1 \sqrt{T/(2\pi^2)} \sin(2\pi t/T) \in E$ with $b_1 = (1, 0, \dots, 0)' \in \mathbb{R}^N$, where $(\cdot)'$ denotes the transposition of a vector. Computing directly, we have

$$\|x_1\|^2 = \|x_1\|_1^2 = 1, \quad \|x_1\|_{L^2}^2 = \frac{T^2}{4\pi^2}. \quad (40)$$

So $x_1 \in S$. It follows from (37) that

$$\begin{aligned} \varphi_K(rx_1) &= \frac{1}{2} r^2 - \int_0^T V_K(rx_1) dt \\ &\leq \frac{1}{2} r^2 - a_3 r^\lambda \|x_1\|_{L^\lambda}^\lambda + a_4 T \\ &\leq \frac{1}{2} r^2 - a_3 M_1^\lambda r^\lambda \|x_1\|_{L^2}^\lambda + a_4 T \\ &\leq \frac{1}{2} r^2 - a_3 M_1^\lambda r^\lambda \frac{T^\lambda}{(2\pi)^\lambda} + a_4 T = h(r), \end{aligned} \quad (41)$$

where M_1 denotes the constant such that $\|x\|_{L^\lambda} \geq M_1 \|x\|_{L^2}$. Since $\lambda > 2$, $h(r) \rightarrow -\infty$ as $r \rightarrow +\infty$. Then there exists $M_2 > 0$ independent of K such that $h(r) \leq M_2$. Thus

$$\begin{aligned} c_K &= \inf_{x \in \mathcal{M}} \varphi_K(x) = \inf_{x \in S} \max_{r \in \mathbb{R}^+} \varphi_K(rx) \\ &\leq \max_{r \in \mathbb{R}^+} \varphi_K(rx_1) \leq \max_{r \in \mathbb{R}^+} h(r) \leq M_2. \end{aligned} \quad (42)$$

Hence

$$\begin{aligned} M_2 &\geq c_K = \varphi_K(x_K) \\ &= \int_0^T \left[\frac{1}{2} (V'_K(x_K), x_K) - V_K(x_K) \right] dt \\ &\geq \frac{\lambda - 2}{2} \int_0^T V_K(x_K) dt. \end{aligned} \quad (43)$$

Consequently, we have

$$\begin{aligned} c_{\infty}^{-2} \|x\|_{L^{\infty}}^2 &\leq \|x\|^2 = 2\varphi_K(x_K) \\ &\quad + 2 \int_0^T V_K(x_K) dt \leq \frac{2\lambda}{\lambda - 2} M_2. \end{aligned} \quad (44)$$

It follows that there exists $K_0 > 0$ independent of K such that $\|x_K\|_{L^{\infty}} \leq K_0$. If $K > K_0$, by the definition of V_K , $V_K = V$ and hence x_K is a nonconstant T -periodic solution of (1) with $A = 0$. $\square$

Proof of Theorem 3. Since V is even, so do φ and Ψ . Since $\inf_{x \in S} \Psi(x) = c_0 > 0$ and Ψ satisfies the (PS) condition, then Lemma 6 yields that Ψ has infinitely many pairs of critical points. According to Lemma 13, φ has infinitely many pairs of critical points. Thus (1) has infinitely many pairs of periodic solutions. $\square$

Example 15. Let $A = 0$ and let V be

$$V(x) = |x|^2 \left[\ln(1 + |x|^2) \right]^2. \quad (45)$$

It is easy to check that V does not satisfy (V8) or even (V7) (cf. [23]). However, V satisfies (V1)–(V6). Applying Theorems 1 and 3, for any $T > 0$, (1) possesses infinitely many pairs of T -periodic solutions and has at least one solution with T as its minimal period.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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