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Research Article

Discussion on Generalized-($\alpha\psi$, $\beta\varphi$)-Contractive Mappings via Generalized Altering Distance Function and Related Fixed Point Theorems

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We extend the notion of ($\alpha\psi$, $\beta\varphi$)-*contractive mapping*, a very recent concept by Berzig and Karapinar. This allows us to consider contractive conditions that generalize a wide range of nonexpansive mappings in the setting of metric spaces provided with binary relations that are not necessarily neither partial orders nor preorders. Thus, using this kind of contractive mappings, we show some related fixed point theorems that improve some well known recent results and can be applied in a variety of contexts.

1. Introduction and Preliminaries

After the appearance of the pioneering *Banach contractive mapping principle* and due to its possible applications, *fixed point theory* has become one of the most useful branches of *nonlinear analysis*, with applications to very different settings, including, among others, resolution of all kind of equations (differential, integral, matrix, etc.), image recovery, convex minimization and split feasibility, and equilibrium problems.

In the last decades, fixed point theorems in partially ordered metric spaces have attracted much attention, especially after the works of Ran and Reurings [1], Nieto and Rodríguez-López [2], Bhaskar and Lakshmikantham [3], Berinde and Borcut [4, 5], Karapınar [6, 7], Berzig and Samet [8], and Karapınar et al. [9–11], among others. Their results have been extended to contractivity conditions in which *altering distance functions* (a notion introduced by Khan et al. [12]) play an important role. Very recently, Alghamdi and Karapınar [13] used a similar notion in G-metric spaces, and Berzig and Karapinar [14] also considered a more general kind of contractivity conditions using a pair of generalized altering distance functions.

In this paper, by introducing the notion of *generalized-*($\alpha\psi$, $\beta\varphi$)*-contractive mappings*, we collect, improve, and generalize some existing results on this topic in the literature.

Now, we recollect some basic definitions and useful results for the sake of completeness of the paper. First, we recollect the concept of altering distance function as follows.

Definition 1 (Khan et al. [12]). A function $\psi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function if the following properties are satisfied:

(i) $\psi(t)$ is continuous and nondecreasing;
(ii) $\psi(t) = 0$ if and only if $t = 0$.

In what follows, we state the definition of $\mathscr{R}$-preserving mapping which plays crucial roles in the setting of main results.

Definition 2 (see, e.g., [14]). Let X be a set and $\mathscr{R}$ be a binary relation on X. We say that $T : X \to X$ is $\mathscr{R}$-preserving mapping if

$$x, y \in X : x\mathscr{R}y \Longrightarrow Tx\mathscr{R}Ty. \tag{1}$$

Throughout the paper, let $\mathbb{N}$ denote the set of all nonnegative integers, and let $\mathbb{R}$ be the set of all real numbers.

Example 3 (see, e.g., [14]). Let $X = \mathbb{R}$ and a function $T : X \to X$ be defined as $Tx = e^x$. Define $\alpha, \beta : X \times X \to 0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \le y, \\ 2, & \text{otherwise;} \end{cases}$$
$$\beta(x, y) = \begin{cases} 1, & \text{if } x \le y, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Define the first binary relation $\mathscr{R}_1$ by $x\mathscr{R}_1 y$ if and only if $\alpha(x, y) \le 1$, and define the second binary relation by $x\mathscr{R}_2 y$ if and only if $\beta(x, y) \ge 1$. Then, we obtain easily that T is simultaneously $\mathscr{R}_1$-preserving and $\mathscr{R}_2$-preserving.

Definition 4 (see [14]). Let $N \in \mathbb{N}$. We say that $\mathscr{R}$ is N–transitive on X if

$$x_0, x_1, \dots, x_{N+1} \in X : x_i \mathscr{R} x_{i+1}$$
$$\forall i \in \{0, 1, \dots, N\} \Longrightarrow x_0 \mathscr{R} x_{N+1}. \tag{3}$$

The following remark is a consequence of the previous definition.

Remark 5 (see [14]). Let $N \in \mathbb{N}$. We have the following.

(1) If $\mathscr{R}$ is transitive, then it is N-transitive for all $N \in \mathbb{N}$.

(2) If $\mathscr{R}$ is N-transitive, then it is kN-transitive for all $k \in \mathbb{N}$.

Definition 6 (see [14]). Let (X, d) be a metric space and $\mathscr{R}_1$, $\mathscr{R}_2$ be two binary relations on X. We say that (X, d) is $(\mathscr{R}_1, \mathscr{R}_2)$-regular if for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ as $n \to +\infty$ and

$$x_n \mathscr{R}_1 x_{n+1}, \quad x_n \mathscr{R}_2 x_{n+1}, \quad \forall n \in \mathbb{N}, \tag{4}$$

there exists a subsequence $\{x_{n(k)}\}$ such that

$$x_{n(k)} \mathscr{R}_1 x, \quad x_{n(k)} \mathscr{R}_2 x, \quad \forall k \in \mathbb{N}. \tag{5}$$

Definition 7. We say that a subset D of X is $(\mathscr{R}_1, \mathscr{R}_2)$-directed if for all $x, y \in D$, there exists $z \in X$ such that

$$x\mathscr{R}_1 z, \qquad y\mathscr{R}_1 z, \qquad x\mathscr{R}_2 z, \qquad y\mathscr{R}_2 z. \tag{6}$$

Definition 8. Let $T : X \to X$ be a mapping. We say that a subset D of X is $(\mathscr{R}_1, \mathscr{R}_2)$-directed with respect to T if for all $x, y \in D$, there exists $z \in X$ such that

$$\{T^n z\} \text{ is a convergent sequence,}$$
$$x\mathscr{R}_1 z, \qquad y\mathscr{R}_1 z, \qquad x\mathscr{R}_2 z, \qquad y\mathscr{R}_2 z. \tag{7}$$

Remark 9. A subset D of X is an $(\mathscr{R}_1, \mathscr{R}_2)$-directed subset if, and only if, it is an $(\mathscr{R}_1, \mathscr{R}_2)$-directed subset with respect to the identity mapping I_X.

We recall the notion of a pair of generalized altering distance as follows.

Definition 10. We say that the pair of functions (ψ, φ) is a pair of generalized altering distance (where $\psi, \varphi : [0, +\infty) \to 0, +\infty))$ if the following hypotheses hold:

(a1) ψ is continuous;

(a2) ψ is nondecreasing;

(a3) $\lim_{n\to\infty} \varphi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$.

The condition (a3) was introduced by Popescu in [16] and Moradi and Farajzadeh in [15]. Notice that the above conditions do not determine the values $\psi(0)$ and $\varphi(0)$.

Definition 11 (see [14]). Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an $(\alpha\psi, \beta\varphi)$-contractive mapping if there exists a pair of generalized altering distance functions (ψ, φ) and two mappings $\alpha, \beta : X \times X \to [0, +\infty)$ such that

$$\psi(d(Tx, Ty)) \le \alpha(x, y)\, \psi(d(x, y)) - \beta(x, y)\, \varphi(d(x, y)), \quad \forall x, y \in X. \tag{8}$$

2. Main Results

Firstly, we present two technical properties that will be very useful in the proof of our main result.

Lemma 12. *If (ψ, φ) is a pair of generalized altering distance functions and $r, s \in [0, +\infty)$ are such that $\psi(r) \le (\psi - \varphi)(s)$, then one, and only one, of the following conditions holds:*

$$\textit{either } r < s \quad \textit{or} \quad s = 0. \tag{9}$$

Proof. Firstly, notice that both possibilities are not compatible. Suppose that $r \ge s$. Since ψ is nondecreasing and $\varphi \ge 0$,

$$\psi(s) \le \psi(r) \le (\psi - \varphi)(s) = \psi(s) - \varphi(s) \le \psi(s), \tag{10}$$

so $\psi(r) = \psi(s)$ and $\varphi(s) = 0$. Defining $t_n = s$ for all $n \in \mathbb{N}$, we have that $\lim_{n\to\infty} \varphi(t_n) = 0$. By (a3), $s = \lim_{n\to\infty} t_n = 0$. □

Lemma 13. *Let $\{x_n\}$ be a sequence in a metric space (X, d).*

(1) *If $\{x_n\}$ is not Cauchy, then there exists $\varepsilon_0 > 0$ and two subsequences $\{x_{m(k)}\}_{k\in\mathbb{N}}$ and $\{x_{n(k)}\}_{k\in\mathbb{N}}$ verifying that, for all $k \in \mathbb{N}$,*

$$k \le m(k) < n(k), \qquad d(x_{m(k)}, x_{n(k)}) \ge \varepsilon_0,$$
$$d(x_{m(k)}, x_p) < \varepsilon_0, \tag{11}$$
$$\forall p \in \{m(k)+1, m(k)+2, \dots, n(k)-2, n(k)-1\}$$

Furthermore, $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon_0$.

(2) *In addition to this, if $\{x_n\}$ also verifies $\{d(x_n, x_{n+1})\} \to 0$, then*

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon_0,$$
$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+p}) = \varepsilon_0 \quad \forall p \ge 0. \tag{12}$$

Proof. The first part is well-known as $n(k)$ can be chosen to be the lowest integer p that does not verify $d(x_{m(k)}, x_p) \geq \varepsilon_0$, then $d(x_{m(k)}, x_{n(k)-1}) < \varepsilon_0$. The first part of the second item can be proved as follows. For all $k \in \mathbb{N}$,

$$\begin{aligned} d\left(x_{m(k)-1}, x_{n(k)-1}\right) &\leq d\left(x_{m(k)-1}, x_{m(k)}\right) \\ &\quad + d\left(x_{m(k)}, x_{n(k)}\right) + d\left(x_{n(k)}, x_{n(k)-1}\right); \\ d\left(x_{m(k)}, x_{n(k)}\right) &\leq d\left(x_{m(k)}, x_{m(k)-1}\right) + d\left(x_{m(k)-1}, x_{n(k)-1}\right) \\ &\quad + d\left(x_{n(k)-1}, x_{n(k)}\right). \end{aligned} \tag{13}$$

Therefore, taking limit as $k \to \infty$ in

$$\begin{aligned} d\left(x_{m(k)}, x_{n(k)}\right) &- d\left(x_{m(k)-1}, x_{m(k)}\right) - d\left(x_{n(k)-1}, x_{n(k)}\right) \\ &\leq d\left(x_{m(k)-1}, x_{n(k)-1}\right) \\ &\leq d\left(x_{m(k)-1}, x_{m(k)}\right) + d\left(x_{m(k)}, x_{n(k)}\right) \\ &\quad + d\left(x_{n(k)-1}, x_{n(k)}\right), \end{aligned} \tag{14}$$

we deduce that $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon_0$. To prove the second part of the second item, we proceed by induction methodology on $p \in \mathbb{N}$. If $p = 0$, it follows from item (1). Suppose that (12) holds for some $p \geq 0$. On the one hand,

$$\begin{aligned} &d\left(x_{m(k)}, x_{n(k)+p+1}\right) \\ &\quad \leq d\left(x_{m(k)}, x_{n(k)+p}\right) + d\left(x_{n(k)+p}, x_{n(k)+p+1}\right), \end{aligned} \tag{15}$$

and on the other hand,

$$\begin{aligned} d\left(x_{m(k)}, x_{n(k)+p}\right) &\leq d\left(x_{m(k)}, x_{n(k)+p+1}\right) \\ &\quad + d\left(x_{n(k)+p+1}, x_{n(k)+p}\right). \end{aligned} \tag{16}$$

Joining both inequalities,

$$\begin{aligned} d\left(x_{m(k)}, x_{n(k)+p}\right) &- d\left(x_{n(k)+p+1}, x_{n(k)+p}\right) \\ &\leq d\left(x_{m(k)}, x_{n(k)+p+1}\right) \\ &\leq d\left(x_{m(k)}, x_{n(k)+p}\right) \\ &\quad + d\left(x_{n(k)+p}, x_{n(k)+p+1}\right). \end{aligned} \tag{17}$$

Taking limit as $k \to \infty$ and using (12) and $\{d(x_n, x_{n+1})\} \to 0$, we conclude that

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)+p+1}\right) = \varepsilon_0. \tag{18}$$

□

Next we introduce the notion of generalized-$(\alpha\psi, \beta\varphi)$-contractive mappings which is an extension of Definition 11.

Definition 14. Let (X, d) be a metric space and let $T : X \to X$ be a given mapping. We say that T is a generalized-$(\alpha\psi, \beta\varphi)$-contractive mapping if there exists a pair of generalized altering distance functions (ψ, φ) and two mappings $\alpha, \beta : X \times X \to [0, +\infty)$ such that

$$\begin{aligned} \psi\left(d\left(Tx, Ty\right)\right) &\leq \alpha\left(x, y\right)\psi\left(M\left(x, y\right)\right) \\ &\quad - \beta\left(x, y\right)\varphi\left(M\left(x, y\right)\right), \quad \forall x, y \in X, \end{aligned} \tag{19}$$

where $M(x, y)$ is given by one of the following cases:

(i) $M_1(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), (d(x, Ty) + d(y, Tx))/2\}$ (type I);

(ii) $M_2(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ (type II);

(iii) $M_3(x, y) = \max\{d(x, y), (d(x, Tx) + d(y, Ty))/2, (d(x, Ty) + d(y, Tx))/2\}$ (type III);

(iv) $M_4(x, y) = \max\{d(x, y), (d(x, Tx) + d(y, Ty))/2\}$ (type IV);

(v) $M_5(x, y) = d(x, y)$ (type V),

for all $x, y \in X$.

In the sequel, the binary relations $\mathscr{R}_1$ and $\mathscr{R}_2$ are defined as follows.

Definition 15. Let X be a set, and $\alpha, \beta : X \times X \to [0, +\infty)$ are two mappings. We define two binary relations $\mathscr{R}_1$ and $\mathscr{R}_2$ on X by

$$\begin{aligned} x, y \in X &: x\mathscr{R}_1 y \Longleftrightarrow \alpha\left(x, y\right) \leq 1, \\ x, y \in X &: x\mathscr{R}_2 y \Longleftrightarrow \beta\left(x, y\right) \geq 1. \end{aligned} \tag{20}$$

Now we are ready to study the existence and the uniqueness of fixed points.

2.1. Existence of Fixed Points.

We may now state our first main result.

Theorem 16. *Let (X, d) be a complete metric space and $N \in \mathbb{N} \setminus \{0\}$ and $T : X \to X$ be an $(\alpha\psi, \beta\varphi)$-contractive mapping of type I satisfying the following conditions:*

(i) *$\mathscr{R}_1$ and $\mathscr{R}_2$ are N-transitive;*

(ii) *T is $\mathscr{R}_1$-preserving and $\mathscr{R}_2$-preserving;*

(iii) *there exists $x_0 \in X$ such that $x_0\mathscr{R}_1 Tx_0$ and $x_0\mathscr{R}_2 Tx_0$;*

(iv) *T is continuous.*

Then, T has a fixed point; that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Let $x_0 \in X$ such that $x_0\mathscr{R}_i Tx_0$ for $i = 1, 2$. Define the sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n, \quad \forall n \geq 0. \tag{21}$$

If $x_n = x_{n+1}$ for some $n \geq 0$, then $x^* = x_n$ is a fixed point T. Assume that $x_n \neq x_{n+1}$; that is,

$$d\left(x_n, n_{n+1}\right) > 0, \quad \forall n \geq 0. \tag{22}$$

From (ii) and (iii), we have

$$\begin{aligned} x_0\mathcal{R}_1 Tx_0 &\Longrightarrow \alpha(x_0,x_1) \\ &= \alpha(x_0,Tx_0) \le 1 \Longrightarrow \alpha(x_1,x_2) \\ &= \alpha(Tx_0,Tx_1) \le 1; \\ x_0\mathcal{R}_2 Tx_0 &\Longrightarrow \beta(x_0,x_1) \\ &= \beta(x_0,Tx_0) \ge 1 \Longrightarrow \beta(x_1,x_2) \\ &= \beta(Tx_0,Tx_1) \ge 1. \end{aligned} \tag{23}$$

By induction, from (ii) it follows that

$$\alpha(x_n,x_{n+1}) \le 1, \quad \beta(x_n,x_{n+1}) \ge 1 \quad \forall n \ge 0. \tag{24}$$

Substituting $x = x_n$ and $y = x_{n+1}$ in (19), we obtain

$$\begin{aligned} \psi(d(Tx_n,Tx_{n+1})) \le{}& \alpha(x_n,x_{n+1})\,\psi(M_1(x_n,x_{n+1})) \\ & - \beta(x_n,x_{n+1})\,\varphi(M_1(x_n,x_{n+1})). \end{aligned} \tag{25}$$

So, by (24) it follows that

$$\psi(d(x_{n+1},x_{n+2})) \le (\psi-\varphi)(M_1(x_n,x_{n+1})), \tag{26}$$

where

$$\begin{aligned} & M_1(x_n,x_{n+1}) \\ &= \max\left(d(x_n,x_{n+1}), d(x_n,Tx_n), d(x_{n+1},Tx_{n+1}), \frac{d(x_n,Tx_{n+1}) + d(x_{n+1},Tx_n)}{2}\right) \\ &= \max\left(d(x_n,x_{n+1}), d(x_n,x_{n+1}), d(x_{n+1}x_{n+2}), \frac{d(x_n,x_{n+2}) + d(x_{n+1},x_{n+1})}{2}\right) \\ &= \max\left(d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), \frac{d(x_n,x_{n+2})}{2}\right) \\ &= \max(d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})) \end{aligned} \tag{27}$$

(the last equality follows from $d(x_n,x_{n+2}) \le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2})$). By Lemma 12, either $d(x_{n+1},x_{n+2}) < M_1(x_n,x_{n+1})$ or $M_1(x_n,x_{n+1}) = 0$, but the second case is impossible by (22). Then, we get

$$\begin{aligned} d(x_{n+1},x_{n+2}) &< \max(d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})) \\ &= M_1(x_n,x_{n+1}), \end{aligned} \tag{28}$$

that is,

$$\begin{gathered} d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1}), \\ M_1(x_n,x_{n+1}) = d(x_n,x_{n+1}) \quad \forall n \ge 0. \end{gathered} \tag{29}$$

From (29), $\{d(x_n,x_{n+1})\}$ is monotone decreasing and, consequently, there exists $r \ge 0$ such that

$$\{d(x_n,x_{n+1})\} \longrightarrow r \quad \text{as } n \longrightarrow \infty. \tag{30}$$

Notice that (26) and (29) imply that, for all $n \ge 0$,

$$\begin{aligned} \psi(d(x_{n+1},x_{n+2})) &\le (\psi-\varphi)(d(x_n,x_{n+1})) \\ &= \psi(d(x_n,x_{n+1})) - \varphi(d(x_n,x_{n+1})) \\ &\le \psi(d(x_n,x_{n+1})). \end{aligned} \tag{31}$$

Letting $n \to +\infty$ in (31) and taking into account that ψ is continuous, we obtain that the sequence $\{\varphi(d(x_n,x_{n+1}))\}$ has finite limit and

$$\psi(r) \le \psi(r) - \lim_{n\to\infty}\varphi(d(x_n,x_{n+1})) \le \psi(r), \tag{32}$$

which implies that $\lim_{n\to\infty}\varphi(d(x_n,x_{n+1})) = 0$. Then, by (a3), we get

$$\{d(x_n,x_{n+1})\} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{33}$$

Next we show that $\{x_n\}$ is a Cauchy sequence reasoning by contradiction. If $\{x_n\}$ is not Cauchy, Lemma 13 assures us that there exists $\varepsilon_0 > 0$ and two subsequences $\{x_{m(k)}\}_{k\in\mathbb{N}}$ and $\{x_{n(k)}\}_{k\in\mathbb{N}}$ verifying that, for all $k \in \mathbb{N}$,

$$\begin{gathered} k \le m(k) < n(k), \qquad d(x_{m(k)},x_{n(k)}) \ge \varepsilon_0, \\ d(x_{m(k)},x_{n(k)-1}) < \varepsilon_0, \end{gathered} \tag{34}$$

and also

$$\begin{gathered} \lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)-1}) = \varepsilon_0, \\ \lim_{k\to\infty} d(x_{m(k)},x_{n(k)+p}) = \varepsilon_0 \quad \forall p \ge 0. \end{gathered} \tag{35}$$

Since $n(k) > m(k)$, consider, for all $k \in \mathbb{N}$, the Euclidean division $(n(k)-m(k)) : N$, whose quotient will be denoted by $\mu_k - 1$ (then $\mu_k \ge 1$) and whose rest will be denoted by $N + 1 - \eta_k$ as follows:

$$\begin{array}{r|l} n(k)-m(k) & N \\ \hline N+1-\eta_k & \mu_k - 1 \end{array} \qquad \begin{cases} n(k)-m(k) \\ \quad = (\mu_k-1)N + (N+1-\eta_k), \\ 0 \le N+1-\eta_k < N, \end{cases} \tag{36}$$

$$\begin{aligned} &\Longleftrightarrow \begin{cases} n(k)-m(k) = N\mu_k + 1 - \eta_k, \\ 1 < \eta_k \le N+1, \end{cases} \\ &\Longleftrightarrow \begin{cases} n(k)+\mu_k = m(k) + N\mu_k + 1, \\ 1 < \eta_k \le N+1. \end{cases} \end{aligned} \tag{37}$$

Notice that μ_k and μ_k are convenient integer numbers such that $\mu_k \ge 1$ and $2 \le \mu_k \le N+1$. Hence, μ_k can only take a finite quantity of integer numbers, in the interval $[2, N+1]$.

Therefore, there exist subsequences of $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ (also verifying (34) and (35)) such that μ_k is constant (it does not depend on k). In order not to complicate the notation, we will suppose that

$$n(k) + \mu = m(k) + N\mu_k + 1, \qquad 1 < \eta \leq N + 1, \tag{38}$$

where $\eta \in 2, [N + 1]$ is constant.

Let define $m'(k) = n(k) + \mu = m(k) + N\mu_k + 1$ for all $k \in \mathbb{N}$. Taking into account item (2) of Remark 5, (24), and (i), we obtain

$$\alpha(x_{m(k)}, x_{m'(k)}) = \alpha(x_{m(k)}, x_{m(k)+\mu_k N+1}) \leq 1, \qquad \beta(x_{m(k)}, x_{m'(k)}) = \beta(x_{m(k)}, x_{m(k)+\mu_k N+1}) \geq 1 \tag{39}$$

for all $k \in \mathbb{N}$. Furthermore, by (35),

$$\lim_{k\to\infty} d(x_{m(k)}, x_{m'(k)}) = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+\mu}) = \varepsilon_0, \qquad \lim_{k\to\infty} d(x_{m'(k)-1}, x_{m(k)}) = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+\mu-1}) = \varepsilon_0. \tag{40}$$

Following the same technique as in Lemma 13, we also deduce that

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{m'(k)}) = \varepsilon_0. \tag{41}$$

Apply the contractivity condition (19) to $x = x_{m(k)-1}$ and $y = x_{m'(k)-1}$, and we get

$$\begin{aligned}\psi(d(Tx_{m(k)-1}, Tx_{m'(k)-1})) &\leq \alpha(x_{m(k)-1}, x_{m'(k)-1}) \\ &\quad \times \psi(M_1(x_{m(k)-1}, x_{m'(k)-1})) \\ &\quad - \beta(x_{m(k)-1}, x_{m'(k)-1}) \\ &\quad \times \varphi(M_1(x_{m(k)-1}, x_{m'(k)-1})).\end{aligned} \tag{42}$$

Now, using (39), we get

$$\psi(d(x_{m(k)}, x_{m'(k)})) \leq (\psi - \varphi)(M_1(x_{m(k)-1}, x_{m'(k)-1})), \tag{43}$$

where, by (22), for all $k \in \mathbb{N}$,

$$\begin{aligned}&M_1(x_{m(k)-1}, x_{m'(k)-1}) \\ &\quad = \max\Big(d(x_{m(k)-1}, x_{m'(k)-1}),\ d(x_{m(k)-1}, Tx_{m(k)-1}), \\ &\qquad d(x_{m'(k)-1}, Tx_{m'(k)-1}), \\ &\qquad (d(x_{m(k)-1}, Tx_{m'(k)-1}) \\ &\qquad +d(x_{m'(k)-1}, Tx_{m(k)-1}))(2)^{-1}\Big) \\ &\quad = \max\Big(d(x_{m(k)-1}, x_{m'(k)-1}),\ d(x_{m(k)-1}, x_{m(k)}), \\ &\qquad d(x_{m'(k)-1}, x_{m'(k)}), \\ &\qquad (d(x_{m(k)-1}, x_{m'(k)}) \\ &\qquad +d(x_{m'(k)-1}, x_{m(k)}))(2)^{-1}\Big) \\ &\quad > 0.\end{aligned} \tag{44}$$

Lemma 12 shows that $d(x_{m(k)}, x_{m'(k)}) < M_1(x_{m(k)-1}, x_{m'(k)-1})$. Furthermore, by (22), (35), and (41),

$$\begin{aligned}&\lim_{k\to\infty} M_1(x_{m(k)-1}, x_{m'(k)-1}) \\ &\quad = \lim_{k\to\infty} \max\Big(d(x_{m(k)-1}, x_{m'(k)-1}), d(x_{m(k)-1}, x_{m(k)}) \\ &\qquad \times d(x_{m'(k)-1}, x_{m'(k)}), \\ &\qquad (d(x_{m(k)-1}, x_{m'(k)}) \\ &\qquad +d(x_{m'(k)-1}, x_{m(k)}))(2)^{-1}\Big) \\ &\quad = \max\left(\varepsilon_0, 0, 0, \frac{\varepsilon_0 + \varepsilon_0}{2}\right) = \varepsilon_0.\end{aligned} \tag{45}$$

Taking limit as $k \to \infty$ in

$$\begin{aligned}\psi(d(x_{m(k)}, x_{m'(k)})) &\leq \psi(M_1(x_{m(k)-1}, x_{m'(k)-1})) \\ &\quad - \varphi(M_1(x_{m(k)-1}, x_{m'(k)-1})) \\ &\leq \psi(M_1(x_{m(k)-1}, x_{m'(k)-1})),\end{aligned} \tag{46}$$

we deduce that $\lim_{k\to\infty}\varphi(M_1(x_{m(k)-1}, x_{m'(k)-1})) = 0$. By (a3), $\lim_{k\to\infty}M_1(x_{m(k)-1}, x_{m'(k)-1}) = 0$, which contradicts (45) and the fact that $\varepsilon_0 > 0$. This contradiction implies that $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is a complete metric space, then there exists $x^* \in X$ such that $\{x_n\} \to x^*$ as $n \to \infty$. From the continuity of T, it follows that $\{x_{n+1} = Tx_n\} \to Tx^*$ as $n \to \infty$. Due to the uniqueness of the limit, we derive that $Tx^* = x^*$; that is, x^* is a fixed point of T. □

Theorem 17. *In Theorem 16, if we replace the continuity of T by the $(\mathcal{R}_1, \mathcal{R}_2)$-regularity of X, then the conclusion of Theorem 16 holds.*

Proof. Following the lines of the proof of Theorem 16, we get that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, then there exists $x^* \in X$ such that $\{x_n\} \to x^*$ as $n \to \infty$. Furthermore, the sequence $\{x_n\}$ satisfies (24); that is,

$$x_n \mathscr{R}_1 x_{n+1}, \quad x_n \mathscr{R}_2 x_{n+1} \quad \forall n \in \mathbb{N}. \tag{47}$$

Now, since (X, d) is $(\mathscr{R}_1, \mathscr{R}_2)$-regular, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \mathscr{R}_1 x^*$, that is, $\alpha(x_{n(k)}, x^*) \le 1$, and $x_{n(k)} \mathscr{R}_2 x^*$, that is, $\beta(x_{n(k)}, x^*) \ge 1$, for all $k \in \mathbb{N}$. By setting $x = x_{n(k)}$ and $y = x^*$ in (19), we obtain, for all $k \in \mathbb{N}$,

$$\begin{aligned} \psi\left(d\left(x_{n(k)+1}, Tx^*\right)\right) \le{}& \alpha\left(x_{n(k)}, x^*\right) \psi\left(M_1\left(x_{n(k)}, x^*\right)\right) \\ & - \beta\left(x_{n(k)}, x^*\right) \varphi\left(M_1\left(x_{n(k)}, x^*\right)\right), \end{aligned} \tag{48}$$

that is,

$$\psi\left(d\left(x_{n(k)+1}, Tx^*\right)\right) \le (\psi - \varphi)\left(M_1\left(x_{n(k)}, x^*\right)\right) \quad \forall k \in \mathbb{N}, \tag{49}$$

where

$$\begin{aligned} & M_1\left(x_{n(k)}, x^*\right) \\ & = \max\left(d\left(x_{n(k)}, x^*\right), d\left(x_{n(k)}, Tx_{n(k)}\right), d\left(x^*, Tx^*\right), \frac{d\left(x_{n(k)}, Tx^*\right) + d\left(x^*, Tx_{n(k)}\right)}{2} \right) \\ & = \max\left(d\left(x_{n(k)}, x^*\right), d\left(x_{n(k)}, x_{n(k)+1}\right), d\left(x^*, Tx^*\right), \frac{d\left(x_{n(k)}, Tx^*\right) + d\left(x^*, Tx_{n(k)+1}\right)}{2} \right). \end{aligned} \tag{50}$$

We prove that $Tx^* = x^*$ reasoning by contradiction. If $d(x^*, Tx^*) > 0$, then $M_1(x_{n(k)}, x^*) \ge d(x^*, Tx^*) > 0$, for all $k \in \mathbb{N}$. By Lemma 12,

$$\begin{aligned} & d\left(x_{n(k)+1}, Tx^*\right) \\ & < M_1\left(x_{n(k)}, x^*\right) \\ & = \max\left(d\left(x_{n(k)}, x^*\right), d\left(x_{n(k)}, x_{n(k)+1}\right), d\left(x^*, Tx^*\right) \frac{d\left(x_{n(k)}, Tx^*\right) + d\left(x^*, x_{n(k)+1}\right)}{2} \right). \end{aligned} \tag{51}$$

Futhermore,

$$\begin{aligned} & \lim_{k \to \infty} M_1\left(x_{n(k)}, x^*\right) \\ & = \lim_{k \to \infty} \max\left(d\left(x_{n(k)}, x^*\right), d\left(x_{n(k)}, x_{n(k)+1}\right), d\left(x^*, Tx^*\right), \frac{d\left(x_{n(k)}, Tx^*\right) + d\left(x^*, x_{n(k)+1}\right)}{2} \right) \\ & = \max\left(d\left(x^*, x^*\right), d\left(x^*, x^*\right), d\left(x^*, Tx^*\right), \frac{d\left(x^*, Tx^*\right) + d\left(x^*, x^*\right)}{2} \right) \\ & = d\left(x^*, Tx^*\right). \end{aligned} \tag{52}$$

By (49), for all $k \in \mathbb{N}$,

$$\begin{aligned} \psi\left(d\left(x_{n(k)+1}, Tx^*\right)\right) & \le \psi\left(M_1\left(x_{n(k)}, x^*\right)\right) - \varphi\left(M_1\left(x_{n(k)}, x^*\right)\right) \\ & \le \psi\left(M_1\left(x_{n(k)}, x^*\right)\right). \end{aligned} \tag{53}$$

Using the continuity of ψ and letting $k \to \infty$ in the above inequality, we get

$$\lim_{k \to \infty} \varphi\left(M_1\left(x_{n(k)}, x^*\right)\right) = 0. \tag{54}$$

By (a3) and (52),

$$d\left(x^*, Tx^*\right) = \lim_{k \to \infty} M_1\left(x_{n(k)}, x^*\right) = 0, \tag{55}$$

which contradicts that $d(x^*, Tx^*) > 0$. This contradiction concludes that x^* is a fixed point of T. □

Taking into account that

$$\begin{gathered} M_5(x, y) \le M_4(x, y) \le M_3(x, y) \le M_1(x, y), \\ M_2(x, y) \le M_1(x, y), \quad \forall x, y \in X, \\ M_i(x, y) > 0, \quad \text{if } x \ne y \ \ (i \in \{1, 2, 3, 4, 5\}), \end{gathered} \tag{56}$$

the same proofs of the above theorems can be followed point by point to demonstrate the next result.

Corollary 18. *Theorems 16 and 17 also hold if T is a generalized $(\alpha\psi, \beta\varphi)$-contractive mapping of type I, II, III, IV, or V.*

2.2. Uniqueness. The uniqueness of the fixed point is studied in the following result.

Theorem 19. *Adding to the hypotheses of Theorem 16 (resp., Theorem 17) that X is $(\mathscr{R}_1, \mathscr{R}_2)$-directed and T is of type III, IV, or V, we obtain unicity of the fixed point of T.*

Proof. Assume that T is of type III; that is, $M = M_3$ (the other cases are similar). Suppose that x^* and y^* are any two fixed points of T. Since X is $(\mathscr{R}_1, \mathscr{R}_2)$-directed, there exists $z_0 \in X$ such that $x^* \mathscr{R}_1 z_0$, $y^* \mathscr{R}_1 z_0$, $x^* \mathscr{R}_2 z_0$, and $x^* \mathscr{R}_1 z_0$; that is,

$$\begin{aligned} \alpha(x^*, z_0) \leq 1, \qquad \alpha(y^*, z_0) \leq 1, \\ \beta(x^*, z_0) \geq 1, \qquad \beta(y^*, z_0) \geq 1. \end{aligned} \tag{57}$$

Define $z_n = T^n z_0$ for all $n \in \mathbb{N}$. We claim that $\{z_n\} \to x^*$ and $\{z_n\} \to y^*$. Hence, by the unicity of the limit, we will conclude that $x^* = y^*$. Therefore, it is only necessary to prove that $\{z_n\} \to x^*$.

Indeed, since T is $\mathscr{R}_i$-preserving for $i = 1, 2$, from (57), we get that

$$\begin{aligned} x^* \mathscr{R}_1 z_0 \Longrightarrow Tx^* \mathscr{R}_1 T z_0 \Longrightarrow x^* \mathscr{R}_1 z_1 \Longrightarrow \alpha(x^*, z_1) \leq 1, \\ x^* \mathscr{R}_2 z_0 \Longrightarrow Tx^* \mathscr{R}_2 T z_0 \Longrightarrow x^* \mathscr{R}_2 z_1 \Longrightarrow \beta(x^*, z_1) \geq 1, \end{aligned} \tag{58}$$

and, proceeding by induction, we have

$$\alpha(x^*, z_n) \leq 1, \quad \beta(x^*, z_n) \geq 1 \quad \forall n \in \mathbb{N}. \tag{59}$$

Using (59) and (19), we deduce that

$$\begin{aligned} \psi(d(x^*, z_{n+1})) &= \psi(d(Tx^*, Tz_n)) \\ &\leq \alpha(x^*, z_n)\, \psi(M_3(x^*, z_n)) \\ &\quad - \beta(x^*, z_n)\, \varphi(M_3(x^*, z_n)) \qquad (60) \\ &\leq (\psi - \varphi)(M_3(x^*, z_n)), \qquad (61) \end{aligned}$$

where

$$\begin{aligned} &M_3(x^*, z_n) \\ &= \max\left(d(x^*, z_n), \frac{d(x^*, Tx^*) + d(z_n, Tz_n)}{2}, \frac{d(x^*, Tz_n) + d(z_n, Tx^*)}{2} \right) \\ &= \max\left(d(x^*, z_n), \frac{d(z_n, z_{n+1})}{2}, \frac{d(x^*, z_{n+1}) + d(z_n, x^*)}{2} \right) \\ &= \max\left(d(x^*, z_n), \frac{d(x^*, z_{n+1}) + d(z_n, x^*)}{2} \right) \end{aligned} \tag{62}$$

(the last equality holds because $d(z_n, z_{n+1}) \leq d(x^*, z_{n+1}) + d(z_n, x^*)$). By Lemma 12, either $d(x^*, z_{n+1}) < M_3(x^*, z_n)$ or $M_3(x^*, z_n) = 0$. If $d(x^*, z_{n+1}) < M_3(x^*, z_n)$, then $d(x^*, z_{n+1}) < d(x^*, z_n)$. The second case yields to $x^* = z_n = z_{n+1}$. In any case, we deduce that

$$\begin{aligned} M_3(x^*, z_n) = d(x^*, z_n), \\ d(x^*, z_{n+1}) \leq d(x^*, z_n), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{63}$$

Since $\{d(x^*, z_n)\}$ is a bounded below, nonincreasing sequence, there exists $r \geq 0$ such that $\{d(x^*, z_n)\} \to r$. By (61),

$$\begin{aligned} \psi(d(x^*, z_{n+1})) &\leq (\psi - \varphi)(M_3(x^*, z_n)) \\ &= \psi(d(x^*, z_n)) - \varphi(d(x^*, z_n)) \\ &\leq \psi(d(x^*, z_n)) \end{aligned} \tag{64}$$

for all $n \in \mathbb{N}$. By the continuity of ψ and taking limit as $n \to \infty$, we deduce that $\lim_{k \to \infty} \varphi(d(x^*, z_n)) = 0$. Using (a3), we have $r = \lim_{k \to \infty} d(x^*, z_n) = 0$; that is, $\{z_n\} \to x^*$. This finishes the proof. □

Now, we derive a particular condition which ensures the uniqueness of the fixed point for the mappings T of type I, II, III, IV, or V as follows:

(C): if $r, s \in [0, +\infty)$ are such that $\psi(r) \leq (\psi - \varphi)(s)$, then either $r < 1/2\ s$ or $s = 0$.

For instance, if $k \in (1/2, 1)$ and we consider $\psi(t) = t$ and $\varphi(t) = kt$ for all $t \geq 0$, then ψ and φ verify condition (C).

Theorem 20. *Adding to the hypotheses of Theorem 16 (resp., Theorem 17) that X is $(\mathscr{R}_1, \mathscr{R}_2)$-directed and T is of type I, II, III, IV, or V, we obtain the unicity of the fixed point of T whenever condition (C) is satisfied.*

Proof. Following the lines of the proof of Theorem 19, we will prove that $\{z_n\} \to x^*$. Since X is $(\mathscr{R}_1, \mathscr{R}_2)$-directed with respect to T, there exists $z_0 \in X$ such that the sequence $\{z_n = T^n z_0\}_{n \in \mathbb{N}}$ converges (to some $z^* \in X$) and also $x^* \mathscr{R}_1 z_0$, $y^* \mathscr{R}_1 z_0$, $x^* \mathscr{R}_2 z_0$, and $x^* \mathscr{R}_1 z_0$; that is,

$$\begin{aligned} \alpha(x^*, z_0) \leq 1, \qquad \alpha(y^*, z_0) \leq 1, \\ \beta(x^*, z_0) \geq 1, \qquad \beta(y^*, z_0) \geq 1. \end{aligned} \tag{65}$$

Now we will prove that $z^* = x^*$. By induction, we have that $\alpha(x^*, z_n) \leq 1$ and $\beta(x^*, z_n) \geq 1$ for all $n \in \mathbb{N}$. Substituting $x = x^*$ and $y = z_n$ in (19), we get

$$\begin{aligned} \psi(d(x^*, z_{n+1})) &= \psi(d(Tx^*, Tz_n)) \\ &\leq \alpha(x^*, z_n)\, \psi(M_1(x^*, z_n)) \\ &\quad - \beta(x^*, z_n)\, \varphi(M_1(x^*, z_n)), \end{aligned} \tag{66}$$

that is,

$$\psi(d(Tx^*, Tz_n)) \leq (\psi - \varphi)(M_1(x^*, z_n)), \tag{67}$$

where

$$\begin{aligned} &M_1(x^*, z_n) \\ &= \max\Big(d(x^*, z_n), d(x^*, Tx^*), d(z_n, Tz_n), \end{aligned}$$

$$\left.\frac{d(x^*,Tz_n)+d(z_n,Tx^*)}{2}\right)$$

$$= \max\Big(d(x^*,z_n), d(z_n,z_{n+1}), \frac{d(x^*,z_{n+1})+d(z_n,x^*)}{2}\Big)$$

$$\leq 2\max\left(d(x^*,z_n), d(x^*,z_{n+1})\right). \tag{68}$$

Now from inequality (67) and the condition (C), it follows that, for all $n \in \mathbb{N}$,

$$\text{either } d(x^*,z_{n+1}) < \frac{1}{2}M_1(x^*,z_n) \quad \text{or} \quad M_1(x^*,z_n) = 0. \tag{69}$$

If there is some $n_0 \in \mathbb{N}$ such that $M_1(x^*,z_{n_0}) = 0$, the proof is finished (because $d(z_{n_0},Tz_{n_0}) = 0$). On the contrary, assume that $d(x^*,z_{n+1}) < (1/2)M_1(x^*,z_n)$ for all $n \in \mathbb{N}$. If $M_1(x^*,z_n) \leq 2d(x^*,z_{n+1})$, then

$$\begin{aligned} d(x^*,z_{n+1}) &< \frac{1}{2}M_1(x^*,z_n) \\ &\leq \frac{1}{2}2d(x^*,z_{n+1}) = d(x^*,z_{n+1}), \end{aligned} \tag{70}$$

which is a contradiction. Hence, necessarily, $M_1(x^*,z_n) \leq 2d(x^*,z_n)$ for all $n \in \mathbb{N}$, and then

$$\begin{aligned} d(x^*,z_{n+1}) &< \frac{1}{2}M_1(x^*,z_n) \\ &\leq \frac{1}{2}2d(x^*,z_n) = d(x^*,z_n) \quad \forall n \in \mathbb{N}. \end{aligned} \tag{71}$$

Thus, we deduce that $\{d(x^*,z_n)\}$ is a nonincreasing, bounded below sequence, so there exists $r \geq 0$ such that $\{d(x^*,z_n)\} \to r$. Therefore,

$$\begin{aligned} &\lim_{n\to\infty} M_1(x^*,z_n) \\ &= \lim_{n\to\infty}\max\Big(d(x^*,z_n), d(z_n,z_{n+1}), \frac{d(x^*,z_{n+1})+d(z_n,x^*)}{2}\Big) \\ &= \max\left(r, 0, \frac{r+r}{2}\right) = r. \end{aligned} \tag{72}$$

By (67),

$$\begin{aligned} \psi(d(x^*,z_{n+1})) &\leq (\psi-\varphi)(M_1(x^*,z_n)) \\ &= \psi(M_1(x^*,z_n)) - \varphi(M_1(x^*,z_n)) \\ &\leq \psi(M_1(x^*,z_n)) \end{aligned} \tag{73}$$

for all $n \in \mathbb{N}$. By the continuity of ψ and taking limit as $n \to \infty$, we deduce that $\lim_{k\to\infty}\varphi(M_1(x^*,z_n)) = 0$. Using (a3), we have $r = \lim_{k\to\infty}M_1(x^*,z_n) = 0$; that is, $\{z_n\} \to x^*$. This completes the proof. □

Theorem 21. *Adding to the hypotheses of Theorem 16 (resp., Theorem 17) that X is $(\mathscr{R}_1,\mathscr{R}_2)$-directed with respect to T and T is of type I, II, III, IV, or V, we obtain the unicity of the fixed point of T.*

Proof. Following the lines of the proof of Theorem 19, we will prove that $\{z_n\} \to x^*$. Since X is $(\mathscr{R}_1,\mathscr{R}_2)$-directed with respect to T, there exists $z_0 \in X$ such that the sequence $\{z_n = T^n z_0\}_{n\in\mathbb{N}}$ converges (to some $z^* \in X$) and also $x^*\mathscr{R}_1 z_0$, $y^*\mathscr{R}_1 z_0$, $x^*\mathscr{R}_2 z_0$, and $x^*\mathscr{R}_1 z_0$; that is,

$$\begin{aligned} \alpha(x^*,z_0) &\leq 1, \qquad \alpha(y^*,z_0) \leq 1, \\ \beta(x^*,z_0) &\geq 1, \qquad \beta(y^*,z_0) \geq 1. \end{aligned} \tag{74}$$

Now we will prove that $z^* = x^*$. By induction, we have $\alpha(x^*,z_n) \leq 1$ and $\beta(x^*,z_n) \geq 1$, for all $n \in \mathbb{N}$. Substituting $x = x^*$ and $y = z_n$ in (19), we get

$$\begin{aligned} \psi(d(x^*,z_{n+1})) &= \psi(d(Tx^*,Tz_n)) \\ &\leq \alpha(x^*,z_n)\psi(M_1(x^*,z_n)) \\ &\quad - \beta(x^*,z_n)\varphi(M_1(x^*,z_n)) \\ &\leq (\psi-\varphi)(M_1(x^*,z_n)), \end{aligned} \tag{75}$$

where

$$\begin{aligned} &M_1(x^*,z_n) \\ &= \max\Big(d(x^*,z_n), d(x^*,Tx^*), d(z_n,Tz_n), \frac{d(x^*,Tz_n)+d(z_n,Tx^*)}{2}\Big) \\ &= \max\Big(d(x^*,z_n), d(z_n,z_{n+1}), \frac{d(x^*,z_{n+1})+d(z_n,x^*)}{2}\Big). \end{aligned} \tag{76}$$

Notice that

$$\begin{aligned} &\lim_{n\to\infty} M_1(x^*,z_n) \\ &= \lim_{n\to\infty}\max\Big(d(x^*,z_n), d(z_n,z_{n+1}), \frac{d(x^*,z_{n+1})+d(z_n,x^*)}{2}\Big) \\ &= \max\left(d(x^*,z^*), 0, \frac{d(x^*,z^*)+d(x^*,z^*)}{2}\right) \\ &= d(x^*,z^*). \end{aligned} \tag{77}$$

Taking into account that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \psi(d(x^*,z_{n+1})) &\leq \psi(M_1(x^*,z_n)) - \varphi(M_1(x^*,z_n)) \\ &\leq \psi(M_1(x^*,z_n)), \end{aligned} \tag{78}$$

and taking limit as $n \to \infty$, we deduce that the sequence $\{\varphi(M_1(x^*, z_n))\}_{n\in\mathbb{N}}$ has finite limit and

$$\psi(d(x^*, z^*)) \le \psi(d(x^*, z^*)) - \lim_{n\to\infty}\varphi(M_1(x^*, z_n)) \le \psi(d(x^*, z^*)), \tag{79}$$

so $\lim_{n\to\infty}\varphi(M_1(x^*, z_n)) = 0$. By (a3), we conclude that $d(x^*, z^*) = \lim_{n\to\infty} M_1(x^*, z_n) = 0$; that is, $x^* = z^*$. □

3. Applications

Very recently, a mapping satisfying contraction on metric spaces endowed with a binary relation has been introduced by Samet and Turinici in [17]; therefore, this work has been extended and improved in [14, 18]. In this section, using our main results, we derive some consequences on metric spaces endowed with N-transitive binary relation, as on metric spaces endowed with a partial order. Furthermore, we establish a fixed point results for cyclic mappings.

3.1. Fixed Point Results on Metric Spaces Endowed with N-Transitive Binary Relation. In this section, we establish a fixed point theorem on metric space endowed with N-transitive binary relation $\mathcal{S}$. Therefore, we denote by $x\mathcal{S}y$ if x is $\mathcal{S}$-related to y.

Definition 22. We say that (X, d) is $\mathcal{S}$-regular if for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$, and

$$x_n \mathcal{S} x_{n+1} \quad \forall n \in \mathbb{N}, \tag{80}$$

there exists a subsequence $\{x_{n(k)}\}$ such that

$$x_{n(k)} \mathcal{S} x \quad \forall k \in \mathbb{N}. \tag{81}$$

Definition 23. We say that a subset D of X is $\mathcal{S}$-directed if for every $x, y \in D$, there exists $z \in X$ such that $x\mathcal{S}z$ and $y\mathcal{S}z$.

Corollary 24. *Let X be a nonempty set endowed with a binary relation $\mathcal{S}$. Suppose that there is a metric d on X such that (X, d) is complete. Let $T : X \to X$ satisfy the $\mathcal{S}$-weakly (ψ, φ)-contractive conditions; that is,*

$$\psi(d(Tx, Ty)) \le \psi(M(x, y)) - \varphi(M(x, y)) \quad \forall x\mathcal{S}y, \tag{82}$$

where ψ, φ are altering distance functions and M is given by Definition 14. Suppose also that the following conditions hold:

(i) *$\mathcal{S}$ is N-transitive ($N > 0$);*

(ii) *T is a $\mathcal{S}$-preserving mapping;*

(iii) *there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$;*

(iv) *T is continuous or (X, d) is $\mathcal{S}$-regular.*

Then T has a fixed point. Moreover, if we suppose that X is $\mathcal{S}$-directed with respect to I_X or T, then we have the uniqueness of the fixed point.

Proof. In view to link this theorem to the main result, we define the mapping $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x\mathcal{S}y \text{ or } x\mathcal{M}_i y; \\ 1 + \dfrac{\psi(d(Tx, Ty))}{\psi(M(x, y))} + \dfrac{\varphi(M(x, y))}{\varphi(M(x, y)) + \psi(M(x, y))}, & \text{otherwise}; \end{cases} \tag{83}$$

and we define the mapping $\beta : X \times X \to [0, +\infty)$ by

$$\beta(x, y) = \begin{cases} 1, & \text{if } x\mathcal{S}y \text{ or } x\mathcal{M}_i y; \\ \dfrac{\psi(M(x, y))}{\varphi(M(x, y)) + \psi(M(x, y))}, & \text{otherwise}, \end{cases} \tag{84}$$

where $x\mathcal{M}_i y$ for $i = 1, \ldots, 5$ are defined by

(a) $x\mathcal{M}_1 y$ if $(x = y) \wedge (x = Tx) \wedge (y = Ty) \wedge (x = Ty) \wedge (y = Tx)$;

(b) $x\mathcal{M}_2 y$ if $(x = y) \wedge (x = Tx) \wedge (y = Ty)$;

(c) $x\mathcal{M}_3 y$ if $(x = y) \wedge (x = Tx) \wedge (y = Ty) \wedge (x = Ty) \wedge (y = Tx)$;

(d) $x\mathcal{M}_4 y$ if $(x = y) \wedge (x = Tx) \wedge (y = Ty)$;

(e) $x\mathcal{M}_5 y$ if $(x = y)$.

In case x is neither $\mathcal{S}$-related nor $\mathcal{M}_i$-related to y, the functions α and β are well defined, since $\varphi(M(x, y)) \ne 0$ and $\psi(M(x, y)) \ne 0$.

We can verify easily that $\mathcal{R}_1$ and $\mathcal{R}_2$ are N-transitive.

Next, we claim that T is a $(\alpha\psi, \beta\varphi)$-contractive mapping. Indeed, in case $x\mathcal{S}y$, we get easily

$$\psi(d(Tx, Ty)) \le \alpha(x, y)\psi(M(x, y)) - \beta(x, y)\varphi(M(x, y)), \tag{85}$$

and in case x is neither $\mathcal{S}$-related nor $\mathcal{M}_i$-related to y, we have

$$\begin{aligned} \alpha(x, y)\psi(M(x, y)) - \beta(x, y)\varphi(M(x, y)) &= \psi(M(x, y)) + \psi(d(Tx, Ty)) \\ &\ge \psi(d(Tx, Ty)), \end{aligned} \tag{86}$$

hence, our claim holds.

Moreover, since T is $\mathcal{S}$-preserving, we get

$$\begin{aligned} x, y \in X, \quad x\mathcal{R}_1 y &\Longrightarrow \alpha(x, y) \le 1 \\ &\Longrightarrow x\mathcal{S}y \Longrightarrow Tx\mathcal{S}Ty \Longrightarrow \alpha(Tx, Ty) \le 1 \\ &\Longrightarrow Tx\mathcal{R}_1 Ty, \end{aligned} \tag{87}$$

and similarly, we have

$$\begin{aligned} x, y \in X, \quad x\mathcal{R}_2 y &\Longrightarrow \beta(x, y) \ge 1 \\ &\Longrightarrow x\mathcal{S}y \Longrightarrow Tx\mathcal{S}Ty \Longrightarrow \beta(Tx, Ty) \ge 1 \\ &\Longrightarrow Tx\mathcal{R}_2 Ty. \end{aligned} \tag{88}$$

Thus, T is $\mathscr{R}_i$-preserving for $i = 1, 2$. Now, if condition (iii) is satisfied; that is, T is continuous, the existence of a fixed point follows from Theorem 16. Suppose now that the (X, d) is $\mathscr{S}$-regular; hence, let $\{x_n\}$ be a nondecreasing sequence in X such that $x_n \mathscr{S} x_{n+1}$; that is, $\alpha(x_n, x_{n+1}) \leq 1$ and $\beta(x_n, x_{n+1}) \geq 1$, for all n. Suppose also that $x_n \to x \in X$ as $n \to \infty$. Since (X, d) is $\mathscr{S}$-regular, there exists a subsequence $\{x_{n(k)}\}$ such that $x_{n(k)} \mathscr{S} x$ for all k. This implies from the definition of α and β that $\alpha(x_{n(k)}, x) \leq 1$ and $\beta(x_{n(k)}, x) \geq 1$, for all k, which implies that $x_{n(k)} \mathscr{R}_i x$ for $i = 1, 2$ and for all k. In this case, the existence of a fixed point follows from Theorem 17.

To show the uniqueness, suppose that X is $\mathscr{S}$-directed with respect to I_X (resp., T); that is, for all $x, y \in X$, there exists a $z \in X$ such that $x \mathscr{S} z$ and $y \mathscr{S} z$ (resp., with $\{T^n z\}$ being a convergent sequence), which implies from the definition of α and β that $(\alpha(x, z) \leq 1) \wedge (\alpha(y, z) \leq 1)$ and $(\beta(x, z) \geq 1) \wedge (\beta(y, z) \geq 1)$); that is, X is $(\mathscr{R}_1, \mathscr{R}_2)$-directed with respect to I_X (resp., T). Hence, Theorem 20 or 19 (resp., Theorem 21) gives us the uniqueness of this fixed point. □

3.2. Fixed Point Results in Partially Ordered Metric Spaces. We start by defining the binary relations $\mathscr{R}_i$ for $i = 1, 2$ and the concept of $\leq$-directed.

Definition 25. Let $(X, \leq)$ be a partially ordered set.

(1) We define two binary relations $\mathscr{R}_1$ and $\mathscr{R}_2$ on X by

$$x, y \in X : x \mathscr{R}_i y \Longleftrightarrow x \leq y \quad \text{for } i = 1, 2. \tag{89}$$

(2) We say that X is $\leq$-directed if every $x, y \in X$ have a common upper bound; that is, there exists $z \in X$ such that $x \leq z$ and $y \leq z$.

The following definition is useful later.

Definition 26. Let $(X, \leq)$ be a partially ordered set and d be a metric on X. We say that (X, d) is $\leq$-regular if for every nondecreasing sequence $\{x_n\}$ in X such that $\{x_n\} \to x \in X$, there exists a subsequence $\{x_{n(k)}\}$ such that $x_{n(k)} \leq x$ for all $k \geq 0$.

Notice that, by the transitivity condition of $\leq$, in such a case, we have $x_n \leq x$ for all $n \geq 0$.

Corollary 27. *Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Suppose that the mapping $T : X \to X$ is weakly contractive; that is,*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \quad \forall x \leq y, \tag{90}$$

where ψ and φ are altering distance functions and M is given by Definition 14. Suppose also that the following conditions hold:

(i) *T is a nondecreasing mapping;*

(ii) *there exists $x_0 \in X$ with $x_0 \leq Tx_0$;*

(iii) *T is continuous or (X, d) is $\leq$-regular.*

Then T has a fixed point. Moreover, if X is $\leq$-directed with respect to I_X or T, we have the uniqueness of the fixed point.

Proof. The proof follows immediately from the previous proof, since $\leq$ is a binary, 1-transitive relation. □

3.3. Fixed Point Results for Cyclic Contractive Mappings. The main result of Kirk et al. in [19] is as follows.

Theorem 28 (see [19]). *For $i \in \{1, \ldots, N\}$, let A_i be a nonempty closed subset of a complete metric space (X, d) and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

(i) *$T(A_i) \subseteq A_{i+1}$ for all $i \in \{1, \ldots, N\}$ with $A_{N+1} := A_1$;*

(ii) *there exists $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leq k d(x, y). \tag{91}$$

Then T has a unique fixed point in $\cap_{i=1}^N A_i$.

Let us define the binary relations $\mathscr{R}_1$ and $\mathscr{R}_2$ as follows.

Definition 29. Let X be a nonempty set and let A_i, $i \in \{1, \ldots, N\}$ be nonempty closed subsets of X. We define two binary relations $\mathscr{R}_k$ for $k = 1, 2$ by

$$\begin{aligned} x, y \in X : x \mathscr{R}_k y \Longleftrightarrow (x, y) \in \Gamma \\ := \cup_{i=1}^N (A_i \times A_{i+1}) \quad \text{with } A_{N+1} := A_1. \end{aligned} \tag{92}$$

Now, based on Theorem 17 we will derive a more general result for cyclic mappings.

Corollary 30. *For $i \in \{1, \ldots, N\}$, let A_i be nonempty closed subsets of a complete metric space (X, d) and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

(i) *$T(A_i) \subseteq A_{i+1}$ for all $i \in \{1, \ldots, N\}$ with $A_{N+1} := A_1$;*

(ii) *there exists two altering distance functions ψ and φ such that*

$$\begin{aligned} \psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \\ \forall (x, y) \in A_i \times A_{i+1} \quad \forall i \in \{1, \ldots, N\}. \end{aligned} \tag{93}$$

Then T has a unique fixed point in $\cap_{i=1}^N A_i$.

Proof. Let $Y := \cup_{i=1}^N A_i$. For all $i \in \{1, \ldots, N\}$, we have by assumption that each A_i is nonempty closed subset of the complete metric space X, which implies that (Y, d) is complete.

Define the mapping $\alpha : Y \times Y \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \Gamma \text{ or } x \mathscr{M}_i y; \\ 1 + \dfrac{\psi(d(Tx, Ty))}{\psi(M(x, y))} \\ \quad + \dfrac{\varphi(M(x, y))}{\varphi(M(x, y)) + \psi(M(x, y))}, & \\ \quad \text{otherwise}, \end{cases} \tag{94}$$

and define the mapping $\beta : Y \times Y \to [0, +\infty)$ by

$$\beta(x,y) = \begin{cases} 1, & \\ \quad \text{if } (x,y) \in \Gamma \text{ or } x\mathscr{M}_i y; & \\ \dfrac{\psi(M(x,y))}{\varphi(M(x,y)) + \psi(M(x,y))} & \\ \quad \text{otherwise.} & \end{cases} \tag{95}$$

Hence, Definition 15 is equivalent to Definition 29.

We start by checking that $\mathscr{R}_1$ and $\mathscr{R}_2$ are N-transitive. Indeed, let $x_0, \ldots, x_{N+1} \in Y$ such that $x_k \mathscr{R}_1 x_{k+1}$ and $x_k \mathscr{R}_2 x_{k+1}$ for all $k \in \{0, \ldots, N\}$; that is, $\alpha(x_k, x_{k+1}) \leq 1$ and $\beta(x_k, x_{k+1}) \geq 1$ for all $k \in \{0, \ldots, N\}$ such that

$$\begin{aligned} x_0 \in A_i, \quad x_1 \in A_{i+1}, \ldots, x_k \in A_{i+k}, \ldots, \\ x_{N+1} \in A_{i+N+1} = A_{i+1}, \end{aligned} \tag{96}$$

which implies that $(x_0, x_{N+1}) \in A_i \times A_{i+1} \subseteq Y$. Hence, we obtain $\alpha(x_0, x_{N+1}) \leq 1$ and $\beta(x_0, x_{N+1}) \geq 1$, that is, $x_0 \mathscr{R}_1 x_{N+1}$ and $x_0 \mathscr{R}_2 x_{N+1}$, which implies that $\mathscr{R}_1$ and $\mathscr{R}_2$ are N-transitive.

Next, from (ii) and the definition of α and β, we can write

$$\begin{aligned} \psi(d(Tx, Ty)) \leq \alpha(x,y)\, \psi(M(x,y)) \\ - \beta(x,y)\, \varphi(M(x,y)), \end{aligned} \tag{97}$$

for all $x, y \in Y$. Thus, T is $(\alpha\psi, \beta\varphi)$-contractive mapping.

We claim next that T is $\mathscr{R}_1$-preserving and $\mathscr{R}_2$-preserving. Indeed, let $x, y \in Y$ such that $x\mathscr{R}_1 y$ and $x\mathscr{R}_2 y$; that is, $\alpha(x,y) \leq 1$ and $\beta(x,y) \geq 1$; hence, there exists $i \in \{1, \ldots, N\}$ such that $x \in A_i$, $y \in A_{i+1}$. Thus, $(Tx, Ty) \in A_{i+1} \times A_{i+2} \subseteq \Gamma$; then $\alpha(Tx, Ty) \leq 1$ and $\beta(Tx, Ty) \geq 1$, that is, $Tx\mathscr{R}_1 Ty$ and $Tx\mathscr{R}_2 Ty$. Hence, our claim holds.

Also, from (i), for any $x_0 \in A_i$ for all $i \in \{1, \ldots, N\}$, we have $(x_0, Tx_0) \in A_i \times A_{i+1}$, which implies that $\alpha(x_0, Tx_0) \leq 1$ and $\beta(x_0, Tx_0) \geq 1$, that is, $x_0 \mathscr{R}_1 Tx_0$ and $x_0 \mathscr{R}_2 Tx_0$.

Now, we claim that Y is $(\mathscr{R}_1, \mathscr{R}_2)$-regular. Let $\{x_n\}$ be a sequence in Y such that $x_n \to x \in Y$ as $n \to \infty$, and

$$x_n \mathscr{R}_1 x_{n+1}, \quad x_n \mathscr{R}_2 x_{n+1} \quad \forall n, \tag{98}$$

that is,

$$\alpha(x_n, x_{n+1}) \leq 1, \quad \beta(x_n, x_{n+1}) \geq 1 \quad \forall n. \tag{99}$$

It follows that there exist $i, j \in \{1, \ldots, N\}$ such that

$$x_n \in A_{i+n} \quad \forall n \in \mathbb{N},\ x \in A_j, \tag{100}$$

so

$$x_{(j-i-1+N)+kN} \in A_{j-1+(k+1)N} = A_{j-1} \quad \forall k \in \mathbb{N}. \tag{101}$$

By letting

$$n(k) := (j - i - 1 + N) + kN \quad \forall k \in \mathbb{N}, \tag{102}$$

we conclude that the subsequence $\{x_{n(k)}\}$ satisfies

$$(x_{n(k)}, x) \in A_{j-1} \times A_j \subseteq \Gamma \quad \forall k \in \mathbb{N}, \tag{103}$$

hence $\alpha(x_{n(k)}, x) \leq 1$ and $\beta(x_{n(k)}, x) \geq 1$ for all k, that is, $x_{n(k)} \mathscr{R}_1 x$ and $x_{n(k)} \mathscr{R}_2 x$, which proves our claim.

Hence, all the hypotheses of Theorem 17 are satisfied on (Y, d), and we deduce that T has a fixed point x^* in Y. Since $x^* \in A_i$ for some $i \in \{1, \ldots, N\}$ and $x^* = Tx^* \in A_{i+1}$ for all $i \in \{1, \ldots, N\}$, then $x^* \in \cap_{i=1}^N A_i$.

Moreover, it is easy to check that X is $(\mathscr{R}_1, \mathscr{R}_2)$-directed with respect to I_X (resp., T). Indeed, let $x, y \in Y$ with $x \in A_i$, $y \in A_j$, $i, j \in \{1, \ldots, N\}$. For $z = x^* \in Y$, we have $((\alpha(x,z) \leq 1) \wedge (\alpha(y,z) \leq 1))$, $((\beta(x,z) \geq 1) \wedge (\beta(y,z) \geq 1))$, and $\{T^n z\}$ is a convergent sequence. Thus, X is $(\mathscr{R}_1, \mathscr{R}_2)$-directed with respect to I_X or T. Hence, the uniqueness follows by Theorem 19 or Theorem 20 (resp., Theorem 21). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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