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Research Article

Some Interesting Bifurcations of Nonlinear Waves for the Generalized Drinfel'd-Sokolov System

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We study the bifurcations of nonlinear waves for the generalized Drinfel'd-Sokolov system $u_t+(v^m)_x=0$, $v_t+a(v^n)_{xxx}+bu_xv+cuv_x=0$ called $D(m,n)$ system. We reveal some interesting bifurcation phenomena as follows. (1) For $D(2,1)$ system, the fractional solitary waves can be bifurcated from the trigonometric periodic waves and the elliptic periodic waves, and the kink waves can be bifurcated from the solitary waves and the singular waves. (2) For $D(1,2)$ system, the compactons can be bifurcated from the solitary waves, and the peakons can be bifurcated from the solitary waves and the singular cusp waves. (3) For $D(2,2)$ system, the solitary waves can be bifurcated from the smooth periodic waves and the singular periodic waves.

1. Introduction

The $D(m,n)$ system [1, 2] is read as

$$\begin{aligned} &u_t+\left(v^m\right)_x=0,\\ &v_t+a\left(v^n\right)_{xxx}+bu_xv+cuv_x=0, \end{aligned}\tag{1}$$

where a, b, c, m, and n are constants. Through some transformations, Xie and Yan [3] got some compacton and solitary pattern solutions of system (1) with $m=n-1$, including

$$\begin{aligned} &v_a(\xi)=\left[\frac{2n(\lambda-cd_0)}{(n+1)a\alpha_m}\cos^2\left(\frac{(n-1)\sqrt{\alpha_m}}{2n}\xi\right)\right]^{1/(n-1)},\\ &u_a(\xi)=\frac{v_a^{n-1}(\xi)}{\lambda}+d_0, \end{aligned}\tag{2}$$

$$\begin{aligned} &v_b(\xi)=\left[\frac{2n(cd_0-\lambda)}{(n+1)a\alpha_m}\sinh^2\left(\frac{(n-1)\sqrt{\alpha_m}}{2n}\xi\right)\right]^{1/(n-1)},\\ &u_b(\xi)=\frac{v_b^{n-1}(\xi)}{\lambda}+d_0, \end{aligned}\tag{3}$$

where $\lambda\neq 0$, d_0 are constants,

$$\xi=x-\lambda t,\qquad \alpha_m=\frac{bm+c}{a\lambda(m+1)}.\tag{4}$$

Deng et al. [4], by using the Weierstrass elliptic function method, presented many solutions of system (1) with $n=m+1$. It also includes the above solutions (2) and (3). Zhang et al. [5] showed some solutions of system (1) under the special parameters via employing the bifurcation method. By means of the complete discrimination system for polynomial method, many solutions of system (1) were acquired in [6].

In [2], the system

$$\begin{aligned} &u_t+\left(v^2\right)_x=0,\\ &v_t+av_{xxx}+bu_xv+cuv_x=0 \end{aligned}\tag{5}$$

was introduced. Wang [7] gave recursion, Hamiltonian, symplectic and cosymplectic operator, roots of symmetries, and scaling symmetry for system (5). Wazwaz [8], by using the tanh method and the sine-cosine method, obtained many solutions with compact and noncompact structures of system (5), including

$$v_c(\xi)=\lambda\sqrt{-\frac{6}{2b+c}}\operatorname{sech}\left(\sqrt{\frac{\lambda}{a}}\xi\right),\qquad u_c(\xi)=\frac{1}{\lambda}v_c^2(\xi),$$

$$\left(\frac{\lambda}{a}>0\right),$$

$$v_d(\xi) = \lambda\sqrt{-\frac{3}{2b+c}}\tanh\left(\sqrt{-\frac{\lambda}{2a}}\xi\right), \qquad u_d(\xi) = \frac{1}{\lambda}v_d^2(\xi), \quad \left(\frac{\lambda}{a}<0\right),$$

$$v_e(\xi) = \lambda\sqrt{-\frac{3}{2b+c}}\coth\left(\sqrt{-\frac{\lambda}{2a}}\xi\right), \qquad u_e(\xi) = \frac{1}{\lambda}v_e^2(\xi), \quad \left(\frac{\lambda}{a}<0\right). \tag{6}$$

Biazar and Ayati [9] obtained some solutions of system (5) through Exp-function method and modification of Exp-function method. Zhang et al. [10], by employing the complex method, gained all meromorphic exact solutions of system (5). Applying the auxiliary equation method, some exact solutions of system (5) were given in [11]. El-Wakil and Abdou [12] got some new exact solutions of system (5) by means of modified extended tanh-function method.

The other generalized Drinfel'd-Sokolov system [13]

$$\begin{aligned} &u_t + a_1uu_x + b_1u_{xxx} + c_0(v^\sigma)_x = 0,\\ &v_t + a_2uv_x + b_2v_{xxx} = 0 \end{aligned} \tag{7}$$

is considered in [14–22]. In [23–31], many exact solutions of system (7) were obtained. Clearly, system (1) and system (7) are two different systems.

In this paper, we are interested in system (1). We study the bifurcations of nonlinear waves for system (1).

Under the transformations

$$\xi = x - \lambda t, \qquad u(x,t) = \psi(\xi), \qquad v(x,t) = \varphi(\xi), \tag{8}$$

system (1) is reduced to

$$\begin{aligned} &\lambda\psi' - (\varphi^m)' = 0,\\ &-\lambda\varphi' + a(\varphi^n)''' + b\psi'\varphi + c\psi\varphi' = 0. \end{aligned} \tag{9}$$

Integrating the first equation of system (9) once, we have

$$\psi = \frac{1}{\lambda}\varphi^m + d, \tag{10}$$

where d is an integral constant. Substituting (10) into the second equation of system (9) and integrating it once yield the following equation:

$$\begin{aligned} &g + (cd-\lambda)\varphi + \frac{bm+c}{\lambda(m+1)}\varphi^{m+1}\\ &\quad + a\left[n(n-1)\varphi^{n-2}(\varphi')^2 + n\varphi^{n-1}\varphi''\right] = 0, \end{aligned} \tag{11}$$

where g is another integral constant. Letting $\varphi'(\xi) = y$, we obtain the planar system

$$\begin{aligned} &\frac{d\varphi}{d\xi} = y,\\ &\frac{dy}{d\xi} = -\frac{n(n-1)\varphi^{n-2}y^2 + \alpha_m\varphi^{m+1} + \beta\varphi + \gamma}{n\varphi^{n-1}}, \end{aligned} \tag{12}$$

where α_m is given in (4) and

$$\begin{aligned} &\beta = \frac{cd-\lambda}{a},\\ &\gamma = \frac{g}{a}. \end{aligned} \tag{13}$$

Letting $n\varphi^{n-1}d\tau = d\xi$, system (12) becomes

$$\begin{aligned} &\frac{d\varphi}{d\tau} = n\varphi^{n-1}y,\\ &\frac{dy}{d\tau} = -n(n-1)\varphi^{n-2}y^2 - \alpha_m\varphi^{m+1} - \beta\varphi - \gamma, \end{aligned} \tag{14}$$

which is called $P(m,n)$ system. Clearly, systems (12) and (14) possess the same first integral

$$\varphi^n\left(\frac{n}{2}\varphi^{n-2}y^2 + \frac{\alpha_m}{m+n+1}\varphi^{m+1} + \frac{\beta}{n+1}\varphi + \frac{\gamma}{n}\right) = h. \tag{15}$$

Employing (14) and (15), we reveal some interesting bifurcation phenomena listed in the above abstract.

In Section 2, we will consider $D(2,1)$ system. Firstly, we will show that the fractional solitary waves can be bifurcated from the trigonometric periodic waves and the elliptic periodic waves. Secondly, we will demonstrate that the kink waves can be bifurcated from the smooth solitary waves and the singular waves. In Section 3, we will consider $D(1,2)$ system. Firstly, we will confirm that the compactons can be bifurcated from the smooth solitary waves. Secondly, we will clarify that the peakons can be bifurcated from the solitary waves and the singular cusp waves. In Section 4, $D(2,2)$ system will be considered. We will verify that the solitary waves can be bifurcated from the smooth periodic waves and the periodic singular waves. A short conclusion will be given in Section 5.

2. The Bifurcations of Solitary Waves and Kink Waves for $D(2,1)$ System

When $m=2$ and $n=1$, system (1) becomes $D(2,1)$ system:

$$\begin{aligned} &u_t + \left(v^2\right)_x = 0,\\ &v_t + av_{xxx} + bu_xv + cuv_x = 0. \end{aligned} \tag{16}$$

We will reveal two kinds of interesting bifurcation phenomena to system (16). The first phenomenon is that fractional solitary waves can be bifurcated from two types of smooth periodic waves: trigonometric periodic waves and elliptic periodic waves. The second phenomenon is that the kink waves can be bifurcated from the smooth solitary waves and the singular waves. We state these results and give proof as follows.

Note that $P(2,1)$ system is read as

$$\begin{aligned} &\frac{d\varphi}{d\tau} = y,\\ &\frac{dy}{d\tau} = -\alpha_2\varphi^3 - \beta\varphi - \gamma, \end{aligned} \tag{17}$$

where

$$\alpha_2 = \frac{2b+c}{3a\lambda}, \tag{18}$$

and β, γ are given in (13). When $27\alpha_2\gamma^2 > -4\beta^3$, let

$$\begin{aligned}
\varphi_{01} &= \frac{1}{\alpha_2}\left(\frac{\Omega}{18}\right)^{1/3} - \beta\left(\frac{2}{3\Omega}\right)^{1/3},\\
\varphi_{02} &= \beta\left(1-\sqrt{3}i\right)\left(\frac{1}{12\Omega}\right)^{1/3} - \frac{1+\sqrt{3}i}{2\alpha_2}\left(\frac{\Omega}{18}\right)^{1/3},\\
\varphi_{03} &= \beta\left(1+\sqrt{3}i\right)\left(\frac{1}{12\Omega}\right)^{1/3} - \frac{1-\sqrt{3}i}{2\alpha_2}\left(\frac{\Omega}{18}\right)^{1/3},\\
\Omega &= \sqrt{12\alpha_2^3\beta^3 + 81\alpha_2^4\gamma^2} - 9\alpha_2^2\gamma,
\end{aligned} \tag{19}$$

$$\gamma_1(\beta) = \frac{2}{3\sqrt{3}}\left(-\frac{\beta^3}{\alpha_2}\right)^{1/2}. \tag{20}$$

Proposition 1. *For given $\alpha_2 > 0$ and $\beta < 0$, if $-\gamma_1(\beta) < \gamma < 0$, then φ_{01}, φ_{02}, and φ_{03} are real and system* (16) *has two types of special periodic wave solutions which become the fractional solitary wave solution*

$$v_k(\xi) = \sqrt{-\frac{3\beta}{\alpha_2}\frac{9+2\beta\xi^2}{9-6\beta\xi^2}}, \qquad u_k(\xi) = \frac{1}{\lambda}v_k^2(\xi) + d, \tag{21}$$

when $\gamma \to -\gamma_1(\beta)+0$. These two types of special periodic wave solutions are as follows.

(1) *Trigonometric periodic wave solution:*

$$\begin{aligned}
v_1(\xi) &= \varphi_{03} + \frac{2A_1}{\sqrt{\Delta_1}\cos(\eta_1\xi) - B_1},\\
u_1(\xi) &= \frac{1}{\lambda}v_1^2(\xi) + d,
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
A_1 &= (\varphi_1 - \varphi_{03})(\varphi_{03} - \varphi_2),\\
B_1 &= \varphi_1 + \varphi_2 - 2\varphi_{03},\\
\Delta_1 &= B_1^2 + 4A_1,\\
\eta_1 &= \sqrt{-\frac{A_1\alpha_2}{2}},\\
\varphi_1 &= -\varphi_{03} + \sqrt{-\frac{2\beta}{\alpha_2} - 2\varphi_{03}^2},\\
\varphi_2 &= -\varphi_{03} - \sqrt{-\frac{2\beta}{\alpha_2} - 2\varphi_{03}^2}.
\end{aligned} \tag{23}$$

(2) *Elliptic periodic wave solution:*

$$\begin{aligned}
v_2(\xi) &= \frac{\varphi_3(\varphi_4-\varphi_6) + \varphi_6(\varphi_3-\varphi_4)\,sn^2(\eta_2\xi,k_1)}{(\varphi_4-\varphi_6) + (\varphi_3-\varphi_4)\,sn^2(\eta_2\xi,k_1)},\\
u_2(\xi) &= \frac{1}{\lambda}v_2^2(\xi) + d,
\end{aligned} \tag{24}$$

where

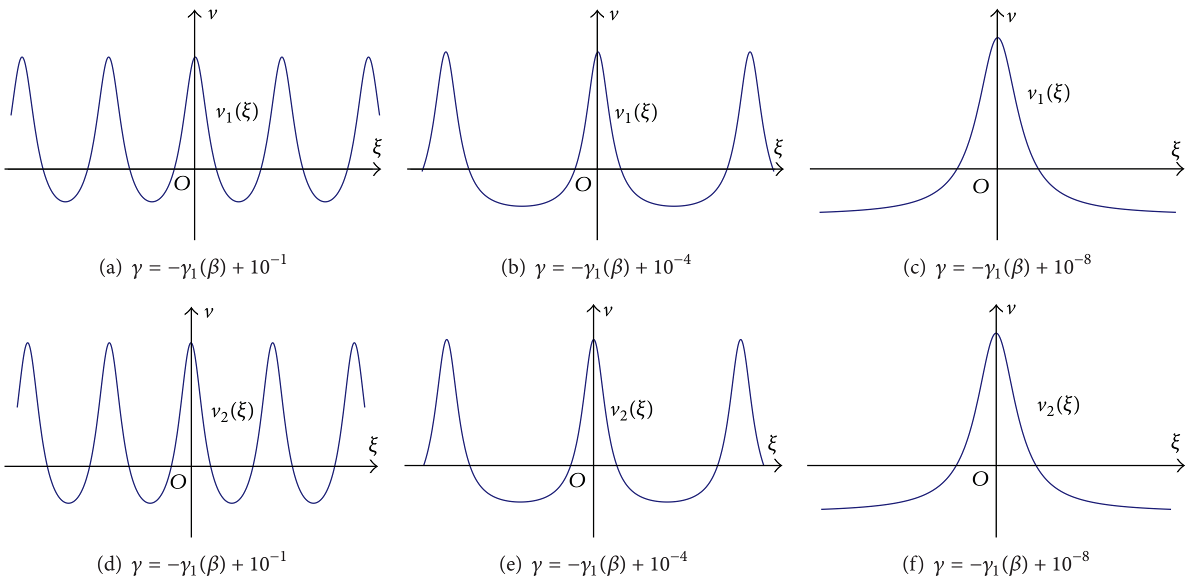
$$\begin{aligned}
\eta_2 &= \sqrt{\frac{\alpha_2}{8}(\varphi_3-\varphi_5)(\varphi_4-\varphi_6)},\\
k_1^2 &= \frac{(\varphi_3-\varphi_4)(\varphi_5-\varphi_6)}{(\varphi_3-\varphi_5)(\varphi_4-\varphi_6)},\\
\varphi_3 &= \frac{1}{2}\left(-\varphi_5-\varphi_6+\sqrt{-8\frac{\beta}{\alpha_2} - 3\varphi_5^2 - 2\varphi_5\varphi_6 - 3\varphi_6^2}\right),\\
\varphi_4 &= \frac{1}{2}\left(-\varphi_5-\varphi_6-\sqrt{-8\frac{\beta}{\alpha_2} - 3\varphi_5^2 - 2\varphi_5\varphi_6 - 3\varphi_6^2}\right),\\
\varphi_5 &= -\frac{1}{3}\varphi_6 + \left(2\beta + \frac{2}{3}\alpha_2\varphi_6^2\right)\left(\frac{1}{4\theta_2}\right)^{1/3}\left(1+i\sqrt{3}\right)\\
&\quad - \frac{1}{6\alpha_2}\left(\frac{\theta_2}{2}\right)^{1/3}\left(1-i\sqrt{3}\right),\\
&\varphi_{03}^* < \varphi_6 < \varphi_{03},\\
\varphi_{03}^* &= -\varphi_{02} - \sqrt{-\frac{2\beta}{\alpha_2} - 2\varphi_{02}^2},\\
\theta_1 &= -108\alpha_2^2\gamma - 36\alpha_2^2\beta\varphi_6 - 20\alpha_2^3\varphi_6^3,\\
\theta_2 &= \theta_1 + \sqrt{4\left(6\alpha_2\beta + 2\alpha_2^2\varphi_6^2\right)^3 + \theta_1^2}.
\end{aligned} \tag{25}$$

For the varying process, see Figure 1.

Proposition 2. *For given $\alpha_2 > 0$ and $\beta < 0$, if $0 < \gamma < \gamma_1(\beta)$, then φ_{01}, φ_{02}, and φ_{03} are real and system* (16) *has two types of special periodic wave solutions which become a fractional solitary wave solution:*

$$v_y(\xi) = \sqrt{-\frac{3\beta}{\alpha_2}\frac{2\beta\xi^2+9}{6\beta\xi^2-9}}, \qquad u_y(\xi) = \frac{1}{\lambda}v_y^2(\xi) + d, \tag{26}$$

when $\gamma \to \gamma_1(\beta) - 0$. These two types of special periodic wave solutions are as follows.



(a) $\gamma = -\gamma_1(\beta) + 10^{-1}$ (b) $\gamma = -\gamma_1(\beta) + 10^{-4}$ (c) $\gamma = -\gamma_1(\beta) + 10^{-8}$

(d) $\gamma = -\gamma_1(\beta) + 10^{-1}$ (e) $\gamma = -\gamma_1(\beta) + 10^{-4}$ (f) $\gamma = -\gamma_1(\beta) + 10^{-8}$

Figure 1: The solitary wave is bifurcated from two types of periodic waves. The varying process for the figures of $v_1(\xi)$ and $v_2(\xi)$ when $\alpha_2 > 0$, $\beta < 0$, and $\gamma \to -\gamma_1(\beta) + 0$ where $\alpha_2 = 4$ and $\beta = -9$.

(1) *Trigonometric periodic wave solution:*

$$v_3(\xi) = \varphi_{01} + \frac{2A_2}{B_2 - \sqrt{\Delta_2}\cos(\eta_3\xi)},$$
$$u_3(\xi) = \frac{1}{\lambda}v_3^2(\xi) + d, \tag{27}$$

where

$$A_2 = (\varphi_7 - \varphi_{01})(\varphi_{01} - \varphi_8),$$
$$B_2 = 2\varphi_{01} - \varphi_7 - \varphi_8,$$
$$\Delta_2 = B_2^2 + 4A_2,$$
$$\eta_3 = \sqrt{-\frac{A_2\alpha_2}{2}},$$
$$\varphi_7 = -\varphi_{01} + \sqrt{-\frac{2\beta}{\alpha_2} - 2\varphi_{01}^2},$$
$$\varphi_8 = -\varphi_{01} - \sqrt{-\frac{2\beta}{\alpha_2} - 2\varphi_{01}^2}. \tag{28}$$

(2) *Elliptic periodic wave solution:*

$$v_4(\xi) = \frac{\varphi_{12}(\varphi_9 - \varphi_{11}) + \varphi_9(\varphi_{11} - \varphi_{12})\,sn^2(\eta_4\xi, k_2)}{(\varphi_9 - \varphi_{11}) + (\varphi_{11} - \varphi_{12})\,sn^2(\eta_4\xi, k_2)},$$
$$u_4(\xi) = \frac{1}{\lambda}v_4^2(\xi) + d, \tag{29}$$

where

$$\eta_4 = \sqrt{\frac{\alpha_2}{8}(\varphi_9 - \varphi_{11})(\varphi_{10} - \varphi_{12})},$$
$$k_2^2 = \frac{(\varphi_9 - \varphi_{10})(\varphi_{11} - \varphi_{12})}{(\varphi_9 - \varphi_{11})(\varphi_{10} - \varphi_{12})},$$
$$\varphi_{01}^* = -\varphi_{02} + \sqrt{-\frac{2\beta}{\alpha_2} - 2\varphi_{02}^2},$$
$$\varphi_{01} < \varphi_9 < \varphi_{01}^*,$$
$$\theta_3 = -20v_9^3\alpha_2^3 - 36v_9\alpha_2^2\beta - 108\alpha_2^2\gamma,$$
$$\theta_4 = \theta_3 + \sqrt{4(2v_9^2\alpha_2^2 + 6\alpha_2\beta)^3 + \theta_3^2},$$
$$\varphi_{10} = -\frac{\varphi_9}{3} - \left(\frac{2}{3}\varphi_9^2\alpha_2 + 2\beta\right)\left(\frac{2}{\theta_4}\right)^{1/3} + \frac{1}{3\alpha_2}\left(\frac{\theta_4}{2}\right)^{1/3},$$
$$\varphi_{11} = \frac{1}{2}\left(-\varphi_9 - \varphi_{10} + \sqrt{-3\varphi_9^2 - 2\varphi_9\varphi_{10} - 3\varphi_{10}^2 - 8\frac{\beta}{\alpha_2}}\right),$$
$$\varphi_{12} = \frac{1}{2}\left(-\varphi_9 - \varphi_{10} - \sqrt{-3\varphi_9^2 - 2\varphi_9\varphi_{10} - 3\varphi_{10}^2 - 8\frac{\beta}{\alpha_2}}\right). \tag{30}$$

For the varying process, see Figure 2.

Proposition 3. *For given $\alpha_2 < 0$ and $\beta > 0$, if $-\gamma_1(\beta) < \gamma < \gamma_1(\beta)$, then system (16) has four nonlinear wave solutions which become two kink wave solutions:*

$$v_\delta^\pm(\xi) = \pm\sqrt{-\frac{\beta}{\alpha_2}}\tanh\left(\sqrt{\frac{\beta}{2}}\xi\right), \qquad u_\delta(\xi) = \frac{1}{\lambda}v_\delta^2(\xi) + d, \tag{31}$$

when $\gamma \to 0$. These four nonlinear wave solutions are as follows:

$$\begin{aligned} v_5(\xi) &= \frac{4A_3\left(\varphi_{01} - \delta e^{-\eta_5\xi}\right) - \varphi_{01}\left(\delta e^{-\eta_5\xi} - B_3\right)^2}{4A_3 - \left(\delta e^{-\eta_5\xi} - B_3\right)^2}, \\ u_5(\xi) &= \frac{1}{\lambda}v_5^2(\xi) + d, \\ v_6(\xi) &= \frac{4A_3\left(\varphi_{01} - \delta e^{\eta_5\xi}\right) - \varphi_{01}\left(\delta e^{\eta_5\xi} - B_3\right)^2}{4A_3 - \left(\delta e^{\eta_5\xi} - B_3\right)^2}, \\ u_6(\xi) &= \frac{1}{\lambda}v_6^2(\xi) + d, \\ v_7(\xi) &= \frac{4A_4\left(\varphi_{03} - \mu e^{\eta_6\xi}\right) - \varphi_{03}\left(\mu e^{\eta_6\xi} - B_4\right)^2}{4A_4 - \left(\mu e^{\eta_6\xi} - B_4\right)^2}, \\ u_7(\xi) &= \frac{1}{\lambda}v_7^2(\xi) + d, \\ v_8(\xi) &= \frac{4A_4\left(\varphi_{03} - \mu e^{-\eta_6\xi}\right) - \varphi_{03}\left(\mu e^{-\eta_6\xi} - B_4\right)^2}{4A_4 - \left(\mu e^{-\eta_6\xi} - B_4\right)^2}, \\ u_8(\xi) &= \frac{1}{\lambda}v_8^2(\xi) + d, \end{aligned} \tag{32}$$

where

$$\begin{aligned} \delta &= 8\sqrt{-\frac{\beta}{\alpha_2}}, \\ A_3 &= 6\varphi_{01}^2 + \frac{2\beta}{\alpha_2}, \\ \eta_5 &= \sqrt{-\frac{A_3\alpha_2}{2}}, \\ B_3 &= 4\varphi_{01}, \\ \mu &= -8\sqrt{-\frac{\beta}{\alpha_2}}, \\ A_4 &= 6\varphi_{03}^2 + \frac{2\beta}{\alpha_2}, \\ \eta_6 &= \sqrt{-\frac{A_4\alpha_2}{2}}, \\ B_4 &= 4\varphi_{03}. \end{aligned} \tag{33}$$

These four nonlinear wave solutions possess the following properties.

(a) *If $0 < \gamma < \gamma_1(\beta)$, then $v_5(\xi)$ and $v_6(\xi)$ represent two solitary waves which tend to two kink waves $v_\delta^\pm(\xi)$ (see Figure 3) when $\gamma \to 0+0$.*

(b) *If $-\gamma_1(\beta) < \gamma < 0$, then $v_5(\xi)$ and $v_6(\xi)$ represent two singular waves which tend to two kink waves $v_\delta^\pm(\xi)$ (see Figure 4) when $\gamma \to 0-0$.*

(c) *If $0 < \gamma < \gamma_1(\beta)$, then $v_7(\xi)$ and $v_8(\xi)$ represent two singular waves which tend to two kink waves $v_\delta^\pm(\xi)$ (see Figure 5) when $\gamma \to 0+0$.*

(d) *If $-\gamma_1(\beta) < \gamma < 0$, then $v_7(\xi)$ and $v_8(\xi)$ represent two solitary waves which tend to two kink waves $v_\delta^\pm(\xi)$ (see Figure 6) when $\gamma \to 0-0$.*

The Derivations of Propositions 1–3. According to the qualitative theory, we obtain the bifurcation phase portraits of system (17) as in Figure 7. Employing some orbits in Figure 7, we derive the results of Propositions 1–3 as follows.

(1) When $\alpha_2 > 0$, $\beta < 0$, and $-\gamma_1(\beta) < \gamma < 0$ (as in Figure 7(a)), the closed curves l_1 and l_2 possess the following expressions:

$$\begin{aligned} l_1: y^2 &= \frac{\alpha_2}{2}(\varphi - \varphi_{03})^2(\varphi_1 - \varphi)(\varphi - \varphi_2) \\ &\qquad (\varphi_{03} < \varphi_2 \le \varphi \le \varphi_1), \\ l_2: y^2 &= \frac{\alpha_2}{2}(\varphi_3 - \varphi)(\varphi - \varphi_4)(\varphi - \varphi_5)(\varphi - \varphi_6) \\ &\qquad (\varphi_6 < \varphi_5 < \varphi_4 \le \varphi \le \varphi_3). \end{aligned} \tag{34}$$

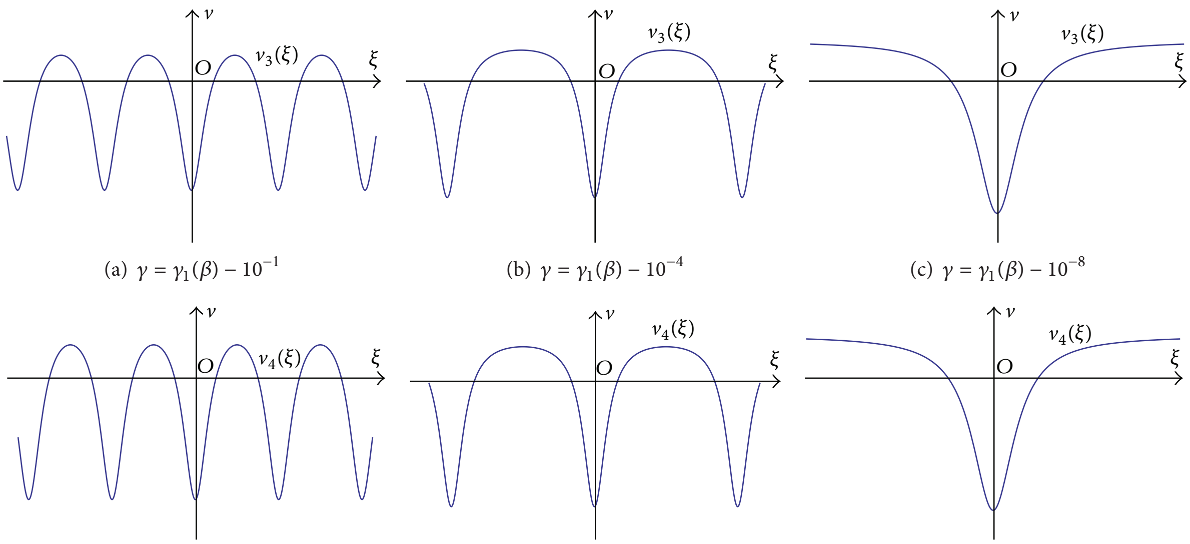
Substituting (34) into $d\varphi/d\xi = y$ and integrating them along l_1 and l_2, respectively, it follows that

$$\begin{aligned} &\int_\varphi^{\varphi_1} \frac{ds}{(s - \varphi_{03})\sqrt{(\varphi_1 - s)(s - \varphi_2)}} = \sqrt{\frac{\alpha}{2}}|\xi| \quad (\text{along } l_1), \\ &\int_\varphi^{\varphi_3} \frac{ds}{\sqrt{(\varphi_3 - s)(s - \varphi_4)(s - \varphi_5)(s - \varphi_6)}} = \sqrt{\frac{\alpha}{2}}|\xi| \\ &\qquad (\text{along } l_2). \end{aligned} \tag{35}$$

Completing the integrals above and solving the equations for φ, we obtain the solutions $v_1(\xi)$, $u_1(\xi)$ (see (22)) and $v_2(\xi)$, $u_2(\xi)$ (see (24)).

(2) When $\alpha_2 > 0$, $\beta < 0$, and $0 < \gamma < \gamma_1(\beta)$ (as in Figure 7(a)), the closed curves l_3 and l_4 possess the following expressions:

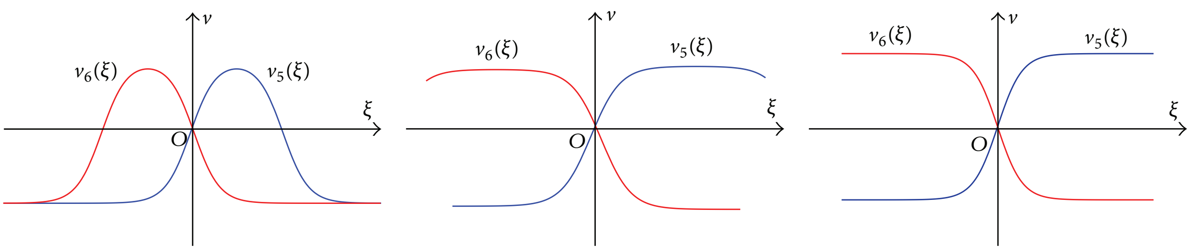
$$\begin{aligned} l_3: y^2 &= \frac{\alpha}{2}(\varphi_{01} - \varphi)^2(\varphi_7 - \varphi)(\varphi - \varphi_8) \\ &\qquad (\varphi_8 \le \varphi \le \varphi_7 < \varphi_{01}), \\ l_4: y^2 &= \frac{\alpha}{2}(\varphi_9 - \varphi)(\varphi_{10} - \varphi)(\varphi_{11} - \varphi)(\varphi - \varphi_{12}) \\ &\qquad (\varphi_{12} \le \varphi \le \varphi_{11} < \varphi_{10} < \varphi_9). \end{aligned} \tag{36}$$



(a) $\gamma = \gamma_1(\beta) - 10^{-1}$ (b) $\gamma = \gamma_1(\beta) - 10^{-4}$ (c) $\gamma = \gamma_1(\beta) - 10^{-8}$

(d) $\gamma = \gamma_1(\beta) - 10^{-1}$ (e) $\gamma = \gamma_1(\beta) - 10^{-4}$ (f) $\gamma = \gamma_1(\beta) - 10^{-8}$

FIGURE 2: The solitary wave is bifurcated from two types of periodic waves. The varying process for the figures of $v_3(\xi)$ and $v_4(\xi)$ when $\alpha_2 > 0$, $\beta < 0$, and $\gamma \rightarrow \gamma_1(\beta) - 0$ where $\alpha_2 = 4$ and $\beta = -9$.



(a) $\gamma = 1 \times 10^{-1}$ (b) $\gamma = 1 \times 10^{-4}$ (c) $\gamma = 1 \times 10^{-8}$

FIGURE 3: Two kink waves are bifurcated from two peak-solitary waves. The varying process for the figures of $v_5(\xi)$ and $v_6(\xi)$ when $\alpha_2 < 0$, $\beta > 0$, and $\gamma \rightarrow 0 + 0$ where $\alpha_2 = -2$ and $\beta = 9$.

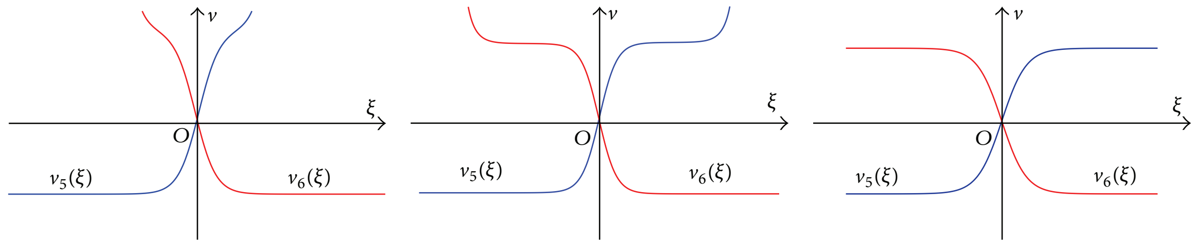
Substituting (36) into $d\varphi/d\xi = y$ and integrating them along l_3 and l_4, respectively, it follows that

$$\int_{\varphi_8}^{\varphi} \frac{ds}{(\varphi_{01} - s)\sqrt{(\varphi_7 - s)(s - \varphi_8)}} = \sqrt{\frac{\alpha}{2}}\,|\xi| \quad (\text{along } l_3),$$
$$\int_{\varphi_{12}}^{\varphi} \frac{ds}{\sqrt{(\varphi_9 - s)(\varphi_{10} - s)(\varphi_{11} - s)(s - \varphi_{12})}} = \sqrt{\frac{\alpha}{2}}\,|\xi| \quad (\text{along } l_4). \tag{37}$$

Completing the integrals above and solving the equations for φ, we obtain the solutions $v_3(\xi)$, $u_3(\xi)$ (see (27)) and $v_4(\xi)$, $u_4(\xi)$ (see (29)).

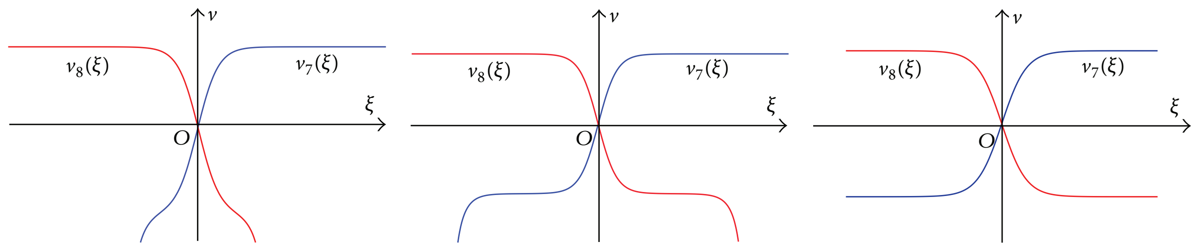
When $\gamma \rightarrow -\gamma_1(\beta) + 0$, it follows that φ_2, φ_{03}, and φ_{03}^* tend to $\varphi_2^* = -\sqrt{-\beta/3\alpha_2}$ and φ_1, φ_3 tend to $\varphi_1^* = \sqrt{-3\beta/\alpha_2}$. Further, we have

$$\varphi_4 \longrightarrow \varphi_2^*, \qquad \varphi_5 \longrightarrow \varphi_2^*, \qquad \varphi_6 \longrightarrow \varphi_2^*,$$
$$A_1 = (\varphi_1 - \varphi_{03})(\varphi_{03} - \varphi_2) \longrightarrow 0,$$
$$B_1 = \varphi_1 + \varphi_2 - 2\varphi_{03} \longrightarrow \varphi_1^* - \varphi_2^*,$$
$$\eta_1 = \sqrt{-\frac{A_1\alpha_2}{2}} \longrightarrow 0,$$
$$\eta_2 = \sqrt{\frac{\alpha_2}{8}(\varphi_3 - \varphi_5)(\varphi_4 - \varphi_6)} \longrightarrow 0,$$
$$\sqrt{\Delta_1} = \sqrt{B_1^2 + 4A_1} = B_1 + \frac{2A_1}{B_1} + \cdots,$$



(a) $\gamma = -1 \times 10^{-1}$ (b) $\gamma = -1 \times 10^{-4}$ (c) $\gamma = -1 \times 10^{-8}$

FIGURE 4: Two kink waves are bifurcated from two singular waves. The varying process for the figures of $v_5(\xi)$ and $v_6(\xi)$ when $\alpha_2 < 0$, $\beta > 0$, and $\gamma \to 0 - 0$ where $\alpha_2 = -2$ and $\beta = 9$.



(a) $\gamma = 1 \times 10^{-1}$ (b) $\gamma = 1 \times 10^{-4}$ (c) $\gamma = 1 \times 10^{-8}$

FIGURE 5: Two kink waves are bifurcated from two singular waves. The varying process for the figures of $v_7(\xi)$ and $v_8(\xi)$ when $\alpha_2 < 0$, $\beta > 0$, and $\gamma \to 0 + 0$ where $\alpha_2 = -2$ and $\beta = 9$.

$$\cos(\eta_1\xi) = 1 - \frac{(\eta_1\xi)^2}{2!} + \frac{(\eta_1\xi)^4}{4!} + \cdots,$$
$$\operatorname{sn}(\eta_2\xi, k_1) = \eta_2\xi - (1 + k_1^2)\frac{(\eta_2\xi)^3}{3!} + \cdots. \tag{38}$$

Thus, we have

$$\begin{aligned}
&\lim_{\gamma\to-\gamma_1+0} v_1(\xi) \\
&= \lim_{\gamma\to-\gamma_1+0} \varphi_{03} + \frac{2A_1}{\sqrt{\Delta_1}\cos(\eta_1\xi) - B_1} \\
&= \lim_{\gamma\to-\gamma_1+0} \varphi_{03} \\
&\quad + 2\left(A_1 + o\left(A_1^2\right)\right) \\
&\quad \times \left(\left(B_1 + \frac{2A_1}{B_1} + o\left(A_1^2\right)\right)\right. \\
&\qquad \left.\times \left(1 + \frac{\alpha_2 A_1 \xi^2}{4} + o\left(A_1^2\right)\right) - B_1\right)^{-1} \\
&= \lim_{\gamma\to-\gamma_1+0} \varphi_{03} + \frac{2A_1 + o\left(A_1^2\right)}{(2A_1/B_1) + (\alpha_2 A_1 B_1 \xi^2/4) + o\left(A_1^2\right)} \\
&= \varphi_2^* + \frac{8(\varphi_1^* - \varphi_2^*)}{\alpha_2(\varphi_1^* - \varphi_2^*)^2\xi^2 + 8} \\
&= \sqrt{-\frac{3\beta}{\alpha_2}}\frac{9 + 2\beta\xi^2}{9 - 6\beta\xi^2} \\
&= v_k(\xi) \quad (\text{see } (21)),
\end{aligned}$$

$$\begin{aligned}
&\lim_{\gamma\to-\gamma_1+0} v_2(\xi) \\
&= \lim_{\gamma\to-\gamma_1+0} \frac{\varphi_3(\varphi_4 - \varphi_6) + \varphi_6(\varphi_3 - \varphi_4)\operatorname{sn}^2(\eta_2\xi, k_1)}{(\varphi_4 - \varphi_6) + (\varphi_3 - \varphi_4)\operatorname{sn}^2(\eta_2\xi, k_1)} \\
&= \lim_{\gamma\to-\gamma_1+0} \left(\left(\varphi_3(\varphi_4 - \varphi_6) + \varphi_6(\varphi_3 - \varphi_4)\right.\right. \\
&\qquad \times \left[\sqrt{\frac{\alpha_2}{8}(\varphi_3 - \varphi_5)(\varphi_4 - \varphi_6)}\xi\right. \\
&\qquad \left.\left. + o(\varphi_4 - \varphi_6)\right]^2\right) \\
&\quad \times \left((\varphi_4 - \varphi_6) + (\varphi_3 - \varphi_4)\right. \\
&\qquad \times \left[\sqrt{\frac{\alpha_2}{8}(\varphi_3 - \varphi_5)(\varphi_4 - \varphi_6)}\xi\right. \\
&\qquad \left.\left.\left. + o(\varphi_4 - \varphi_6)\right]^2\right)^{-1}\right)
\end{aligned}$$

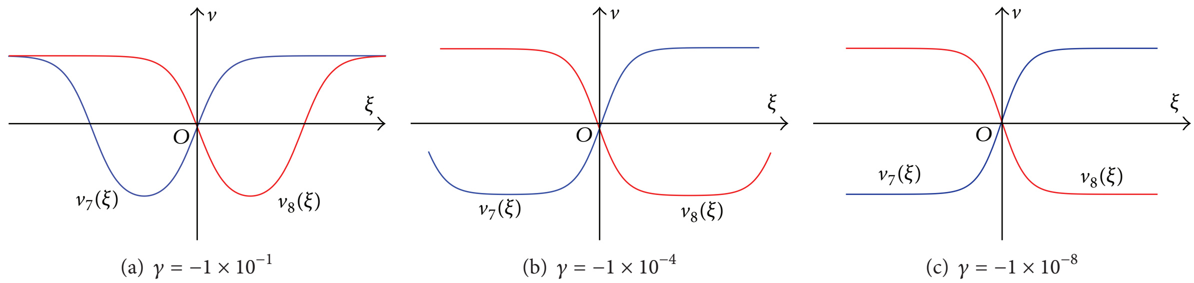


FIGURE 6: Two kink waves are bifurcated from two valley-solitary waves. The varying process for the figures of $v_7(\xi)$ and $v_8(\xi)$ when $\alpha_2 < 0$, $\beta > 0$, and $\gamma \to 0 - 0$ where $\alpha_2 = -2$ and $\beta = 9$.

$$\begin{aligned}
&= \lim_{\gamma \to -\gamma_1 + 0} \Big(\Big(8\varphi_3 (\varphi_4 - \varphi_6) + \varphi_6 (\varphi_3 - \varphi_4)(\varphi_3 - \varphi_5) \\
&\qquad \times (\varphi_4 - \varphi_6)\alpha_2 \xi^2 + o\big((\varphi_4 - \varphi_6)^{3/2}\big)\Big) \\
&\qquad \times \Big(8(\varphi_4 - \varphi_6) + (\varphi_3 - \varphi_4)(\varphi_3 - \varphi_5) \\
&\qquad \times (\varphi_4 - \varphi_6)\alpha_2 \xi^2 + o\big((\varphi_4 - \varphi_6)^{3/2}\big)\Big)^{-1}\Big) \\
&= \frac{8\varphi_1^* + \varphi_2^* (\varphi_1^* - \varphi_2^*)^2 \alpha_2 \xi^2}{8 + (\varphi_1^* - \varphi_2^*)^2 \alpha_2 \xi^2} \\
&= \sqrt{-\frac{3\beta}{\alpha_2}} \frac{9 + 2\beta\xi^2}{9 - 6\beta\xi^2} \\
&= v_k(\xi) \quad (\text{see } (21)).
\end{aligned} \tag{39}$$

Similarly, we can prove the limit property of $v_3(\xi)$ and $v_4(\xi)$ when $\gamma \to \gamma_1(\beta) - 0$.

(3) When $\alpha_2 < 0$, $\beta > 0$, and $-\gamma_1(\beta) < \gamma < \gamma_1(\beta)$ (as in Figure 7(b)), the curve connecting with $(\varphi_{01}, 0)$ embraces the expression

$$y^2 = -\frac{\alpha_2}{2}(\varphi - \varphi_{01})^2 \left(\varphi^2 + 2\varphi_{01}\varphi + 3\varphi_{01}^2 + \frac{2\beta}{\alpha_2}\right). \tag{40}$$

Substituting (40) into $d\varphi/d\xi = y$ and integrating it, we have

$$\int_p^{\varphi} \frac{ds}{\sqrt{(s - \varphi_{01})^2 (s^2 + 2\varphi_{01}s + 3\varphi_{01}^2 + (2\beta/\alpha_2))}} = \sqrt{-\frac{\alpha_2}{2}}\xi, \tag{41}$$

where

$$p = \varphi_{01} - \sqrt{-\frac{\beta}{\alpha_2}} \frac{24\alpha_2\varphi_{01}^2 + 8\beta}{\alpha_2\varphi_{01}^2 - 8\varphi_{01}\sqrt{-\alpha_2\beta} + 9\beta}. \tag{42}$$

Completing the integral above and solving the equation for φ, we obtain

$$v_5(\xi) = \frac{4A_3\left(\varphi_{01} - \delta e^{-\eta_5\xi}\right) - \varphi_{01}\left(\delta e^{-\eta_5\xi} - B_3\right)^2}{4A_3 - \left(\delta e^{-\eta_5\xi} - B_3\right)^2}. \tag{43}$$

From $v_5(\xi)$, we get

$$v_6(\xi) = v_5(-\xi) = \frac{4A_3\left(\varphi_{01} - \delta e^{\eta_5\xi}\right) - \varphi_{01}\left(\delta e^{\eta_5\xi} - B_3\right)^2}{4A_3 - \left(\delta e^{\eta_5\xi} - B_3\right)^2}. \tag{44}$$

(4) When $\alpha_2 < 0$, $\beta > 0$, and $-\gamma_1(\beta) < \gamma < \gamma_1(\beta)$ (as in Figure 7(b)), the curve connecting with $(\varphi_{03}, 0)$ embraces the expression

$$y^2 = -\frac{\alpha_2}{2}(\varphi_{03} - \varphi)^2 \left(\varphi^2 + 2\varphi_{03}\varphi + 3\varphi_{03}^2 + \frac{2\beta}{\alpha_2}\right). \tag{45}$$

Substituting (45) into $d\varphi/d\xi = y$ and integrating it, we have

$$\int_q^{\varphi} \frac{ds}{\sqrt{(\varphi_{03} - s)^2 (s^2 + 2\varphi_{03}s + 3\varphi_{03}^2 + (2\beta/\alpha_2))}} = \sqrt{-\frac{\alpha_2}{2}}\xi, \tag{46}$$

where

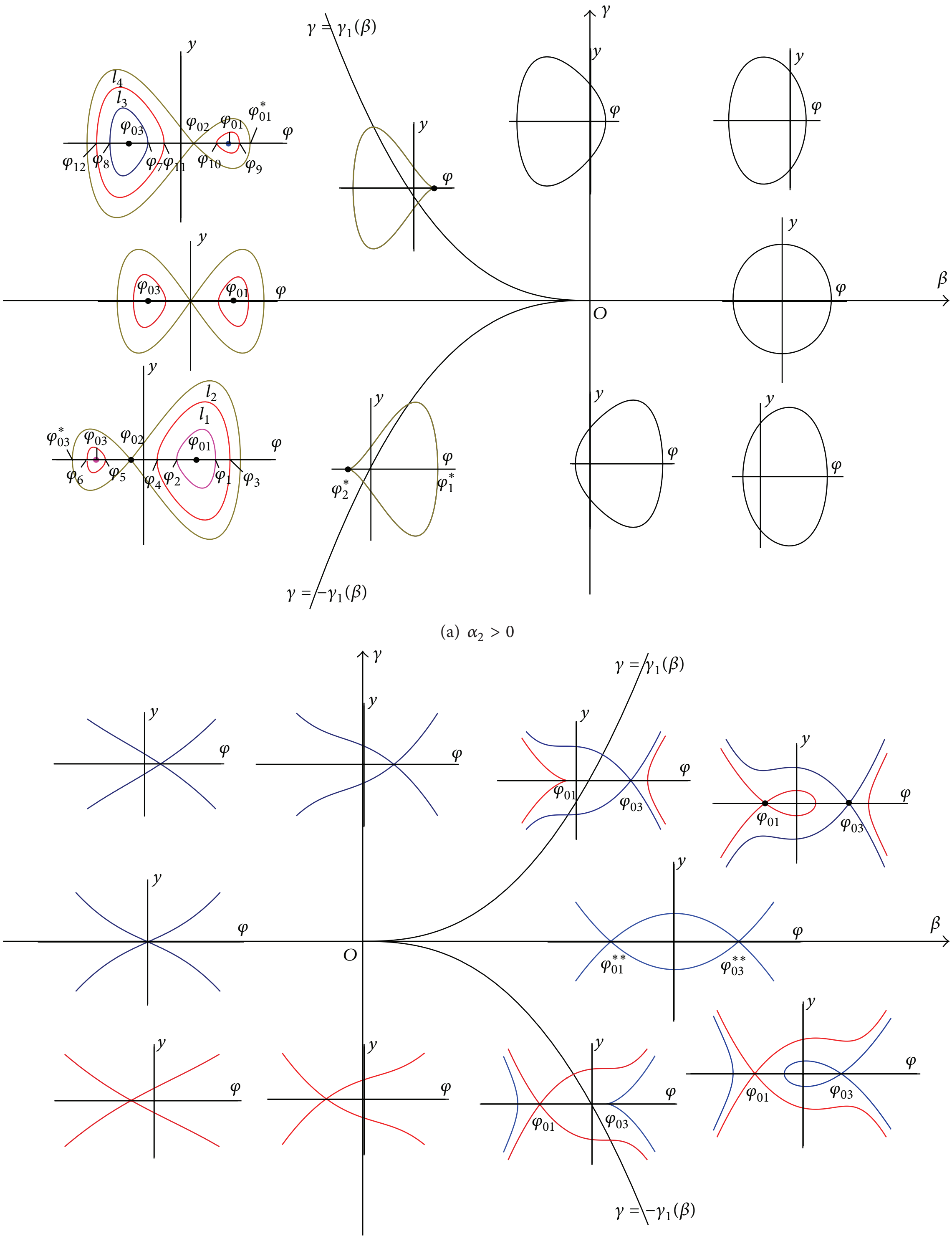
$$q = \varphi_{03} + \sqrt{-\frac{\beta}{\alpha_2}} \frac{24\alpha_2\varphi_{03}^2 + 8\beta}{\alpha_2\varphi_{03}^2 + 8\varphi_{03}\sqrt{-\alpha_2\beta} + 9\beta}. \tag{47}$$

Completing the integral above and solving the equation for φ, we obtain

$$v_7(\xi) = \frac{4A_4\left(\varphi_{03} - \mu e^{\eta_6\xi}\right) - \varphi_{03}\left(\mu e^{\eta_6\xi} - B_4\right)^2}{4A_4 - \left(\mu e^{\eta_6\xi} - B_4\right)^2}. \tag{48}$$

From $v_7(\xi)$, we get

$$\begin{aligned}
v_8(\xi) &= v_7(-\xi) \\
&= \frac{4A_4\left(\varphi_{03} - \mu e^{-\eta_6\xi}\right) - \varphi_{03}\left(\mu e^{-\eta_6\xi} - B_4\right)^2}{4A_4 - \left(\mu e^{-\eta_6\xi} - B_4\right)^2}.
\end{aligned} \tag{49}$$



(a) $\alpha_2 > 0$

(b) $\alpha_2 < 0$

Figure 7: The bifurcation phase portraits of system (17).

Note that when $\gamma \to 0$, it follows that $\varphi_{01} \to \varphi_{01}^{**} = -\sqrt{-\beta/\alpha_2}$, $A_3 \to -4\beta/\alpha_2$, $B_3 \to -4\sqrt{-\beta/\alpha_2}$, and $\eta_5 \to \sqrt{-2\beta}$. Thus, we have

$$\begin{aligned}\lim_{\gamma\to 0} v_5(\xi) &= \lim_{\gamma\to 0} \frac{4A_3\left(\varphi_{01} - \delta e^{-\eta_5\xi}\right) - \varphi_{01}\left(\delta e^{-\eta_5\xi} - B_3\right)^2}{4A_3 - \left(\delta e^{-\eta_5\xi} - B_3\right)^2} \\ &= \lim_{\gamma\to 0} \frac{\delta\varphi_{01}e^{-2\eta_5\xi} + \left(4A_3 - 2\varphi_{01}B_3\right)e^{-\eta_5\xi}}{\delta e^{-2\eta_5\xi} - 2B_3 e^{-\eta_5\xi}} \\ &= \frac{\delta\sqrt{-\alpha_2\beta}e^{-\sqrt{2\beta}\xi} - 8\beta}{\delta\alpha_2 e^{-\sqrt{2\beta}\xi} - 8\sqrt{-\alpha_2\beta}} \\ &= \sqrt{-\frac{\beta}{\alpha_2}}\,\frac{e^{\sqrt{\beta/2}\xi} - e^{-\sqrt{\beta/2}\xi}}{e^{\sqrt{\beta/2}\xi} + e^{-\sqrt{\beta/2}\xi}}\end{aligned}$$

$$= \sqrt{-\frac{\beta}{\alpha_2}} \tanh\left(\sqrt{\frac{\beta}{2}}\xi\right)$$
$$= v_\delta^+(\xi) \quad (\text{see } (31)). \tag{50}$$

Similarly, we can also get $v_6(\xi) \to v_\delta^-(\xi)$, $v_7(\xi) \to v_\delta^+(\xi)$, and $v_8(\xi) \to v_\delta^-(\xi)$ when $\gamma \to 0$. These complete the derivations of Propositions 1–3.

3. The Bifurcations of Compactons and Peakons for $D(1,2)$ System

When $m = 1$ and $n = 2$, system (1) becomes

$$\begin{gathered} u_t + v_x = 0, \\ v_t + a(v^2)_{xxx} + bu_x v + cuv_x = 0. \end{gathered} \tag{51}$$

We will reveal two kinds of interesting bifurcation phenomena to system (51). The first phenomenon is that the smooth solitary waves can turn into the compactons. The second phenomenon is that the peakons can be bifurcated from the singular cusp waves and the solitary waves. The concrete results are stated as follows.

Note that $P(1,2)$ system is read as

$$\begin{aligned} \frac{d\varphi}{d\tau} &= 2\varphi y, \\ \frac{dy}{d\tau} &= -2y^2 - \alpha_1\varphi^2 - \beta\varphi - \gamma, \end{aligned} \tag{52}$$

where

$$\alpha_1 = \frac{b+c}{2a\lambda}, \tag{53}$$

and β, γ are given in (13).

For fixed α_1, put

$$\begin{aligned} \gamma_2(\beta) &= \frac{2\beta^2}{9\alpha_1}, \\ \gamma_3(\beta) &= \frac{\beta^2}{4\alpha_1}. \end{aligned} \tag{54}$$

Proposition 4. *For given $\alpha_1 > 0$ and $\beta \neq 0$, if $0 < \gamma < \gamma_3(\beta)$, then system (51) has a family of solitary wave solutions which become the compacton solution*

$$\begin{gathered} v_o(\xi) = \begin{cases} -\dfrac{4\beta}{3\alpha_1}\cos^2\left(\dfrac{\sqrt{\alpha_1}}{4}\xi\right), & |\xi| < \dfrac{2\pi}{\sqrt{\alpha_1}}, \\ 0, & |\xi| \geq \dfrac{2\pi}{\sqrt{\alpha_1}}, \end{cases} \\ u_o(\xi) = \frac{1}{\lambda}v_o(\xi) + d, \end{gathered} \tag{55}$$

when $\gamma \to 0 + 0$. The solitary wave solutions are as follows.

(1) *When $\beta < 0$, the solitary wave solutions possess the expressions $v_9(\xi) = \varphi(\xi)$ and $u_9(\xi) = (1/\lambda)v_9(\xi) + d$, and $\varphi(\xi)$ is given by the implicit function*

$$\frac{\sqrt{\alpha_1}|\xi| - \pi}{2} = \arcsin\left(\frac{r_1 + r_3 - 2\varphi}{r_1 - r_3}\right) - \frac{r_2}{\sqrt{A_5}}\ln|\Phi_1|, \tag{56}$$

where

$$\begin{aligned} r_1 &= -\frac{1}{3\alpha_1}\left(3\alpha_1 r_2 + 2\beta - \sqrt{6\alpha_1\beta r_2 + 4\beta^2}\right), \\ r_2 &= -\frac{1}{2\alpha_1}\left(\beta + \sqrt{\beta^2 - 4\alpha_1\gamma}\right), \\ r_3 &= -\frac{1}{3\alpha_1}\left(3\alpha_1 r_2 + 2\beta + \sqrt{6\alpha_1\beta r_2 + 4\beta^2}\right), \\ A_5 &= (r_1 - r_2)(r_2 - r_3), \\ B_5 &= r_1 + r_3 - 2r_2, \\ \Phi_1 &= \frac{(r_1 - r_3)(\varphi - r_2)}{2\sqrt{A_5(r_1 - \varphi)(\varphi - r_3)} + B_5(\varphi - r_2) + 2A_5}. \end{aligned} \tag{57}$$

(2) *When $\beta > 0$, the solitary wave solutions possess the expressions $v_{10}(\xi) = \varphi(\xi)$ and $u_{10}(\xi) = (1/\lambda)v_{10}(\xi) + d$, and $\varphi(\xi)$ is given by the implicit function*

$$\frac{\pi - \sqrt{\alpha_1}|\xi|}{2} = \arcsin\left(\frac{r_6 + r_4 - 2\varphi}{r_6 - r_4}\right) + \frac{r_5}{\sqrt{A_6}}\ln|\Phi_2|, \tag{58}$$

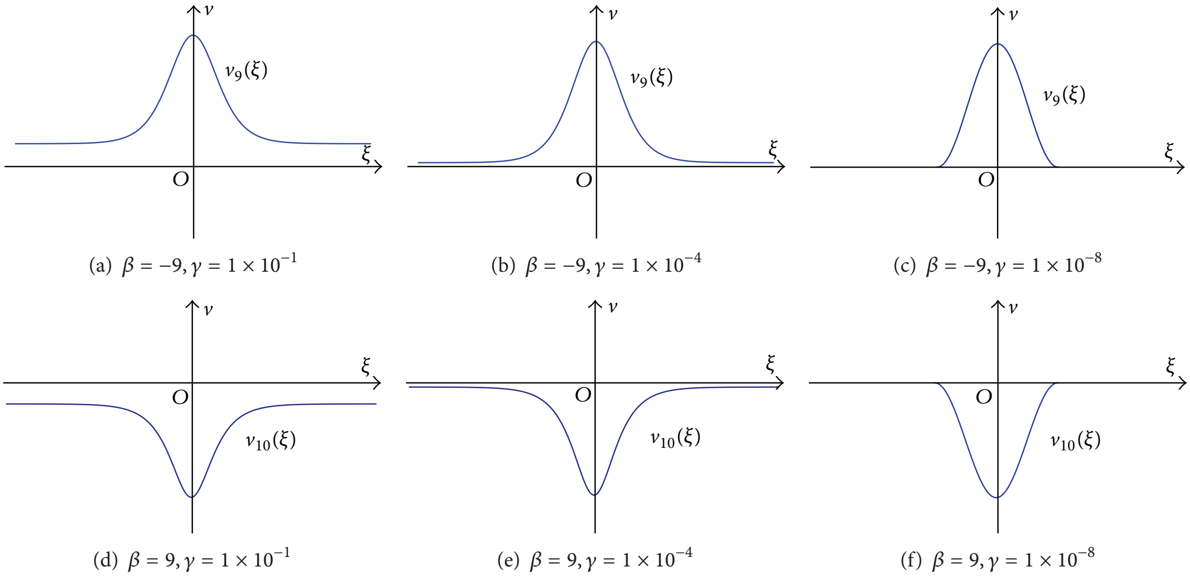
where

$$\begin{aligned} r_4 &= -\frac{1}{3\alpha_1}\left(3\alpha_1 r_5 + 2\beta + \sqrt{6\alpha_1\beta r_5 + 4\beta^2}\right), \\ r_5 &= -\frac{1}{2\alpha_1}\left(\beta - \sqrt{\beta^2 - 4\alpha_1\gamma}\right), \\ r_6 &= -\frac{1}{3\alpha_1}\left(3\alpha_1 r_5 + 2\beta - \sqrt{6\alpha_1\beta r_5 + 4\beta^2}\right), \\ A_6 &= (r_6 - r_5)(r_5 - r_4), \\ B_6 &= 2r_5 - r_6 - r_4, \\ \Phi_2 &= \frac{2\sqrt{A_6(r_6 - \varphi)(\varphi - r_4)} + B_6(r_5 - \varphi) + 2A_6}{(r_6 - r_4)(r_5 - \varphi)}. \end{aligned} \tag{59}$$

For the varying process, see Figure 8.

Proposition 5. *For given $\alpha_1 < 0$ and $\beta \neq 0$, if $\gamma_3(\beta) < \gamma < \gamma_2(\beta)$, then system (51) has two types of nonlinear wave solutions which tend to the peakon solution*

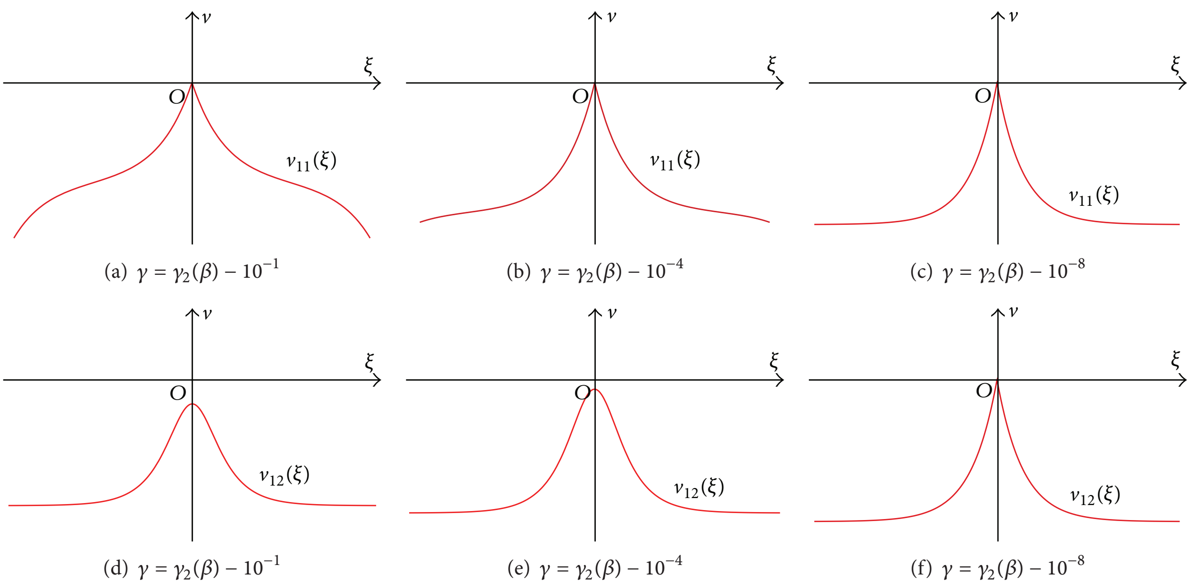
$$\begin{aligned} v_m(\xi) &= \frac{2\beta}{3\alpha_1}\left(e^{-\sqrt{|\alpha_1|/4}|\xi|} - 1\right), \\ u_m(\xi) &= \frac{1}{\lambda}v_m(\xi) + d, \end{aligned} \tag{60}$$



(a) $\beta = -9, \gamma = 1 \times 10^{-1}$ (b) $\beta = -9, \gamma = 1 \times 10^{-4}$ (c) $\beta = -9, \gamma = 1 \times 10^{-8}$

(d) $\beta = 9, \gamma = 1 \times 10^{-1}$ (e) $\beta = 9, \gamma = 1 \times 10^{-4}$ (f) $\beta = 9, \gamma = 1 \times 10^{-8}$

Figure 8: The solitary waves turn into the compactons. The varying process for the figures of $v_9(\xi)$ and $v_{10}(\xi)$ when $\alpha_1 > 0$, $\beta \neq 0$, and $\gamma \to 0 + 0$ where $\alpha_1 = 2$.



(a) $\gamma = \gamma_2(\beta) - 10^{-1}$ (b) $\gamma = \gamma_2(\beta) - 10^{-4}$ (c) $\gamma = \gamma_2(\beta) - 10^{-8}$

(d) $\gamma = \gamma_2(\beta) - 10^{-1}$ (e) $\gamma = \gamma_2(\beta) - 10^{-4}$ (f) $\gamma = \gamma_2(\beta) - 10^{-8}$

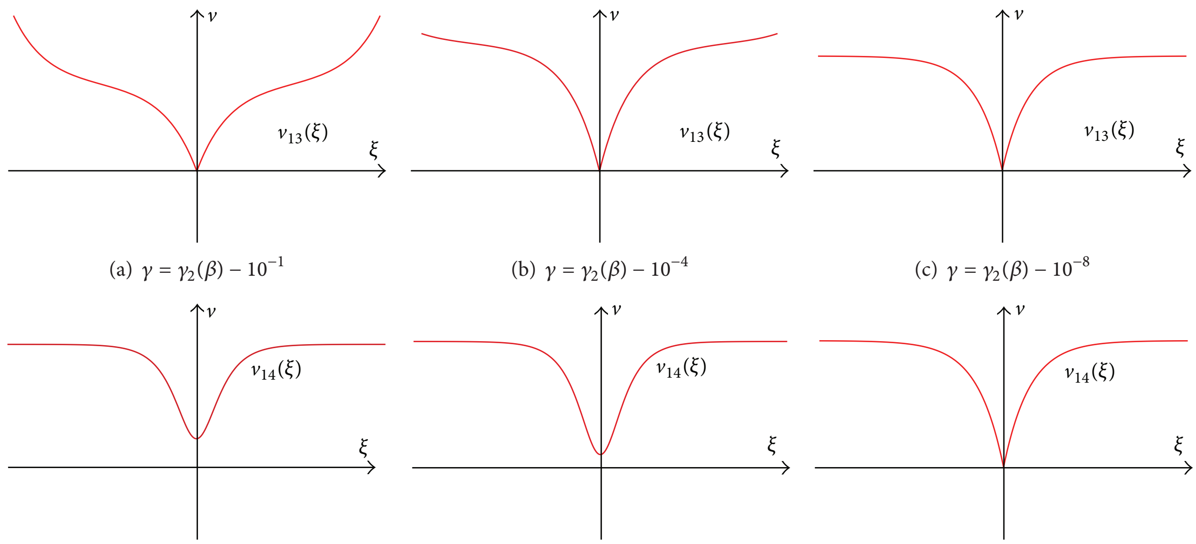
Figure 9: The peakon is bifurcated from the singular cusp wave and the solitary wave. The varying process for the figures of $v_{11}(\xi)$ and $v_{12}(\xi)$ when $\alpha_1 < 0$, $\beta < 0$, and $\gamma \to \gamma_2(\beta) - 0$ where $\alpha_1 = -2$ and $\beta = -9$.

when $\gamma \to \gamma_2(\beta) - 0$. For the varying process, see Figures 9 and 10. These two types of nonlinear wave solutions are singular cusp wave solutions and solitary wave solutions of the following expressions.

(1) *When $\beta < 0$, the singular cusp wave solutions possess the expression*

$$v_{11}(\xi) = \left[\left(\frac{\beta}{3\alpha_1} - \sqrt{\frac{\gamma}{2\alpha_1}}\right) - \left(\frac{\beta}{3\alpha_1} + \sqrt{\frac{\gamma}{2\alpha_1}}\right) e^{-\sqrt{|\alpha_1|/4}|\xi|}\right] \times \left(e^{\sqrt{|\alpha_1|/4}|\xi|} - 1\right),$$
$$u_{11}(\xi) = \frac{1}{\lambda} v_{11}(\xi) + d, \tag{61}$$

and the solitary wave solutions possess the expressions $v_{12}(\xi) = \varphi(\xi)$ and $u_{12}(\xi) = (1/\lambda)v_{12}(\xi) + d$, and $\varphi(\xi)$ is given by the implicit function



(a) $\gamma = \gamma_2(\beta) - 10^{-1}$ (b) $\gamma = \gamma_2(\beta) - 10^{-4}$ (c) $\gamma = \gamma_2(\beta) - 10^{-8}$

(d) $\gamma = \gamma_2(\beta) - 10^{-1}$ (e) $\gamma = \gamma_2(\beta) - 10^{-4}$ (f) $\gamma = \gamma_2(\beta) - 10^{-8}$

FIGURE 10: The peakon is bifurcated from the singular cusp wave and the solitary wave. The varying process for the figures of $v_{13}(\xi)$ and $v_{14}(\xi)$ when $\alpha_1 < 0$, $\beta > 0$, and $\gamma \to \gamma_2(\beta) - 0$ where $\alpha_1 = -2$ and $\beta = 9$.

$$|\xi| = -\sqrt{\frac{4}{|\alpha_1|}}\left(\ln\left|\frac{2\sqrt{\Phi_3} - M_1}{r_7 - r_8}\right| + \frac{r_9}{\sqrt{A_7}}\ln\left|\frac{2\sqrt{A_7\Phi_3} + B_7(r_9 - \varphi) + 2A_7}{(r_7 - r_8)(r_9 - \varphi)}\right|\right), \tag{62}$$

where

$$\begin{aligned}
\Phi_3 &= (r_7 - \varphi)(r_8 - \varphi),\\
M_1 &= 2\varphi - r_7 - r_8,\\
r_7 &= -\frac{1}{3\alpha_1}\left(3\alpha_1 r_9 + 2\beta + \sqrt{6\alpha_1\beta r_9 + 4\beta^2}\right),\\
r_8 &= -\frac{1}{3\alpha_1}\left(3\alpha_1 r_9 + 2\beta - \sqrt{6\alpha_1\beta r_9 + 4\beta^2}\right),\\
r_9 &= -\frac{1}{2\alpha_1}\left(\beta - \sqrt{\beta^2 - 4\alpha_1\gamma}\right),\\
A_7 &= (r_8 - r_9)(r_7 - r_9),\\
B_7 &= r_7 + r_8 - 2r_9.
\end{aligned} \tag{63}$$

(2) *When $\beta > 0$, the singular cusp wave solutions possess the expression*

$$\begin{aligned}
v_{13}(\xi) &= \left[\left(\frac{\beta}{3\alpha_1} - \sqrt{\frac{\gamma}{2\alpha_1}}\right) - \left(\frac{\beta}{3\alpha_1} + \sqrt{\frac{\gamma}{2\alpha_1}}\right)e^{\sqrt{|\alpha_1|/4}|\xi|}\right]\\
&\quad\times\left(e^{-\sqrt{|\alpha_1|/4}|\xi|} - 1\right),\\
u_{13}(\xi) &= \frac{1}{\lambda}v_{13}(\xi) + d,
\end{aligned} \tag{64}$$

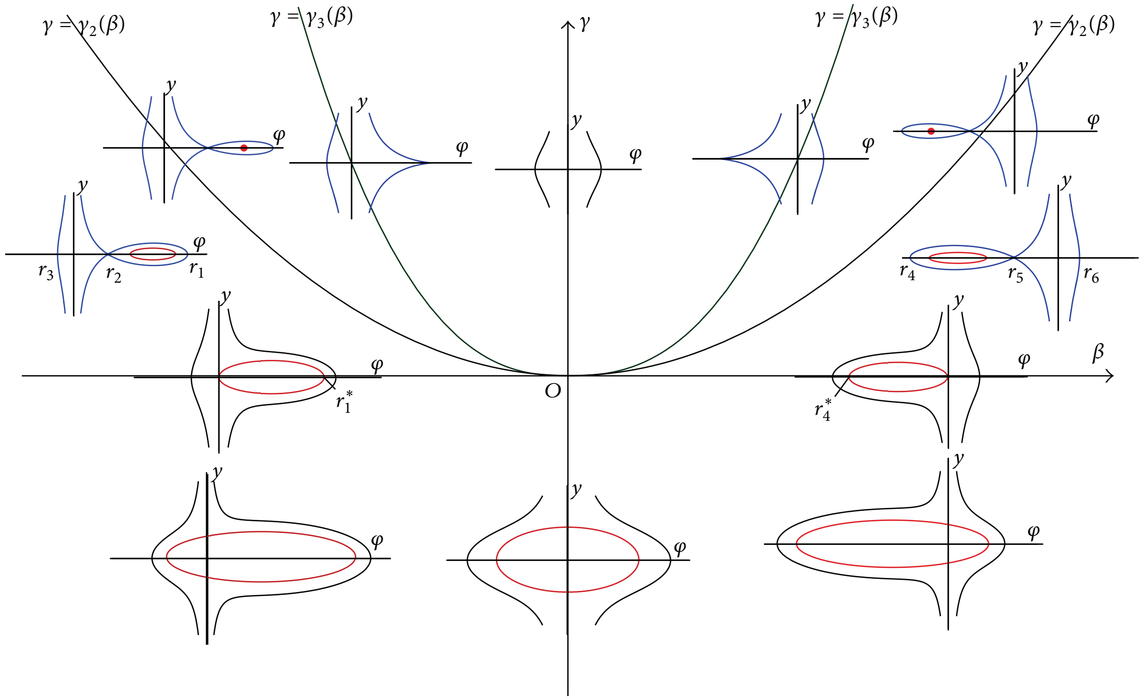
and the solitary wave solutions possess the expressions $v_{14}(\xi) = \varphi(\xi)$ and $u_{14}(\xi) = (1/\lambda)v_{14}(\xi) + d$, and $\varphi(\xi)$ is given by the implicit function

$$|\xi| = -\sqrt{\frac{4}{|\alpha_1|}}\left(\ln\left|\frac{2\sqrt{\Phi_4} + M_2}{r_{11} - r_{12}}\right| - \frac{r_{10}}{\sqrt{A_8}}\ln\left|\frac{2\sqrt{A_8\Phi_4} + B_8(\varphi - r_{10}) + 2A_8}{(r_{11} - r_{12})(\varphi - r_{10})}\right|\right), \tag{65}$$

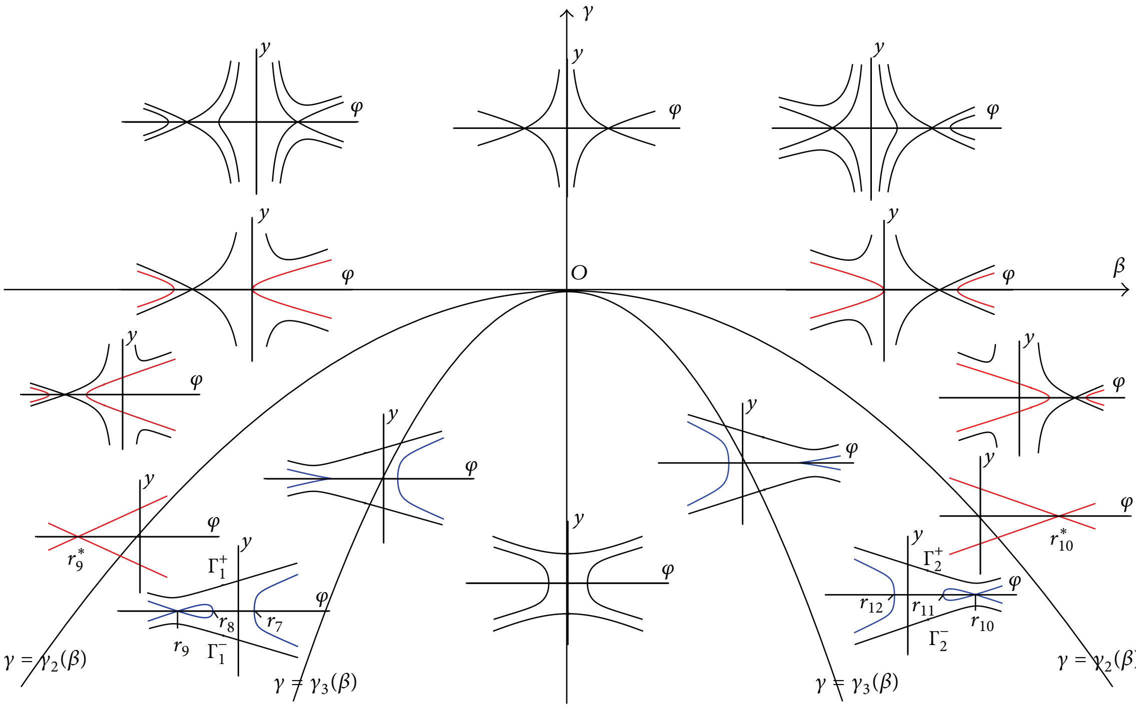
where

$$\begin{aligned}
\Phi_4 &= (\varphi - r_{11})(\varphi - r_{12}),\\
M_2 &= 2\varphi - r_{11} - r_{12},\\
r_{10} &= -\frac{1}{2\alpha_1}\left(\beta + \sqrt{\beta^2 - 4\alpha_1\gamma}\right),\\
r_{11} &= -\frac{1}{3\alpha_1}\left(3\alpha_1 r_{10} + 2\beta + \sqrt{6\alpha_1\beta r_{10} + 4\beta^2}\right),\\
r_{12} &= -\frac{1}{3\alpha_1}\left(3\alpha_1 r_{10} + 2\beta - \sqrt{6\alpha_1\beta r_{10} + 4\beta^2}\right),\\
A_8 &= (r_{10} - r_{11})(r_{10} - r_{12}),\\
B_8 &= 2r_{10} - r_{11} - r_{12}.
\end{aligned} \tag{66}$$

The Derivations of Propositions 4 and 5. According to the qualitative theory, we obtain the bifurcation phase portraits of system (52) as in Figure 11. Through some orbits in Figure 11, we derive the results of Propositions 4 and 5 as follows.



(a) $\alpha_1 > 0$



(b) $\alpha_1 < 0$

FIGURE 11: The bifurcation phase portraits of system (52).

(1) When $\alpha_1 > 0$, $\beta < 0$, and $0 < \gamma < \gamma_3(\beta)$ (as in Figure 11(a)), the homoclinic orbit owns the expression

$$y^2 = \frac{\alpha_1}{4\varphi^2}(\varphi - r_2)^2 (r_1 - \varphi)(\varphi - r_3) \quad (r_3 < r_2 < \varphi \le r_1). \tag{67}$$

Substituting (67) into $d\varphi/d\xi = y$ and integrating it, we have

$$|\xi| = \int_{\varphi}^{r_1} \sqrt{\frac{4s^2}{\alpha_1 (s - r_2)^2 (r_1 - s)(s - r_3)}} ds. \tag{68}$$

Completing the integral above, we get

$$|\xi| = \sqrt{\frac{4}{\alpha_1}}\left[\arcsin\left(\frac{r_1+r_3-2\varphi}{r_1-r_3}\right)+\frac{\pi}{2}-\frac{r_2}{\sqrt{A_5}}\ln|\Phi_1|\right]. \tag{69}$$

(2) When $\alpha_1 > 0$, $\beta > 0$, and $0 < \gamma < \gamma_3(\beta)$ (as in Figure 11(a)), the homoclinic orbit is expressed by

$$y^2 = \frac{\alpha_1}{4\varphi^2}(r_5-\varphi)^2(\varphi-r_4)(r_6-\varphi) \quad (r_4 \le \varphi < r_5 < r_6). \tag{70}$$

Substituting (70) into $d\varphi/d\xi = y$ and integrating it, we have

$$|\xi| = \int_{r_4}^{\varphi}\sqrt{\frac{4s^2}{\alpha_1(r_5-s)^2(s-r_4)(r_6-s)}}. \tag{71}$$

Completing the integral above, we get

$$|\xi| = -\sqrt{\frac{4}{\alpha_1}}\left[\arcsin\left(\frac{r_6+r_4-2\varphi}{r_6-r_4}\right)-\frac{\pi}{2}+\frac{r_5}{\sqrt{A_6}}\ln|\Phi_2|\right]. \tag{72}$$

Note that when $\beta < 0$ and $\gamma \to 0+0$, it follows that $r_1 \to r_1^* = -4\beta/3\alpha_1$, $r_2 \to 0$, and $r_3 \to 0$. Further, we have

$$B_5 = r_1 + r_3 - 2r_2 \longrightarrow r_1^*,$$

$$A_5 = (r_1-r_2)(r_2-r_3) \longrightarrow 0,$$

$$\begin{aligned}
&\lim_{\gamma\to0+0}\frac{r_3}{r_2}\\
&\quad= \lim_{\gamma\to0+0} -\frac{3\alpha_1 r_2 + 2\beta + \sqrt{6\alpha_1\beta r_2 + 4\beta^2}}{3\alpha_1 r_2}\\
&\quad= \lim_{\gamma\to0+0} -\frac{1}{3\alpha_1 r_2}\\
&\qquad\times\left[3\alpha_1 r_2 + 2\beta - 2\beta\left(1+\frac{3\alpha_1 r_2}{4\beta}+o\left(r_2^2\right)\right)\right]\\
&\quad= \lim_{\gamma\to0+0} -\frac{1}{3\alpha_1 r_2}\left(3\alpha_1 r_2 - \frac{3\alpha_1 r_2}{2}+o\left(r_2^2\right)\right) = -\frac{1}{2},\\
&\lim_{\gamma\to0+0}\frac{r_2}{\sqrt{A_5}} = \lim_{\gamma\to0+0}\frac{r_2}{\sqrt{(r_1-r_2)(r_2-r_3)}}\\
&\quad= \lim_{\gamma\to0+0}\sqrt{\frac{r_2}{r_1+r_3-r_2-r_1(r_3/r_2)}}\\
&\quad= \sqrt{\frac{0}{(3/2)r_1^*}} = 0,\\
&\lim_{\gamma\to0+0}\Phi_1\\
&\quad= \lim_{\gamma\to0+0}\frac{(r_1-r_3)(\varphi-r_2)}{2\sqrt{A_5(r_1-\varphi)(\varphi-r_3)}+B_5(\varphi-r_2)+2A_5}\\
&\quad= \frac{r_1^*\varphi}{r_1^*\varphi} = 1.
\end{aligned} \tag{73}$$

Thus, from

$$\begin{aligned}
&|\xi|\\
&= \lim_{\gamma\to0+0}\sqrt{\frac{4}{\alpha_1}}\left[\arcsin\left(\frac{r_1+r_3-2\varphi}{r_1-r_3}\right)+\frac{\pi}{2}-\frac{r_2}{\sqrt{A_5}}\ln|\Phi_1|\right],
\end{aligned} \tag{74}$$

we have

$$|\xi| = \sqrt{\frac{4}{\alpha_1}}\left[\arcsin\left(\frac{r_1^*-2\varphi}{r_1^*}\right)+\frac{\pi}{2}\right]. \tag{75}$$

Solving (75) for φ, we can get $\varphi = v_o(\xi)$ (see (55)), which is the same as (2).

Similarly, from (72), we can also get solution $\varphi \to v_o(\xi)$ when $\beta > 0$ and $\gamma \to 0+0$.

(3) When $\alpha_1 < 0$, $\beta \neq 0$, and $\gamma_3(\beta) < \gamma < \gamma_2(\beta)$ (as in Figure 11(b)), the curves $\Gamma_1^\pm$ and $\Gamma_2^\pm$ own the expression

$$y^2 = -\frac{\alpha_1}{4}\left(\varphi^2+\frac{4\beta}{3\alpha_1}\varphi+\frac{2\gamma}{\alpha_1}\right). \tag{76}$$

Substituting (76) into $d\varphi/d\xi = y$ and integrating it, we have

$$\begin{aligned}
&\int_{\varphi}^{0}\frac{ds}{\sqrt{s^2+(4\beta/3\alpha_1)s+(2\gamma/\alpha_1)}} = \sqrt{-\frac{\alpha_1}{4}}|\xi| \quad (\beta<0),\\
&\int_{0}^{\varphi}\frac{ds}{\sqrt{s^2+(4\beta/3\alpha_1)s+(2\gamma/\alpha_1)}} = \sqrt{-\frac{\alpha_1}{4}}|\xi| \quad (\beta>0).
\end{aligned} \tag{77}$$

Completing the integrals above and solving the equations for φ, we get $\varphi = v_{11}(\xi)$ (see (61)) and $\varphi = v_{13}(\xi)$ (see (64)).

(4) When $\alpha_1 < 0$, $\beta \neq 0$, and $\gamma_3(\beta) < \gamma < \gamma_2(\beta)$ (as in Figure 11(b)), two homoclinic orbits can be expressed as

$$y^2 = -\frac{\alpha_1}{4\varphi^2}(\varphi-r_9)^2(r_7-\varphi)(r_8-\varphi) \quad (\beta<0), \tag{78}$$

$$y^2 = -\frac{\alpha_1}{4\varphi^2}(r_{10}-\varphi)^2(\varphi-r_{11})(\varphi-r_{12}) \quad (\beta>0). \tag{79}$$

Substituting (78) and (79) into $d\varphi/d\xi = y$ and integrating it, we have

$$|\xi| = \int_{\varphi}^{r_8} \sqrt{\frac{4s^2}{|\alpha_1|(s-r_9)^2(r_7-s)(r_8-s)}}ds,$$
$$|\xi| = \int_{r_{11}}^{\varphi} \sqrt{\frac{4s^2}{|\alpha_1|(r_{10}-s)^2(s-r_{11})(s-r_{12})}}ds. \quad (80)$$

Completing the integrals above, we get (62) and (65).

Note that when $\beta < 0$ and $\gamma \to \gamma_2(\beta) - 0$, it follows that $\sqrt{\gamma/2\alpha_1} \to \beta/3\alpha_1$, $r_7 \to 0$, $r_8 \to 0$, $r_9 \to r_9^* = -2\beta/3\alpha_1$, and

$$\begin{aligned} M_1 &= 2\varphi - r_7 - r_8 \longrightarrow 2\varphi, \\ B_7 &= r_8 + r_7 - 2r_9 \longrightarrow -2r_9^*, \\ \sqrt{A_7} &= \sqrt{(r_8-r_9)(r_7-r_9)} \longrightarrow -r_9^*, \\ \sqrt{\Phi_3} &= \sqrt{(r_7-\varphi)(r_8-\varphi)} \longrightarrow -\varphi. \end{aligned} \quad (81)$$

Thus, we have

$$\begin{aligned} &\lim_{\gamma\to\gamma_2-0} v_{13}(\xi) \\ &= \lim_{\gamma\to\gamma_2-0} \left[\left(\frac{\beta}{3\alpha_1} - \sqrt{\frac{\gamma}{2\alpha_1}}\right) - \left(\frac{\beta}{3\alpha_1} + \sqrt{\frac{\gamma}{2\alpha_1}}\right)e^{-\sqrt{|\alpha_1|/4}|\xi|}\right] \\ &\quad \times \left(e^{\sqrt{|\alpha_1|/4}|\xi|} - 1\right) \\ &= \frac{2\beta}{3\alpha_1}\left(e^{-\sqrt{|\alpha_1|/4}|\xi|} - 1\right) \\ &= v_m(\xi) \quad (\text{see (60)}), \end{aligned}$$
$$\begin{aligned} |\xi| &= \lim_{\gamma\to\gamma_2-0} -\sqrt{\frac{4}{|\alpha_1|}} \\ &\quad \times \left(\ln\left|\frac{2\sqrt{\Phi_3} - M_1}{r_7 - r_8}\right| + \frac{r_9}{\sqrt{A_7}}\ln\left|\frac{2\sqrt{A_7\Phi_3} + B_7(r_9-\varphi) + 2A_7}{(r_7-r_8)(r_9-\varphi)}\right|\right) \\ &= \lim_{\gamma\to\gamma_2-0} -\sqrt{\frac{4}{|\alpha_1|}} \\ &\quad \times \ln\left|\left(\frac{2\sqrt{\Phi_3} - M_1}{r_7 - r_8}\right) \times \left(\frac{2\sqrt{A_7\Phi_3} + B_7(r_9-\varphi) + 2A_7}{(r_7-r_8)(r_9-\varphi)}\right)^{r_9/\sqrt{A_7}}\right| \\ &= -\sqrt{\frac{4}{|\alpha_1|}}\ln\left|\frac{-4\varphi(r_9^*-\varphi)}{4r_9^*\varphi}\right| \\ &= -\sqrt{\frac{4}{|\alpha_1|}}\ln\left|\frac{r_9^*-\varphi}{r_9^*}\right|. \end{aligned} \quad (82)\ (83)$$

Solving (78) for φ, we obtain $\varphi = v_m(\xi)$ (see (60)).

Similarly, from (65), we can also get $\varphi = v_m(\xi)$ (see (60)) when $\beta > 0$ and $\gamma \to \gamma_2(\beta) - 0$. These complete the derivations of Propositions 4 and 5.

4. The Bifurcations of Solitary Waves for $D(2,2)$ System

When $m = 2$ and $n = 2$, system (1) becomes

$$\begin{aligned} u_t + \left(v^2\right)_x &= 0, \\ v_t + a\left(v^2\right)_{xxx} + bu_x v + cuv_x &= 0. \end{aligned} \quad (84)$$

We will reveal the interesting bifurcation phenomenon to system (84). That is, the solitary waves can be bifurcated from the smooth periodic waves and the singular periodic waves. The concrete results are stated as follows.

Note that $P(2,2)$ system is read as

$$\begin{aligned} \frac{d\varphi}{d\tau} &= 2\varphi y, \\ \frac{dy}{d\tau} &= -2y^2 - \alpha_2\varphi^3 - \beta\varphi - \gamma, \end{aligned} \quad (85)$$

where α_2 is as (18) and β, γ are given in (13). Let

$$\begin{aligned} r_a &= \frac{1}{3}\left[\frac{1}{\alpha_2}\left(\frac{5Q}{4}\right)^{1/3} - \beta\left(\frac{100}{Q}\right)^{1/3}\right], \\ r_b &= \frac{\beta}{3}\left(\frac{25}{2Q}\right)^{1/3}\left(1 - i\sqrt{3}\right) - \frac{1}{6\alpha_2}\left(\frac{5Q}{4}\right)^{1/3}\left(1 + i\sqrt{3}\right), \\ r_c &= \frac{\beta}{3}\left(\frac{25}{2Q}\right)^{1/3}\left(1 + i\sqrt{3}\right) - \frac{1}{6\alpha_2}\left(\frac{5Q}{4}\right)^{1/3}\left(1 - i\sqrt{3}\right), \\ Q &= -27\alpha_2^2\gamma + \sqrt{80\alpha_2^3\beta^3 + 729\alpha_2^4\gamma^2}, \\ \gamma_4(\beta) &= \frac{4\sqrt{5}}{27}\left(-\frac{\beta^3}{\alpha_2}\right)^{1/2}, \end{aligned} \quad (86)$$

and let $\gamma_1(\beta)$ be as (20).

Proposition 6. *For fixed α_2 and β, system (84) has solitary wave solutions*

$$v_i^{\pm}(\xi) = \pm\sqrt{-\frac{5\beta}{\alpha_2}\left[\frac{2}{3} - \tanh^2\left(\sqrt[4]{-\frac{\alpha_2\beta}{80}}\xi\right)\right]},$$
$$u_i(\xi) = \frac{1}{\lambda}v_i^2(\xi) + d, \tag{87}$$

which can be bifurcated from the following two types of nonlinear wave solutions.

(1) *Smooth periodic wave solution:*

$$v_{15}(\xi) = r_a - (r_a - r_b)\,sn^2(\eta_7\xi, k_3),$$
$$u_{15}(\xi) = \frac{1}{\lambda}v_{15}^2(\xi) + d, \tag{88}$$

where

$$-\gamma_4(\beta) < \gamma < \gamma_4(\beta),$$
$$\eta_7 = \sqrt{\frac{\alpha_2}{20}(r_a - r_c)},$$
$$k_3 = \sqrt{\frac{r_a - r_b}{r_a - r_c}}. \tag{89}$$

(2) *Singular periodic wave solutions:*

$$v_{16}(\xi) = \frac{(A_9 + r_a)\,cn(\eta_8\xi, k_4) - A_9 + r_a}{cn(\eta_8\xi, k_4) + 1},$$
$$u_{16}(\xi) = \frac{1}{\lambda}v_{16}^2(\xi) + d, \quad (\alpha_2 > 0, \beta < 0), \tag{90}$$

$$v_{17}(\xi) = \frac{A_9 + r_a - (A_9 - r_a)\,cn(\eta_8\xi, k_5)}{1 + cn(\eta_8\xi, k_5)},$$
$$u_{17}(\xi) = \frac{1}{\lambda}v_{17}^2(\xi) + d, \quad (\alpha_2 < 0, \beta > 0), \tag{91}$$

where

$$-\gamma_1(\beta) < \gamma < -\gamma_4(\beta),$$
$$A_9^2 = (b_0 - r_a)^2 + a_0^2,$$
$$a_0^2 = -\frac{1}{4}(r_b - r_c)^2,$$
$$b_0 = \frac{1}{2}(r_b + r_c),$$
$$\eta_8 = \sqrt{\frac{1}{5}A_9|\alpha_2|},$$
$$k_4^2 = \frac{A_9 - b_0 + r_a}{2A_9},$$
$$k_5^2 = \frac{A_9 + b_0 - r_a}{2A_9}. \tag{92}$$

For the varying process, see Figures 12 and 13.

The Derivations of Proposition 6. According to the qualitative theory, we also obtain the bifurcation phase portraits of system (85) as in Figure 14. Employing some orbits in Figure 14, we derive the results of Proposition 6 as follows.

(1) When $\alpha_2 > 0$, $\beta < 0$, and $-\gamma_4(\beta) < \gamma < 0$ (as in Figure 14(a)), the closed orbit owns the expression

$$y^2 = \frac{\alpha_2}{5}(r_a - \varphi)(\varphi - r_b)(\varphi - r_c) \quad (r_c < r_b \le \varphi \le r_a). \tag{93}$$

Substituting (93) into $d\varphi/d\xi = y$ and integrating it, we have

$$\int_{\varphi}^{r_a} \frac{ds}{\sqrt{(r_a - s)(s - r_b)(s - r_c)}} = \sqrt{\frac{\alpha_2}{5}}|\xi|. \tag{94}$$

Completing the integral above and solving the equation for φ, we get $v_{15}(\xi)$ (see (88)).

When $\alpha_2 > 0$ and $-\gamma_1(\beta) < \gamma < -\gamma_4(\beta)$, r_a is real and r_b and r_c become a pair of conjugate complex in (94). Completing the integral (94) and solving the equation for φ, we get $v_{16}(\xi)$ (see (90)).

(2) When $\alpha_2 < 0$, $\beta > 0$, similarly, we get $v_{15}(\xi)$ (see (88)) and $v_{17}(\xi)$ (see (91)).

Note that when $\alpha_2 > 0$ and $\gamma \to -\gamma_4(\beta) + 0$, it follows that $r_a \to r_a^* = (2/3)\sqrt{-5\beta/\alpha_2}$, $r_b \to r_b^* = -(1/3)\sqrt{-5\beta/\alpha_2}$, and $r_c \to r_b^* = -(1/3)\sqrt{-5\beta/\alpha_2}$. Further,

$$k_3 = \sqrt{\frac{r_a - r_b}{r_a - r_c}} \longrightarrow 1,$$
$$\eta_7 = \sqrt{\frac{\alpha_2}{20}(r_a - r_c)} \longrightarrow \sqrt[4]{-\frac{\alpha_2\beta}{80}}. \tag{95}$$

Thus, we have

$$\begin{aligned}\lim_{\gamma\to-\gamma_4+0} v_{15}(\xi) &= \lim_{\gamma\to-\gamma_4+0} r_a - (r_a - r_b)\,\mathrm{sn}^2(\eta_7\xi, k_3) \\ &= r_a^* - (r_a^* - r_b^*)\tanh^2\left(\sqrt[4]{-\frac{\alpha_2\beta}{80}}\xi\right) \\ &= \sqrt{-\frac{5\beta}{\alpha_2}}\left[\frac{2}{3} - \tanh^2\left(\sqrt[4]{-\frac{\alpha_2\beta}{80}}\xi\right)\right] \\ &= v_i^+(\xi) \quad (\text{see } (87)).\end{aligned} \tag{96}$$

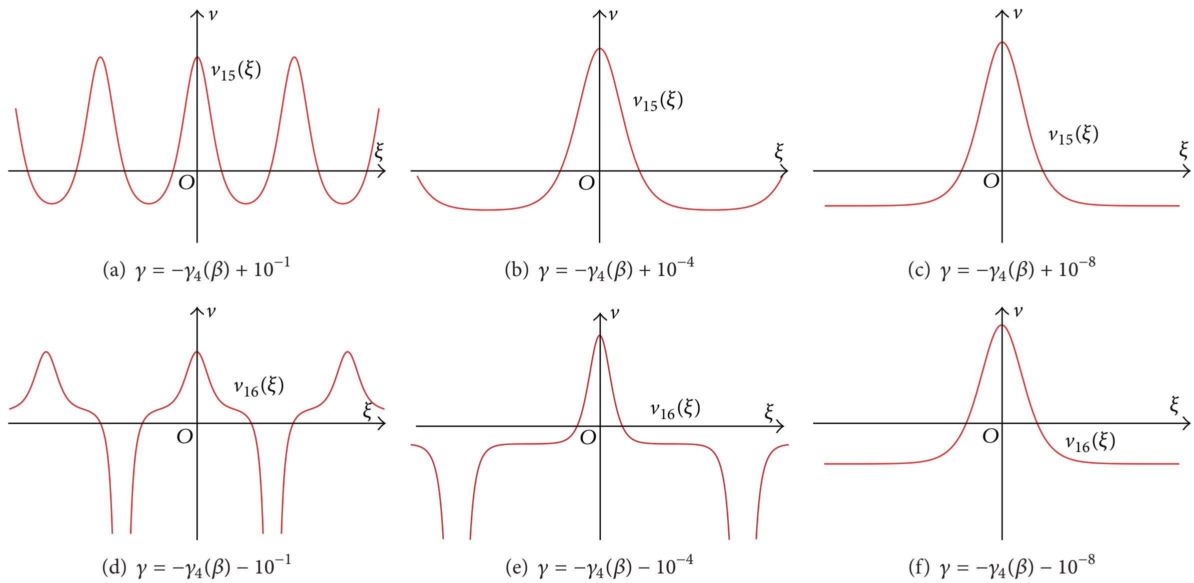


FIGURE 12: The solitary wave is bifurcated from the smooth periodic wave and the singular periodic wave. The varying process for the figures of $v_{15}(\xi)$ and $v_{16}(\xi)$ when $\alpha_2 = 5$, $\beta = -9$, $\gamma \to \gamma_4(\beta) + 0$, or $\gamma \to \gamma_4(\beta) - 0$.

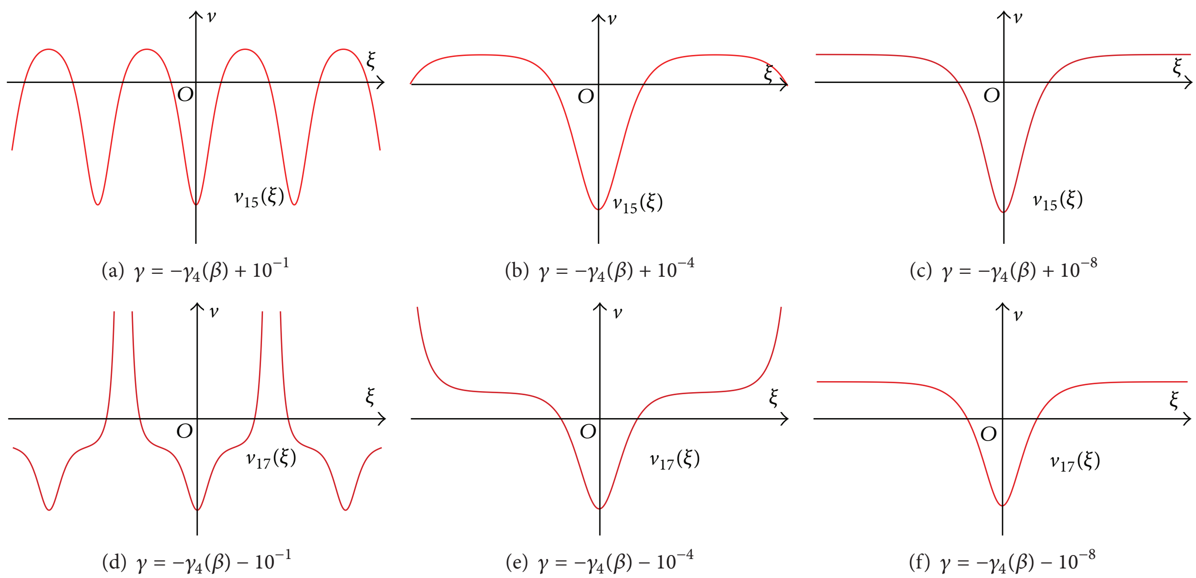


FIGURE 13: The solitary wave is bifurcated from the smooth periodic wave and the singular periodic wave. The varying process for the figures of $v_{15}(\xi)$ and $v_{17}(\xi)$ when $\alpha_2 = -5$, $\beta = 9$, $\gamma \to \gamma_4(\beta) + 0$, or $\gamma \to \gamma_4(\beta) - 0$.

If $\alpha_2 > 0$ and $\gamma \to -\gamma_4(\beta) - 0$, then it follows that $r_a \to r_a^* = (2/3)\sqrt{-5\beta/\alpha_2}$, $r_b \to r_b^* = -(1/3)\sqrt{-5\beta/\alpha_2}$, and $r_c \to r_b^* = -(1/3)\sqrt{-5\beta/\alpha_2}$. Further, it follows that

$$a_0^2 = -\frac{1}{4}(r_b - r_c)^2 \longrightarrow 0,$$

$$b_0 = \frac{1}{2}(r_b + r_c) \longrightarrow r_b^*,$$

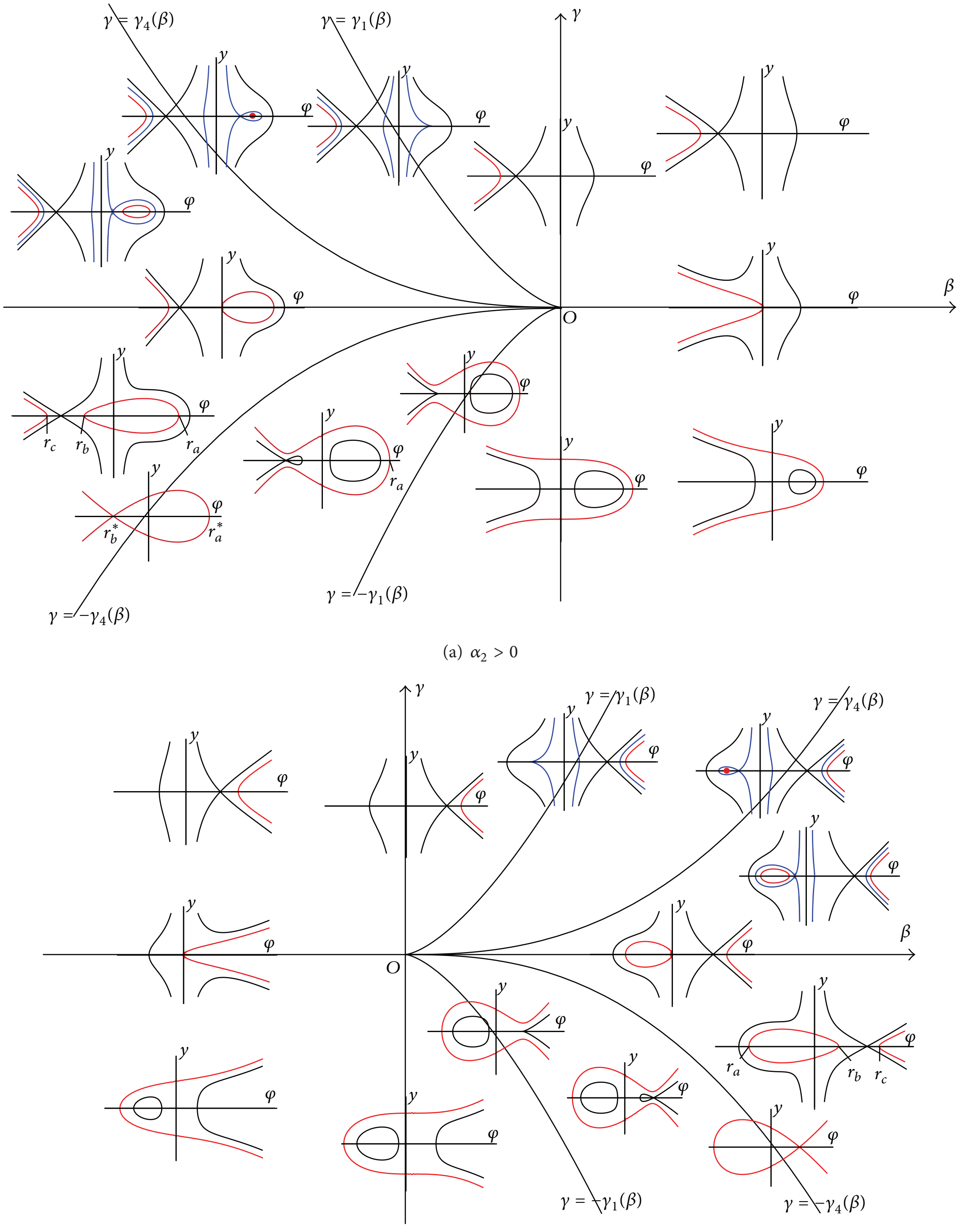
$$A_9 = \sqrt{(b_0 - r_a)^2 + a_0^2} \longrightarrow r_a^* - r_b^*,$$

$$k_4^2 = \frac{A_9 - b_0 + r_a}{2A_9} \longrightarrow 1,$$

$$\eta_8 = \sqrt{\frac{1}{5}A_9\alpha_2} \longrightarrow 2\sqrt[4]{-\frac{\alpha_2\beta}{80}}. \tag{97}$$

Thus, we get

$$\begin{aligned}\lim_{\gamma\to-\gamma_4-0} v_{16}(\xi) &= \lim_{\gamma\to-\gamma_4-0}\frac{(A_9 + r_a)\,\mathrm{cn}(\eta_8\xi, k_4) - A_9 + r_a}{\mathrm{cn}(\eta_8\xi, k_4) + 1}\\ &= \frac{(2r_a^* - r_b^*)\,\mathrm{sech}\left(2\sqrt[4]{-\alpha_2\beta/80}\,\xi\right) + r_b^*}{1 + \mathrm{sech}\left(2\sqrt[4]{-\alpha_2\beta/80}\,\xi\right)}\\ &= \frac{(2r_a^* - r_b^*) + r_b^*\left[2\cosh^2\left(\sqrt[4]{-\alpha_2\beta/80}\,\xi\right) - 1\right]}{2\cosh^2\left(\sqrt[4]{-\alpha_2\beta/80}\,\xi\right)}\end{aligned}$$



(a) $\alpha_2 > 0$

(b) $\alpha_2 < 0$

FIGURE 14: The bifurcation phase portraits of system (85).

$$= (r_a^* - r_b^*)\operatorname{sech}^2\left(\sqrt[4]{-\frac{\alpha_2\beta}{80}}\xi\right) + r_b^*$$

$$= \sqrt{-\frac{5\beta}{\alpha_2}}\left[\frac{2}{3} - \tanh^2\left(\sqrt[4]{-\frac{\alpha_2\beta}{80}}\xi\right)\right]$$

$$= v_i^+(\xi) \quad (\text{see } (87)). \tag{98}$$

Similarly, we can also get $v_{15}(\xi) \to v_i^-(\xi)$ and $v_{17}(\xi) \to v_i^-(\xi)$ when $\alpha_2 < 0$, $\beta > 0$, and $\gamma \to -\gamma_4(\beta) \pm 0$. Hereto, we have completed all of the derivations.

5. Conclusions

In this paper, by employing the bifurcation method and qualitative theory of dynamical systems, we have revealed some interesting bifurcation phenomena of nonlinear waves for the $D(m, n)$ system (1). Firstly, for $D(2, 1)$ system, we have pointed out that the fractional solitary waves can be bifurcated from the trigonometric periodic waves and the elliptic periodic waves (see Figures 1 and 2). In the meantime, the kink waves can be bifurcated from the solitary waves and the singular waves (see Figures 3–6). Secondly, for $D(1, 2)$ system, we have showed that the solitary waves can turn into the compactons (see Figure 8) and the peakons can be bifurcated from the singular cusp waves and the solitary waves (see Figures 9 and 10). Thirdly, for $D(2, 2)$ system, we have confirmed that the solitary waves can be bifurcated from the smooth periodic waves and the singular periodic waves (see Figures 12 and 13).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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