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Research Article

The Generalized Projective Riccati Equations Method for Solving Nonlinear Evolution Equations in Mathematical Physics

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We apply the generalized projective Riccati equations method to find the exact traveling wave solutions of some nonlinear evolution equations with any-order nonlinear terms, namely, the nonlinear Pochhammer-Chree equation, the nonlinear Burgers equation and the generalized, nonlinear Zakharov-Kuznetsov equation. This method presents wider applicability for handling many other nonlinear evolution equations in mathematical physics.

1. Introduction

In the recent years, investigations of exact solutions to nonlinear partial differential equations (NPDEs) play an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and geochemistry. To obtain traveling wave solutions, many powerful methods have been presented, such as the inverse scattering method [1], the tanh-function method [2–8], the Hirota bilinear transform method [9], the truncated Painleve expansion method [10–13], the Backlund transform method [14, 15], the Exp-function method [16–20], the Jacobi elliptic function expansion method [21–23], the generalized Riccati equations method [24–26], the (G'/G)-expansion method [27–33], and the $(G'/G, 1/G)$-expansion method [34–36]. Conte and Musette [37] presented an indirect method to seek more solitary wave solutions of some NPDEs that can be expressed as polynomials in two elementary functions which satisfy a projective Riccati equation [38]. Using this method, many solitary wave solutions of many NPDEs are found [38, 39]. Recently, Yan [40] developed further Conte and Musette's method by introducing more generalized projective Riccati equations.

In this paper, we will use the generalized projective Riccati equations method to construct exact solutions for the following three nonlinear evolution equations with higher-order nonlinear terms:

(i) the nonlinear Pochhammer-Chree equation [41]:

$$u_{tt} - u_{xxtt} - \left(\alpha u + \beta u^{n+1} + \gamma u^{2n+1}\right)_{xx} = 0, \quad n \geq 1, \tag{1}$$

where α, β, and γ are constants and $\gamma < 0$,

(ii) the nonlinear Burgers equation [42]:

$$u_t + a(u^n)_x + bu_{xx} = 0, \quad n > 1, \tag{2}$$

where a and b are constants;

(iii) the nonlinear generalized Zakharov-Kuznetsov equation [43]:

$$u_t + \left(Au^p + Bu^{2p}\right)u_x + C\left(u_{xxx} + u_{xyy}\right) = 0, \quad p > 0, \tag{3}$$

where A, B, and C are nonzero real constants.

Zuo [32] has applied the extended (G'/G)-expansion method and determined the exact solutions of (1), and Hayek [33] has found the exact solutions of (2) using another form of the extended (G'/G)-expansion method, while Zhang [44] has discussed (3) using an algebraic method to find some of its exact solutions. The rest of this paper is organized as follows. In Section 2, we give the description of the generalized projective Riccati equations method. In Section 3, we apply this method to solve (1)–(3). In Section 4, physical explanations of some obtained results are obtained. In Section 5, some conclusions are given.

2. Description of the Generalized Projective Riccati Equations Method

Consider we have the following NPDE:

$$F\left(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots\right) = 0, \tag{4}$$

where F is a polynomial in $u(x,t)$ and its partial derivatives, in which the highest-order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method.

Step 1. We use the wave transformation

$$u(x,t) = u(\xi), \quad \xi = x - ct, \tag{5}$$

where c is a constant, to reduce (4) to the following ODE:

$$P\left(u, u', u'', \ldots\right) = 0, \tag{6}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, such that $' = d/d\xi$.

Step 2. We assume that (6) has the formal solution

$$u(\xi) = A_0 + \sum_{i=1}^{m} \sigma^{i-1}(\xi)\left[A_i\sigma(\xi) + B_i\tau(\xi)\right], \tag{7}$$

where A_0, A_i, and B_i are constants to be determined later. The functions $\sigma(\xi)$ and $\tau(\xi)$ satisfy the ODEs

$$\begin{gathered}\sigma'(\xi) = \varepsilon\sigma(\xi)\tau(\xi), \\ \tau'(\xi) = R + \varepsilon\tau^2(\xi) - \mu\sigma(\xi), \quad \varepsilon = \pm 1,\end{gathered} \tag{8}$$

where

$$\tau^2(\xi) = -\varepsilon\left(R - 2\mu\sigma(\xi) + \frac{\mu^2 - 1}{R}\sigma^2(\xi)\right), \tag{9}$$

where R and μ are nonzero constants.

If $R = \mu = 0$, (6) has the formal solution

$$u(\xi) = \sum_{i=0}^{m} A_i\tau^i(\xi), \tag{10}$$

where $\tau(\xi)$ satisfies the ODE

$$\tau'(\xi) = \tau^2(\xi). \tag{11}$$

Step 3. We determine the positive integer m in (7) by using the homogeneous balance between the highest-order derivatives and the nonlinear terms in (6). In some nonlinear equations the balance number m is not a positive integer. In this case, we make the following transformations.

(a) When $m = q/p$, where q/p is a fraction in the lowest terms, we let

$$u(\xi) = v^{q/p}(\xi), \tag{12}$$

and then we substitute (12) into (6) to get a new equation in the new function $v(\xi)$ with a positive integer balance number.

(b) When m is a negative number, we let

$$u(\xi) = v^m(\xi), \tag{13}$$

and then we substitute (13) into (6) to get a new equation in the new function $v(\xi)$ with a positive integer balance number.

Step 4. Substitute (7) along with (8)-(9) into (6) or (10) along with (11) into (6). Collect all terms of the same order of $\sigma^j(\xi)\tau^i(\xi)$ ($j = 0, 1, \ldots, i = 0, 1$) (or $\tau^j(\xi)$, $j = 0, 1, \ldots$). Setting each coefficient to zero yields a set of algebraic equations which can be solved to find the values of A_0, A_i, B_i, c, μ, and R.

Step 5. It is well known [24] that (8) admits the following solutions.

(i) If $\varepsilon = -1$, $R \neq 0$,

$$\begin{aligned}\sigma_1(\xi) &= \frac{R\operatorname{sech}\left(\sqrt{R}\xi\right)}{\mu\operatorname{sech}\left(\sqrt{R}\xi\right) + 1}, \qquad &\tau_1(\xi) &= \frac{\sqrt{R}\tanh\left(\sqrt{R}\xi\right)}{\mu\operatorname{sech}\left(\sqrt{R}\xi\right) + 1}, \\ \sigma_2(\xi) &= \frac{R\operatorname{csch}\left(\sqrt{R}\xi\right)}{\mu\operatorname{csch}\left(\sqrt{R}\xi\right) + 1}, \qquad &\tau_2(\xi) &= \frac{\sqrt{R}\coth\left(\sqrt{R}\xi\right)}{\mu\operatorname{csch}\left(\sqrt{R}\xi\right) + 1}.\end{aligned} \tag{14}$$

(ii) If $\varepsilon = 1$, $R \neq 0$,

$$\begin{aligned}\sigma_3(\xi) &= \frac{R\sec\left(\sqrt{R}\xi\right)}{\mu\sec\left(\sqrt{R}\xi\right) + 1}, \qquad &\tau_3(\xi) &= \frac{\sqrt{R}\tan\left(\sqrt{R}\xi\right)}{\mu\sec\left(\sqrt{R}\xi\right) + 1}, \\ \sigma_4(\xi) &= \frac{R\csc\left(\sqrt{R}\xi\right)}{\mu\csc\left(\sqrt{R}\xi\right) + 1}, \qquad &\tau_4(\xi) &= -\frac{\sqrt{R}\cot\left(\sqrt{R}\xi\right)}{\mu\csc\left(\sqrt{R}\xi\right) + 1}.\end{aligned} \tag{15}$$

(iii) If $R = \mu = 0$,

$$\sigma_5(\xi) = \frac{C}{\xi}, \qquad \tau_5(\xi) = \frac{1}{\varepsilon\xi}, \tag{16}$$

where C is nonzero constant.

Step 6. Substituting the values of A_0, A_i, B_i, c, μ, and R as well as the solutions (14)–(16) into (7), we obtain the exact solutions of (4).

We close this section with the remark that without loss of generality we take $\varepsilon = -1$ (similarly the case $\varepsilon = 1$ can be done which is omitted here for simplicity).

3. Applications

In this section, we will apply the proposed method described in Section 2 to find the exact traveling wave solutions of the nonlinear equations (1)–(3).

Example 1 (the nonlinear Pochhammer-Chree equation (1)). In this example, we find the exact solutions of (1). To this end, we see that the traveling wave variable (5) permits us to convert (1) into the following ODE:

$$c^2u'' - c^2u'''' - \left(\alpha u + \beta u^{n+1} + \gamma u^{2n+1}\right)'' = 0. \qquad (17)$$

Integrating (17) twice with respect to ξ and vanishing the constants of integration, we get

$$\left(c^2 - \alpha\right) u - c^2u'' - \beta u^{n+1} - \gamma u^{2n+1} = 0. \qquad (18)$$

By balancing u'' with u^{2n+1} in (18) we get $m = 1/n$. According to Step 3, we use the transformation

$$u(\xi) = v^{1/n}(\xi), \qquad (19)$$

where $v(\xi)$ is a new function of ξ. Substituting (19) into (18), we get the new ODE

$$\begin{aligned} &\left(c^2 - \alpha\right) n^2v^2 - c^2nvv'' \\ &\quad - c^2(1-n)\left(v'\right)^2 - \beta n^2v^3 - \gamma n^2v^4 = 0. \end{aligned} \qquad (20)$$

Balancing vv'' with v^4 in (20), we get $m = 1$. Consequently, we get

$$v(\xi) = A_0 + A_1\sigma(\xi) + B_1\tau(\xi), \qquad (21)$$

where A_0, A_1, and B_1 are constants to be determined later.

Substituting (21) into (20) and using (8)-(9) with $\varepsilon = -1$, the left-hand side of (20) becomes a polynomial in σ and τ. Setting the coefficients of this polynomial to be zero yields the following system of algebraic equations:

$$\begin{aligned}
\sigma^4 :\ & -n^2\gamma A_1^4R^2 \\
& + R\left(c^2nA_1^2 - c^2n\mu^2A_1^2 - c^2\mu^2A_1^2\right. \\
& \qquad \left. + c^2A_1^2 - 6\gamma n^2\mu^2A_1^2B_1^2 + 6\gamma n^2A_1^2B_1^2\right) \\
& + 2c^2n\mu^2B_1^2 - c^2n\mu^4B_1^2 - c^2nB_1^2 - c^2\mu^4B_1^2 \\
& + 2c^2\mu^2B_1^2 - c^2B_1^2 - \gamma n^2\mu^4B_1^4 \\
& + 2\gamma n^2\mu^2B_1^4 - \gamma n^2B_1^4 = 0, \\
\sigma^3 :\ & 2c^2\mu^3B_1^2 - 2c^2\mu B_1^2 + 2c^2nA_0A_1 \\
& + 4n^2\gamma\mu^3B_1^4 + 2Rc^2\mu A_1^2 - Rn^2\beta A_1^3 \\
& - 3c^2n\mu B_1^2 - 4n^2\gamma\mu B_1^4 + 3n^2\beta A_1B_1^2 + 3c^2n\mu^3B_1^2 \\
& + 12n^2\gamma A_0A_1B_1^2 - 3n^2\beta\mu^2A_1B_1^2 + Rc^2n\mu A_1^2 \\
& - 4Rn^2\gamma A_0A_1^3 - 2c^2n\mu^2A_0A_1 \\
& + 12Rn^2\gamma\mu A_1^2B_1^2 - 12n^2\gamma\mu^2A_0A_1B_1^2 = 0, \\
\sigma^3\tau :\ & -2c^2n\mu^2A_1B_1 + 2c^2nA_1B_1 - 2c^2\mu^2A_1B_1 \\
& + 2c^2A_1B_1 - 4\gamma n^2\mu^2A_1B_1^3 - 4R\gamma n^2A_1^3B_1 \\
& + 4\gamma n^2A_1B_1^3 = 0, \\
\sigma^2 :\ & R^2\left(-c^2A_1^2 - 6\gamma n^2A_1^2B_1^2\right) \\
& + R\left(3c^2n\mu A_0A_1 + 2c^2nB_1^2 - c^2\mu^2B_1^2\right. \\
& \qquad - 6\gamma n^2\mu^2B_1^4 + 24\gamma n^2\mu A_0A_1B_1^2 \\
& \qquad + 6\beta n^2\mu A_1B_1^2 - 6\gamma n^2A_0^2A_1^2 - 3\beta n^2A_0A_1^2 \\
& \qquad \left. - \alpha n^2A_1^2 + 2\gamma n^2B_1^4 + c^2n^2A_1^2 - 3c^2n\mu^2B_1^2\right) \\
& + c^2n^2\mu^2B_1^2 - c^2n^2B_1^2 - 6\gamma n^2\mu^2A_0^2B_1^2 \\
& - 3\beta n^2\mu^2A_0B_1^2 - \alpha n^2\mu^2B_1^2 + 6\gamma n^2A_0^2B_1^2 \\
& + 3\beta n^2A_0B_1^2 + \alpha n^2B_1^2 = 0, \\
\sigma^2\tau :\ & n^2\beta B_1^3 + 2c^2nA_0B_1 - n^2\beta\mu^2B_1^3 \\
& + 4n^2\gamma A_0B_1^3 + 2Rc^2\mu A_1B_1 - 4n^2\gamma\mu^2A_0B_1^3 \\
& - 3Rn^2\beta A_1^2B_1 - 2c^2n\mu^2A_0B_1 - 12Rn^2\gamma A_0A_1^2B_1 \\
& + 2Rc^2n\mu A_1B_1 + 8Rn^2\gamma\mu A_1B_1^3 = 0, \\
\sigma :\ & 2c^2n^2A_0A_1 - 2n^2\alpha A_0A_1 + 2n^2\alpha\mu B_1^2 \\
& - 3n^2\beta A_0^2A_1 - 4n^2\gamma A_0^3A_1 - 2c^2n^2\mu B_1^2 \\
& - Rc^2nA_0A_1 + 12n^2\gamma\mu A_0^2B_1^2 + Rc^2n\mu B_1^2 \\
& + 4Rn^2\gamma\mu B_1^4 - 3Rn^2\beta A_1B_1^2 + 6n^2\beta\mu A_0B_1^2 \\
& - 12Rn^2\gamma A_0A_1B_1^2 = 0, \\
\sigma\tau :\ & 2c^2n^2A_1B_1 - 2n^2\alpha A_1B_1 + 2n^2\beta\mu B_1^3 \\
& - 12n^2\gamma A_0^2A_1B_1 - Rc^2nA_1B_1 + c^2n\mu A_0B_1 \\
& - 6n^2\beta A_0A_1B_1 - 4Rn^2\gamma A_1B_1^3 + 8n^2\gamma\mu A_0B_1^3 = 0, \\
\tau :\ & 2c^2n^2A_0B_1 - 4\gamma n^2A_0^3B_1 - 3\beta n^2A_0^2B_1 \\
& - 4R\gamma n^2A_0B_1^3 - 2\alpha n^2A_0B_1 - R\beta n^2B_1^3 = 0, \\
\sigma^0 :\ & c^2n^2A_0^2 - n^2\alpha A_0^2 - n^2\beta A_0^3 - n^2\gamma A_0^4 \\
& - Rn^2\alpha B_1^2 + Rc^2n^2B_1^2 - R^2n^2\gamma B_1^4 \\
& - 6Rn^2\gamma A_0^2B_1^2 - 3Rn^2\beta A_0B_1^2 = 0.
\end{aligned} \qquad (22)$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results.

Case 1. We have

$$A_0 = 0, \qquad A_1 = -\frac{\beta(n+1)\left(\mu^2-1\right)}{\mu\gamma R(n+2)},$$
$$B_1 = 0, \qquad c = \pm\frac{\beta n}{\mu(n+2)}\sqrt{-\frac{(n+1)\left(\mu^2-1\right)}{\gamma R}}, \tag{23}$$
$$\mu = \mu, \qquad \alpha = \frac{\left(R-n^2\right)(n+1)\left(\mu^2-1\right)}{\mu^2\gamma R(n+2)^2}.$$

From (14), (19), (21), and (23), we deduce the following exact solutions:

$$u_{11} = \left[-\frac{\beta(n+1)(\mu^2-1)}{\mu\gamma(n+2)}\left(\frac{\operatorname{sech}\left(\sqrt{R}\xi\right)}{\mu\operatorname{sech}\left(\sqrt{R}\xi\right)+1}\right)\right]^{1/n}, \tag{24}$$

$$u_{12} = \left[-\frac{\beta(n+1)(\mu^2-1)}{\mu\gamma(n+2)}\left(\frac{\operatorname{csch}\left(\sqrt{R}\xi\right)}{\mu\operatorname{csch}\left(\sqrt{R}\xi\right)+1}\right)\right]^{1/n}, \tag{25}$$

where $\xi = x \pm (\beta n/\mu(n+2))\sqrt{-((n+1)(\mu^2-1)/\gamma R)}t$.

Case 2. We have

$$A_0 = -\frac{\beta(n+1)}{2\gamma(n+2)}, \qquad A_1 = 0,$$
$$B_1 = \pm\frac{\beta(n+1)}{2\gamma(n+2)\sqrt{R}}, \qquad c = \pm\frac{\beta n}{2(n+2)}\sqrt{-\frac{(n+1)}{\gamma R}}, \tag{26}$$
$$\mu = 0, \qquad \alpha = \frac{\beta^2\left(4R-n^2\right)(n+1)}{4\gamma R(n+2)^2}.$$

In this case, we deduce the following exact solutions:

$$u_{21} = \left[\frac{\beta(n+1)}{2\gamma(n+2)}\left(-1\pm\tanh\left(\sqrt{R}\xi\right)\right)\right]^{1/n}, \tag{27}$$

$$u_{22} = \left[\frac{\beta(n+1)}{2\gamma(n+2)}\left(-1\pm\coth\left(\sqrt{R}\xi\right)\right)\right]^{1/n}, \tag{28}$$

where $\xi = x \pm (\beta n/2(n+2))\sqrt{-((n+1)/\gamma R)}t$.

Case 3. We have

$$A_0 = -\frac{\beta(n+1)}{2\gamma(n+2)}, \qquad A_1 = \pm\frac{\beta(n+1)\sqrt{\mu^2-1}}{2\gamma R(n+2)},$$
$$B_1 = \pm\frac{\beta(n+1)}{2\gamma(n+2)\sqrt{R}}, \qquad \mu = \mu,$$
$$c = \pm\frac{\beta n}{(n+2)}\sqrt{-\frac{(n+1)}{\gamma R}}, \qquad \alpha = \frac{\beta^2\left(R-n^2\right)(n+1)}{\gamma R(n+2)^2}. \tag{29}$$

In this case, we deduce the following exact solutions:

$$u_{31} = \left[\frac{\beta(n+1)}{2\gamma(n+2)} \times\left(-1\pm\frac{\sqrt{\mu^2-1}\operatorname{sech}\left(\sqrt{R}\xi\right)+\tanh\left(\sqrt{R}\xi\right)}{\mu\operatorname{sech}\left(\sqrt{R}\xi\right)+1}\right)\right]^{1/n}, \tag{30}$$

$$u_{32} = \left[\frac{\beta(n+1)}{2\gamma(n+2)} \times\left(-1\pm\frac{\sqrt{\mu^2-1}\operatorname{csch}\left(\sqrt{R}\xi\right)+\coth\left(\sqrt{R}\xi\right)}{\mu\operatorname{csch}\left(\sqrt{R}\xi\right)+1}\right)\right]^{1/n}, \tag{31}$$

where $\xi = x \pm (\beta n/(n+2))\sqrt{-((n+1)/\gamma R)}t$.

Example 2 (the nonlinear Burgers equation (2)). In this example, we study the Burgers equation with power-law nonlinearity (2). To this end, we see that the traveling wave variable (4) permits us to convert (2) into the following ODE:

$$-cu' + a(u^n)' + bu'' = 0. \tag{32}$$

Integrating (32) once with respect ξ and setting the constant of integration to be zero yield

$$-cu + au^n + bu' = 0. \tag{33}$$

By balancing u' with u^n in (33) we get $m = 1/(n-1)$. According to Step 3, we use the transformation

$$u(\xi) = v^{1/(n-1)}(\xi), \tag{34}$$

where $v(\xi)$ is a new function of ξ. Substituting (34) into (33), we get the new ODE

$$-c(n-1)v + a(n-1)v^2 + bv' = 0. \tag{35}$$

Balancing v' with v^2 in (35), we get $m = 1$. Consequently, we get

$$v(\xi) = A_0 + A_1\sigma(\xi) + B_1\tau(\xi), \tag{36}$$

where A_0, A_1, and B_1 are constants to be determined later.

Substituting (36) into (35) and using (8)-(9) with $\varepsilon = -1$, the left-hand side of (35) becomes a polynomial in σ and τ.

Setting the coefficients of this polynomial to be zero yields the following system of algebraic equations:

$$\begin{aligned}
&\sigma^2 : bB_1 + aB_1^2 - a\mu^2 B_1^2 - RaA_1^2 \\
&\qquad - anB_1^2 - b\mu^2 B_1 + an\mu^2 B_1^2 + RanA_1^2 = 0, \\
&\sigma : b\mu B_1 - (n-1)\, cA_1 + 2(n-1)\, aA_0 A_1 \\
&\qquad - 2(n-1)\, a\mu B_1^2 = 0, \\
&\sigma\tau : 2(n-1)\, aA_1 B_1 - bA_1 = 0, \\
&\tau : 2(n-1)\, aA_0 B_1 - (n-1)\, cB_1 = 0, \\
&\sigma^0 : (n-1)\, aA_0^2 - (n-1)\, cA_0 + (n-1)\, RaB_1^2 = 0.
\end{aligned} \tag{37}$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results.

Case 1. We have

$$\begin{aligned}
A_0 &= \pm\frac{b\sqrt{R}}{2a(n-1)}, \qquad A_1 = \pm\frac{b}{2a(n-1)}\sqrt{\frac{\mu^2-1}{R}}, \\
B_1 &= \frac{b}{2a(n-1)}, \qquad c = \pm\frac{b\sqrt{R}}{n-1}.
\end{aligned} \tag{38}$$

From (14), (34), (36), and (38), we deduce the following exact solutions:

$$u_{11} = \left[\frac{b\sqrt{R}}{2a(n-1)} \times \left(\pm 1 + \frac{\tanh\left(\sqrt{R}\xi\right) \pm \sqrt{\mu^2-1}\operatorname{sech}\left(\sqrt{R}\xi\right)}{\mu\operatorname{sech}\left(\sqrt{R}\xi\right)+1}\right)\right]^{1/(n-1)}, \tag{39}$$

$$u_{12} = \left[\frac{b\sqrt{R}}{2a(n-1)} \times \left(\pm 1 + \frac{\coth\left(\sqrt{R}\xi\right) \pm \sqrt{\mu^2-1}\operatorname{csch}\left(\sqrt{R}\xi\right)}{\mu\operatorname{csch}\left(\sqrt{R}\xi\right)+1}\right)\right]^{1/(n-1)}, \tag{40}$$

where $\xi = x \pm (b\sqrt{R}/(n-1))t$.

Case 2. We have

$$\begin{aligned}
A_0 &= \pm\frac{b\sqrt{R}}{a(n-1)}, \qquad A_1 = 0, \\
B_1 &= \frac{b}{a(n-1)}, \qquad c = \pm\frac{2b\sqrt{R}}{n-1}, \qquad \mu = 0.
\end{aligned} \tag{41}$$

In this case, we deduce the following exact solutions:

$$u_{21} = \left[\frac{b\sqrt{R}}{a(n-1)}\left(\tanh(\sqrt{R}\xi) \pm 1\right)\right]^{1/(n-1)}, \tag{42}$$

$$u_{22} = \left[\frac{b\sqrt{R}}{a(n-1)}\left(\coth(\sqrt{R}\xi) \pm 1\right)\right]^{1/(n-1)}, \tag{43}$$

where $\xi = x \pm (2b\sqrt{R}/(n-1))t$.

Example 3 (the nonlinear generalized Zakharov-Kuznetsov equation (3)). In this example, we study the generalized Zakharov-Kuznetsov equation with power-law nonlinearity (3). To this end, we use the traveling wave variable

$$u(x,y,t) = u(\xi), \quad \xi = x + ky + st, \tag{44}$$

where k and s are nonzero constants, to reduce (3) to the following ODE:

$$su' + \left(Au^p + Bu^{2p}\right)u' + C\left(1+k^2\right)u''' = 0. \tag{45}$$

By balancing u''' with $u^{2p}u'$ in (45) we get $m = 1/p$. According to Step 3, we use the transformation

$$u(\xi) = v^{1/p}(\xi), \tag{46}$$

where $v(\xi)$ is a new function of ξ. Substituting (46) into (45), we get the new ODE

$$\begin{aligned}
&Cp^2\left(1+k^2\right)v^2 v''' + 3C\left(-p^2 + p - k^2p^2 + k^2 p\right)vv'v'' \\
&\quad + C\left(1 - 3p - 3k^2 p + 2p^2 + k^2 + 2k^2p^2\right)\left(v'\right)^3 \\
&\quad + \left(Av + s + Bv^2\right)p^2 v^2 v' = 0.
\end{aligned} \tag{47}$$

Balancing v^2v''' with v^4v' in (47), we get $m = 1$. Consequently, we get

$$v(\xi) = A_0 + A_1\sigma(\xi) + B_1\tau(\xi), \tag{48}$$

where A_0, A_1, and B_1 are constants to be determined later.

Substituting (48) into (47) and using (8)-(9) with $\varepsilon = -1$, the left-hand side of (47) becomes a polynomial in σ and τ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in A_0, A_1, B_1, μ, and c, which can be solved using the Maple or Mathematica; we get the following results.

Case 1. We have

$$\begin{aligned}
&A_0 = 0, \qquad A_1 = -\frac{A(2p+1)\left(\mu^2-1\right)}{\mu BR(p+2)}, \\
&B_1 = 0, \qquad \mu = \mu, \qquad s = \frac{A^2(2p+1)\left(\mu^2-1\right)}{\mu^2 B(p+2)^2(p+1)}, \\
&k = \pm\sqrt{-\frac{A^2p^2(2p+1)\left(\mu^2-1\right)}{\mu^2 RBC(p+2)^2(p+1)} - 1}.
\end{aligned} \tag{49}$$

From (14), (46), (48), and (49), we deduce the following exact solutions:

$$u_{11} = \left[-\frac{A(2p+1)(\mu^2-1)}{\mu B(p+2)} \left(\frac{\operatorname{sech}(\sqrt{R}\xi)}{\mu \operatorname{sech}(\sqrt{R}\xi)+1} \right) \right]^{1/p}, \tag{50}$$

$$u_{12} = \left[-\frac{A(2p+1)(\mu^2-1)}{\mu B(p+2)} \left(\frac{\operatorname{csch}(\sqrt{R}\xi)}{\mu \operatorname{csch}(\sqrt{R}\xi)+1} \right) \right]^{1/p}, \tag{51}$$

where $\xi = x + ky + st$, $k = \pm\sqrt{-(A^2p^2(2p+1)(\mu^2-1)/\mu^2 RBC(p+2)^2(p+1)) - 1}$, $s = (A^2(2p+1)(\mu^2-1)/\mu^2 B(p+2)^2(p+1))$.

Case 2. We have

$$\begin{gathered} A_0 = -\frac{A(2p+1)}{2B(p+2)}, \qquad A_1 = 0, \\ B_1 = \pm\frac{A(2p+1)}{2B(p+2)\sqrt{R}}, \qquad \mu = 1, \\ s = \frac{A^2(2p+1)}{B(p+2)^2(p+1)} \\ k = \pm\sqrt{-\frac{A^2p^2(2p+1)}{RBC(p+2)^2(p+1)} - 1}. \end{gathered} \tag{52}$$

In this case, we have the exact solutions

$$u_{21} = \left[\frac{A(2p+1)}{2B(p+2)} \left(-1 \pm \frac{\tanh(\sqrt{R}\xi)}{\operatorname{sech}(\sqrt{R}\xi)+1} \right) \right]^{1/p}, \tag{53}$$

$$u_{22} = \left[\frac{A(2p+1)}{2B(p+2)} \left(-1 \pm \frac{\coth(\sqrt{R}\xi)}{\operatorname{csch}(\sqrt{R}\xi)+1} \right) \right]^{1/p}, \tag{54}$$

where $\xi = x + ky + st$, $k = \pm\sqrt{-(A^2p^2(2p+1)/RBC(p+2)^2(p+1)) - 1}$, $s = (A^2(2p+1)/B(p+2)^2(p+1))$.

Case 3. We have

$$\begin{gathered} A_0 = -\frac{A(2p+1)}{2B(p+2)}, \qquad A_1 = 0, \\ B_1 = \pm\frac{A(2p+1)}{2B(p+2)\sqrt{R}}, \\ \mu = -1, \qquad s = \frac{A^2(2p+1)}{B(p+2)^2(p+1)}, \\ k = \pm\sqrt{-\frac{A^2p^2(2p+1)}{RBC(p+2)^2(p+1)} - 1}. \end{gathered} \tag{55}$$

In this case, we have the exact solutions

$$u_{31} = \left[\frac{A(2p+1)}{2B(p+2)} \left(-1 \pm \frac{\tanh(\sqrt{R}\xi)}{1-\operatorname{sech}(\sqrt{R}\xi)} \right) \right]^{1/p}, \tag{56}$$

$$u_{32} = \left[\frac{A(2p+1)}{2B(p+2)} \left(-1 \pm \frac{\coth(\sqrt{R}\xi)}{1-\operatorname{csch}(\sqrt{R}\xi)} \right) \right]^{1/p}, \tag{57}$$

where $\xi = x + ky + st$, $k = \pm\sqrt{-(A^2p^2(2p+1)/RBC(p+2)^2(p+1)) - 1}$, $s = (A^2(2p+1)/B(p+2)^2(p+1))$.

Case 4. We have

$$\begin{gathered} A_0 = -\frac{A(2p+1)}{2B(p+2)}, \qquad A_1 = 0, \\ B_1 = \pm\frac{A(2p+1)}{2B(p+2)\sqrt{R}}, \\ \mu = 0, \qquad s = \frac{A^2(2p+1)}{B(p+2)^2(p+1)} \\ k = \pm\sqrt{-\frac{A^2p^2(2p+1)}{4RBC(p+2)^2(p+1)} - 1}. \end{gathered} \tag{58}$$

In this case, we have the exact solutions

$$u_{41} = \left[\frac{A(2p+1)}{2B(p+2)} \left(-1 \pm \tanh(\sqrt{R}\xi) \right) \right]^{1/p}, \tag{59}$$

$$u_{42} = \left[\frac{A(2p+1)}{2B(p+2)} \left(-1 \pm \coth(\sqrt{R}\xi) \right) \right]^{1/p}, \tag{60}$$

where $\xi = x + ky + st$, $k = \pm\sqrt{-(A^2p^2(2p+1)/4RBC(p+2)^2(p+1)) - 1}$, $s = (A^2(2p+1)/B(p+2)^2(p+1))$.

Case 5. We have

$$\begin{gathered} A_0 = -\frac{A(2p+1)}{2B(p+2)}, \qquad A_1 = \pm\frac{A(2p+1)\sqrt{\mu^2-1}}{2BR(p+2)}, \\ B_1 = \pm\frac{A(2p+1)}{2B(p+2)\sqrt{R}}, \qquad \mu = \mu, \\ s = \frac{A^2(2p+1)}{B(p+2)^2(p+1)}, \\ k = \pm\sqrt{-\frac{A^2p^2(2p+1)}{RBC(p+2)^2(p+1)} - 1}. \end{gathered} \tag{61}$$

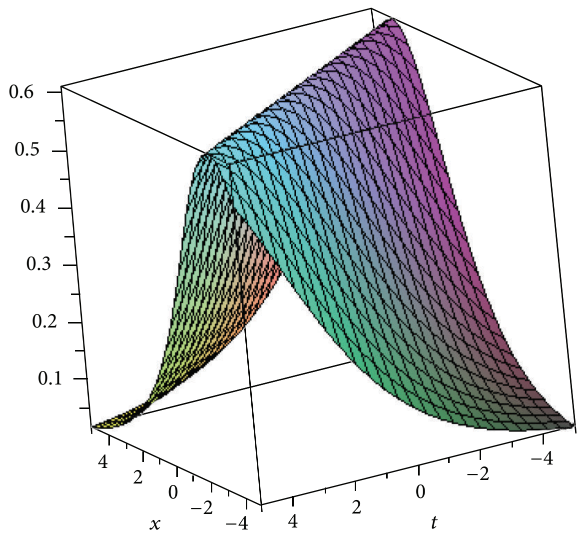

Figure 1: The plot of solution (24) with $\beta = R = 1$, $\mu = n = 2$, and $\gamma = -1$.

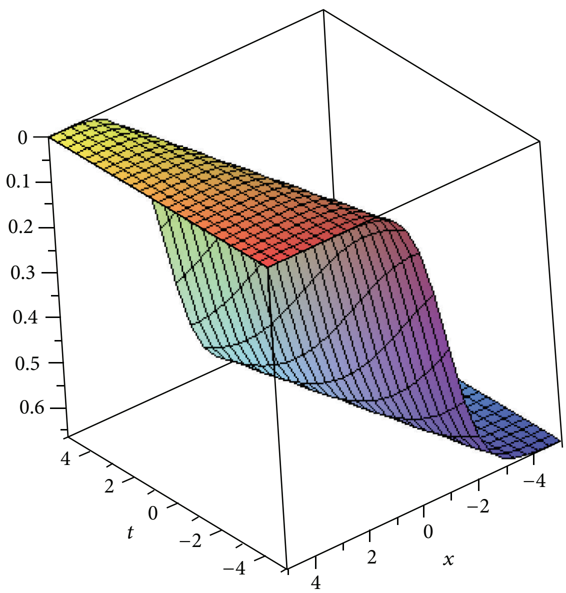

Figure 2: The plot of solution (27) with $\beta = R = n = 1$, and $\gamma = -1$.

In this case, we have the exact solutions

$$u_{51} = \left[\frac{A(2p+1)}{2B(p+2)} \times \left(-1 \pm \frac{\sqrt{\mu^2 - 1}\operatorname{sech}\left(\sqrt{R}\xi\right) + \tanh\left(\sqrt{R}\xi\right)}{\mu \operatorname{sech}\left(\sqrt{R}\xi\right) + 1}\right)\right]^{1/p}, \tag{62}$$

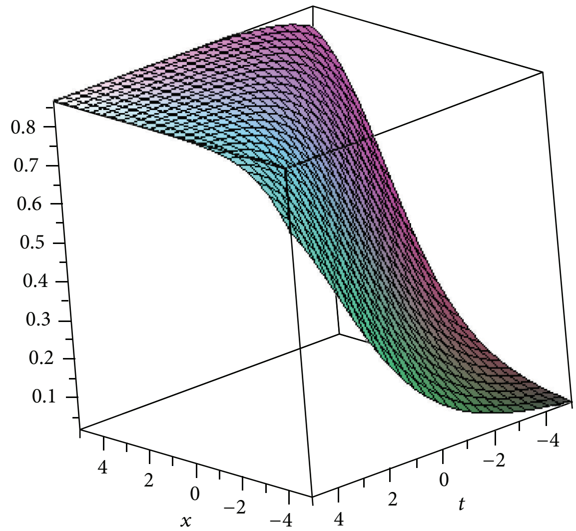

Figure 3: The plot of solution (30) with $\beta = R = 1$, $\mu = n = 2$, and $\gamma = -1$.

$$u_{52} = \left[\frac{A(2p+1)}{2B(p+2)} \times \left(-1 \pm \frac{\sqrt{\mu^2 - 1}\operatorname{csch}\left(\sqrt{R}\xi\right) + \coth\left(\sqrt{R}\xi\right)}{\mu \operatorname{csch}\left(\sqrt{R}\xi\right) + 1}\right)\right]^{1/p}, \tag{63}$$

where $\xi = x + ky + st$, $k = \pm\sqrt{-(A^2p^2(2p+1)/RBC(p+2)^2(p+1)) - 1}$, $s = (A^2(2p+1)/B(p+2)^2(p+1))$.

4. Physical Explanations of Some Obtained Solutions

In this section, we have presented some graphs of the obtained solutions constructed by taking suitable values of involved unknown parameters to visualize the underlying mechanism of the original equation. Using mathematical software Maple, three-dimensional plots of some obtained exact solutions have been shown in Figures 1, 2, 3, 4, 5, 6, 7, and 8.

4.1. The Nonlinear Pochhammer-Chree Equation (1). The obtained solutions for this equation are hyperbolic. From these explicit results it is easy to say that the solution (24) is a bell shaped soliton solution; (25) is a singular bell shaped soliton solution; (27) is a kink shaped soliton solution; (28) is a singular kink shaped soliton solution; (30) is a bell-kink shaped soliton solution; and (31) is a singular bell-kink shaped soliton solution.

4.2. The Nonlinear Burgers Equation (2). From the obtained solutions for the nonlinear Burgers equation (2) we observe

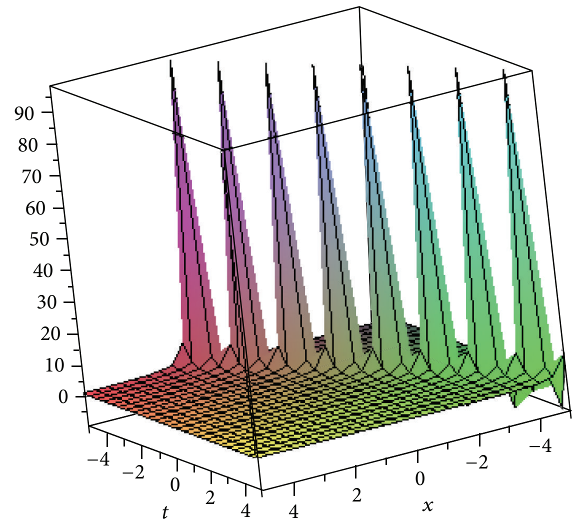

Figure 4: The plot of solution (31) with $\beta = R = 2$, $\mu = 3$, $n = 1$, and $\gamma = -1$.

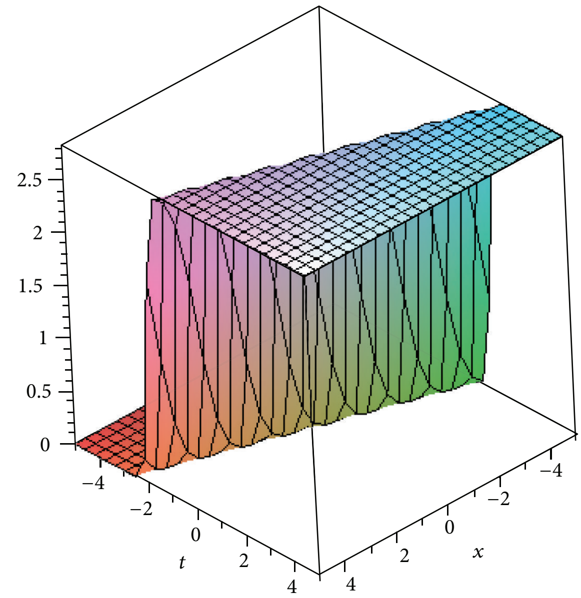

Figure 6: The plot of solution (42) with $a = b = 1$, $R = n = 2$.

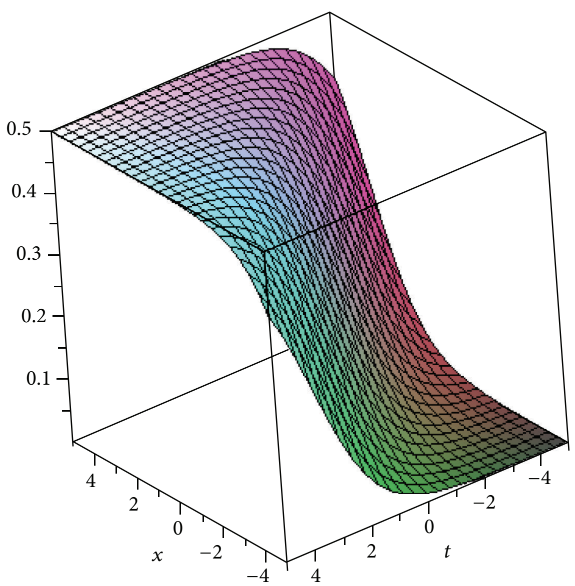

Figure 5: The plot of solution (39) with $a = n = \mu = 2$, $b = R = 1$.

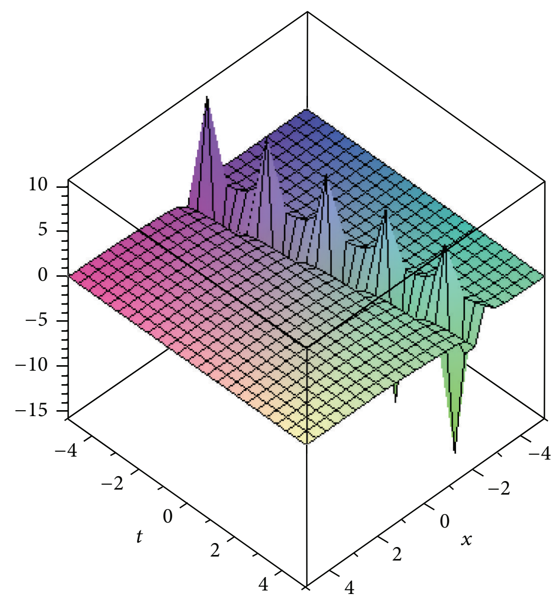

Figure 7: The plot of solution (51) with $p = 1$, $A = R = 2$, $B = \mu = 3$, and $k = 0$.

that the solution (39) is a bell-kink shaped soliton solution; (40) is a singular bell-kink shaped soliton solution; (42) is a kink shaped soliton solution; and (43) is a singular kink shaped soliton solution.

4.3. The Generalized Nonlinear Zakharov-Kuznetsov Equation (3). From the obtained solutions for the generalized nonlinear Zakharov-Kuznetsov equation (3) we can easily conclude that the solution (50) is a bell shaped soliton solution; (51) is a singular bell shaped soliton solution; (53), (56), and (62) are bell-kink shaped solitons solutions; (54), (57), and (63) are singular bell-kink shaped solitons solutions; (59) is a kink shaped soliton solution; and (60) is a singular kink shaped soliton solution.

5. Conclusions

The generalized projective Riccati equations method is used in this paper to obtain some new exact solutions of some nonlinear evolution equations with any-order nonlinear terms, namely, the nonlinear Pochhammer-Chree equation, the nonlinear Burgers equation, and the generalized Zakharov-Kuznetsov equation. On comparing our results in this paper with the well-known results obtained in [32, 33, 41–44] we deduce that our results are different and new and are not published elsewhere. The proposed method of this paper

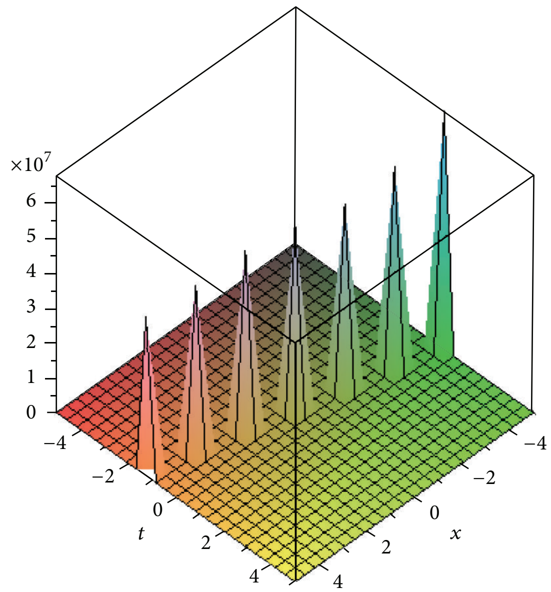

FIGURE 8: The plot of solution (60) with $p = 1$, $A = 6\sqrt{2}$, $R = B = 3$, and $k = 0$.

is effective and can be applied to many other nonlinear equations in mathematical physics.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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