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Research Article

The Hahn Sequence Space Defined by the Cesáro Mean

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The BK-space of all sequences is given as $x = (x_k)$ such that $\Sigma_{k=1}^{\infty} k|x_k - x_{k+1}|$ converges and x_k is a null sequence which is called the Hahn sequence space and is denoted by h. Hahn (1922) defined the h space and gave some general properties. G. Goes and S. Goes (1970) studied the functional analytic properties of this space. The study of Hahn sequence space was initiated by Chandrasekhara Rao (1990) with certain specific purpose in the Banach space theory. In this paper, the matrix domain of the Hahn sequence space determined by the Cesáro mean first order, denoted by C, is obtained, and some inclusion relations and some topological properties of this space are investigated. Also dual spaces of this space are computed and, matrix transformations are characterized.

1. Introduction

By a *sequence space*, we understand a linear subspace of the space $\omega = \mathbb{C}^{\mathbb{N}}$ of all complex sequences which contains ϕ, the set of all finitely nonzero sequences, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. We write ℓ_∞, c, and c_0 for the classical spaces of all bounded, convergent, and null sequences, respectively. Also by bs, cs, ℓ_1, and ℓ_p, we denote the space of all bounded, convergent, absolutely, and p-absolutely convergent series, respectively. Additionally, the spaces $bv, bv(C), \sigma_\infty, \int\lambda$, and $\ell_1(C)$ are defined by

$$bv = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \left| x_k - x_{k-1} \right| < \infty \right\},$$

$$bv(C) = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \left| \frac{x_k}{k+1} - \frac{1}{k.(k+1)} \sum_{j=0}^{k-1} x_j \right| < \infty \right\},$$

$$\sigma_\infty = \left\{ x = (x_k) \in \omega : \sup_n n^{-1} \left| \sum_{k=1}^{n} x_k \right| < \infty \right\},$$

$$\int\lambda = \left\{ x = (x_k) \in \omega : (kx_k) \in \lambda \right\},$$

$$\ell_1(C) = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \left| \frac{1}{k+1} \sum_{j=0}^{k} x_j \right| < \infty \right\}. \tag{1}$$

A coordinate space (or a *K-space*) is a vector space of numerical sequences, where addition and scalar multiplication are defined pointwise. That is, a sequence space λ with a linear topology is called a K-space provided that each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A BK-space is a K-space, which is also a Banach space with continuous coordinate functionals $f_k(x) = x_k$, for all $k \in \mathbb{N}$. If a normed sequence space λ contains a sequence (b_n) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \left\| x - (\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_n b_n) \right\| = 0, \tag{2}$$

then (b_n) is called the *Schauder basis* (or briefly basis) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and it is written as $x = \sum \alpha_k b_k$. An *FK-space* λ is said to have AK property, if $\phi \subset \lambda$ and $\{e^k\}$ is a basis for λ, where e^k is a sequence whose only nonzero term is 1 in the kth place for each $k \in \mathbb{N}$ and $\phi = \operatorname{span}\{e^k\}$, the set of all finitely nonzero sequences. If ϕ is dense in λ, then λ is called an AD-space; thus, AK implies AD.

Let λ and μ be two sequence spaces, and let $A = (a_{nk})$ be an infinite matrix of the complex numbers a_{nk}, where $k, n \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \longrightarrow \mu$ if Ax exists and belongs to μ for every sequence $x = (x_k) \in \lambda$, where $Ax = \{(Ax)_n\}$, the A-transform of x with

$$(Ax)_n = \sum_k a_{nk} x_k \quad \text{for each } n \in \mathbb{N}. \tag{3}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 1 to ∞. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \longrightarrow \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (3) converges for each $n \in \mathbb{N}$ and each $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to l if Ax converges to l which is called the A-limit of x.

The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\} \tag{4}$$

which is a sequence space. If $A = (a_{nk})$ is triangle, that is, $a_{nn} \neq 0$ and $a_{nk} = 0$ for all $k > n$, then one can easily observe that the sequence spaces λ_A and λ are linearly isomorphic; that is, $\lambda_A \cong \lambda$. There are several examples of the matrix domain λ_A of an infinite matrix A in a sequence space λ in Chapter 4 in [1]. By $\mathcal{F}$, we will denote the collection of all finite subsets of $\mathbb{N}$.

Hahn [2] introduced the space $h = \{x : \sum_k k|x_k - x_{k+1}| < \infty \text{ and } \lim_{k\to\infty} x_k = 0\}$ and proved that the following statements hold:

(i) h is a Banach space with the norm $\|x\|_h = \sum_k k|x_k - x_{k+1}| + \sup_{k\in\mathbb{N}} |x_k|$,

(ii) $h \subset \ell_1 \cap \int c_0$,

(iii) $h^\beta = \sigma_\infty$.

2. The New Hahn Sequence Space

Following Hahn [2], we introduce the sequence space $h(C)$ as follows:

$$h(C) = \left\{ x = (x_k) \in \omega : \sum_k k \left| \frac{1}{(k+1)(k+2)} \sum_{j=0}^{k} x_j - \frac{x_{k+1}}{k+2} \right| < \infty, \quad \frac{1}{k+1} \sum_{j=0}^{k} x_j \longrightarrow 0 \ \ (k \longrightarrow \infty) \right\}. \tag{5}$$

With the notation of (4), we may redefine the space $h(C)$ as follows:

$$h(C) = (h)_C. \tag{6}$$

We define a sequence $y = (y_k)$ as the C-transform of a sequence $x = (x_k)$; that is,

$$y_k = \frac{1}{n+1} \sum_{k=0}^{n} x_k \quad \forall n \in \mathbb{N}. \tag{7}$$

Hahn [2] proved that $h \subset \ell_1$. Now, we give some inclusion relations.

Theorem 1. *The following inclusions are strict:*

(a) $h(C) \subset \ell_1(C)$,

(b) $h \subset bv$,

(c) $h(C) \subset bv(C)$.

Proof. (a) It is clear that $h(C) \subset \ell_1(C)$ from $h \subset \ell_1$ [2]. Now, we show that this inclusion is strict. Let us consider the sequence $x = (x_k)$, $x_0 = 1$, and $x_k = (-1)^k(2k+1)/k(k+1)$ $(k \geq 1)$. Then, $y = Cx = \{(-1)^k/(k+1)^2\}$. Since the sequence y is in ℓ_1 but not in h, then $x \in \ell_1(C) \setminus h(C)$.

(b) Since $h \subset \ell_1$ [2] and $\ell_1 \subset bv$, then $h \subset bv$.

(c) We choose the sequence $(x_k) = e = (1, 1, \ldots)$. Since $\lim_{k\to\infty}(1/(k+1)) \sum_{j=0}^{k} x_j = 1$, then $x = e$ is not in $h(C)$, but it is in $bv(C)$. Thus, we see that $h(C) \subset bv(C)$ is strict. □

Theorem 2. *The sequence space $h(C)$ is a BK-space with the norm*

$$\|x\|_{h(C)} = \sum_k k \left| \frac{1}{k+1} \sum_{j=0}^{k} x_j - \frac{1}{k+2} \sum_{j=0}^{k+1} x_j \right| + \sup_{k\in\mathbb{N}} \left| \frac{1}{k+1} \sum_{j=0}^{k} x_j \right|. \tag{8}$$

Proof. Since (6) holds, h is a BK-space with the norm $\|\cdot\|_h$ [2, 3], and the matrix C is triangle matrix, then Theorem 4.3.2 of Wilansky [4] gives the fact that the space $h(C)$ is a BK-space. □

Lemma 3 (see [5]). *The BK-space h has an AK property.*

Since $\{e_k : k \in \mathbb{N}\} \not\subset h(C)$, then one has the following.

Theorem 4. *The BK-space $h(C)$ does not have an AK property.*

Theorem 5. *One has the following:*

$$h(C) = \ell_1(C) \cap \int bv(C) = \ell_1(C) \cap \int bv_0(C). \tag{9}$$

Proof. It is similar to the proof of [3, Theorem 3.2]. □

Theorem 6. *The sequence space $h(C)$ is norm isomorphic to the space h; that is, $h(C) \cong h$.*

Proof. To prove this, we will show the existence of a linear bijection between the spaces $h(C)$ and h. Consider the transformation T defined, with the notation of (7), from $h(C)$

to h by $x \mapsto y = Tx$. The linearity of T is clear. Furthermore, it is trivial that $x = \theta = (0, 0, 0, \ldots)$ whenever $Tx = \theta$, and, hence, T is injective.

Let $y \in h$, and define the sequence $x = (x_k)$ by $x_k = (k+1) y_k - k y_{k-1} (k \in \mathbb{N})$. Then, we have

$$\begin{aligned} \|x\|_{h(C)} &= \sum_k k \left| \frac{1}{k+1} \sum_{j=0}^{k} \left[(j+1) y_j - j y_{j-1} \right] \right. \\ &\qquad \left. - \frac{1}{k+2} \sum_{j=0}^{k+1} \left[(j+1) y_j - j y_{j-1} \right] \right| \\ &= \sum_k k \left| y_k - y_{k+1} \right| = \|y\|_h < \infty. \end{aligned} \tag{10}$$

Consequently, we see from here that T is surjective and is norm preserving. Hence, T is a linear bijection which, therefore, shows that the spaces $h(C)$ and h are norm isomorphic, as desired. □

Theorem 7. *Define a sequence $b^{(k)} = \{b_n^{(k)}\}_{n \in \mathbb{N}}$ of elements of the space $h(C)$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)} = \begin{cases} (-1)^{n-k} (k+1), & k \le n \le k+1, \\ 0, & 0 \le n < k \quad or \quad n > k+1. \end{cases} \tag{11}$$

Then, the sequence $\{b_n^{(k)}\}_{n \in \mathbb{N}}$ is a basis for the space $h(C)$, and any $x \in h(C)$ has a unique representation of the form

$$x = \sum_k \lambda_k b^{(k)}, \tag{12}$$

where $\lambda_k = (C_1 x)_k$ for all $k \in \mathbb{N}$.

Proof. It is clear that $\{b^{(k)}\} \subset h(C)$, since

$$C_1 b^{(k)} = e^k \in \ell \quad (k = 0, 1, 2, \ldots). \tag{13}$$

Let $x \in h(C)$ be given. For every nonnegative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k b^{(k)}. \tag{14}$$

Then, we obtain by applying C_1 to (14) with (13) that

$$\begin{gathered} C_1 x^{[m]} = \sum_{k=0}^{m} \lambda_k C_1 b^{(k)} = \sum_{k=0}^{m} (C_1 x)_k e^k, \\ \left\{ C_1 \left(x - x^{[m]} \right) \right\}_i = \begin{cases} 0, & (0 \le i \le m), \\ (C_1 x)_i, & (i > m), \end{cases} \quad (i, m \in \mathbb{N}). \end{gathered} \tag{15}$$

Given that $\varepsilon > 0$, then there is an integer m_0 such that

$$\left| (C_1 x)_j \right| < \frac{\varepsilon}{2} \tag{16}$$

for all $m \ge m_0$. Hence,

$$\left\| x - x^{[m]} \right\|_{h(C)} = \sup_{n \ge m} \left| (C_1)_n \right| \le \sup_{n \ge m_0} \left| (C_1)_n \right| \le \frac{\varepsilon}{2} < \varepsilon \tag{17}$$

for all $m \ge m_0$, which proves that $x \in h(C)$ is represented as in (12).

To show the uniqueness of this representation, we assume that $x = \sum_k \mu_k b^{(k)}$. Since the linear transformation T, from $h(C)$ to h, used in Theorem 6 is continuous, then we have at this stage that

$$(C_1 x)_n = \sum_k \mu_k \left\{ C_1 b^{(k)} \right\}_n = \sum_k \mu_k e_n^k = \mu_n \quad (n \in \mathbb{N}), \tag{18}$$

which contradicts the fact that $(C_1 x)_n = \lambda_n$ for all $n \in \mathbb{N}$. Hence, the representation (12) of $x \in h(C)$ is unique. □

3. Duals of the Sequence Space $h(C)$

In this section, we state and prove the theorems determining the α-, β-, and γ-duals of the sequence space $h(C)$.

The set $S(\lambda, \mu)$ defined by

$$\begin{aligned} S(\lambda, \mu) = \{ z = (z_k) \in \omega : xz = (x_k z_k) \in \mu \\ \forall x = (x_k) \in \lambda \} \end{aligned} \tag{19}$$

is called the *multiplier space* of the sequence spaces λ and μ. One can easily observe for a sequence space υ with $\lambda \supset \upsilon \supset \mu$ that the inclusions

$$S(\lambda, \mu) \subset S(\upsilon, \mu), \qquad S(\lambda, \mu) \subset S(\lambda, \upsilon) \tag{20}$$

hold. With the notation of (19), the alpha-, beta-, and gamma-duals of a sequence space λ, which are, respectively, denoted by λ^α, λ^β, and λ^γ are defined by

$$\lambda^\alpha = S(\lambda, \ell_1), \qquad \lambda^\beta = S(\lambda, cs), \qquad \lambda^\gamma = S(\lambda, bs). \tag{21}$$

The alpha-, beta-, and gamma-duals of a sequence space are also referred to as the *Köthe-Toeplitz dual*, the *generalized Köthe-Toeplitz dual*, and the *Garling dual* of a sequence space, respectively.

Given an FK-space X containing Φ, its conjugate is denoted by X', and its f-dual or sequential dual is denoted by X^f and is given by $X^f = \{\text{all sequences } (f(e^k)) : f \in X'\}$.

We need the following lemmas.

Lemma 8 (see [6]). *Let $B^U = (b_{nk})$ be defined via a sequence $a = (a_k) \in \omega$, and let the inverse $v = (v_{nk})$ of the triangle matrix $U = (u_{nk})$ be defined by $b_{nk} = \sum_{j=k}^{n} a_j v_{jk}$ for all $k, n \in \mathbb{N}$. Then,*

$$\begin{aligned} \lambda_U^\beta &= \left\{ a = (a_k) \in \omega : B^U \in (\lambda : c) \right\}, \\ \lambda_U^\gamma &= \left\{ a = (a_k) \in \omega : B^U \in (\lambda : \ell_\infty) \right\}. \end{aligned} \tag{22}$$

Lemma 9 (see [5]). *(i) $A \in (h : \ell)$ if and only if*

$$\begin{gathered} \sum_{n=1}^{\infty} |a_{nk}| \text{ converges} \quad (k = 1, 2, \ldots), \\ \sup_k \frac{1}{k} \sum_{n=1}^{\infty} \left| \sum_{v=1}^{k} a_{nv} \right| < \infty. \end{gathered} \tag{23}$$

(ii) $A \in (h : c)$ if and only if

$$\sup_{n,k} \frac{1}{k} \left| \sum_{v=1}^{k} a_{nv} \right| < \infty, \tag{24}$$

$$\lim_{n \to \infty} a_{nk} \text{ exists} \quad (k = 1, 2, \ldots). \tag{25}$$

(iii) $A \in (h : c_0)$ *if and only if*

$$\lim_{n\to\infty} a_{nk} = 0 \tag{26}$$

and (24) *hold.*

(iv) $A \in (h : \ell_\infty)$ *if and only if* (24) *holds.*

(v) $A \in (h : h)$ *if and only if* (26) *holds and*

$$\begin{gathered}\sum_{n=1}^{\infty} n\left|a_{nk} - a_{n+1,k}\right| \text{ converges} \quad (k = 1, 2, \ldots), \\ \sup_k \frac{1}{k}\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}\left(a_{nv} - a_{n+1,v}\right)\right| < \infty.\end{gathered} \tag{27}$$

Theorem 10. *The α-dual of the space $h(C)$ is the set*

$$\begin{aligned}d_1 = \Bigg\{ a = (a_k) \in \omega : \sup_{N,K\in\mathcal{F}} \frac{1}{k} \\ \times \sum_{n\in N}\left|\sum_{k\in K}(-1)^{n-k}(k+1)a_n\right| < \infty\Bigg\}.\end{aligned} \tag{28}$$

Proof. Let $a = (a_n) \in \omega$. We define the matrix $B = (b_{nk})$ via the sequence $a = (a_n)$ by

$$b_{nk} = \begin{cases}(-1)^{n-k}(k+1)a_n & (n-1 \le k \le n+1), \\ 0, & (0 \le k < n-1 \text{ or } k > n),\end{cases} \quad (n, k \in \mathbb{N}). \tag{29}$$

Bearing in mind the relation (7), we immediately derive that

$$a_n x_n = \sum_{k=n-1}^{n}(-1)^{n-k}(k+1)a_n y_k = (By)_n \quad (n \in \mathbb{N}). \tag{30}$$

We, therefore, observe by (30) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in h(C)$ if and only if $By \in \ell_1$ whenever $y \in h$. Then, we derive by Lemma 9 (i) that

$$\sup_{N,K\in\mathcal{F}} \frac{1}{k}\sum_{n\in N}\left|\sum_{k\in K}(-1)^{n-k}(k+1)a_n\right| < \infty, \tag{31}$$

which yields the result that $[h(C)]^\alpha = d_1$. □

Hahn [2] proved that $h^\beta = \sigma_\infty$.

Theorem 11. *Consider the following:*

$$[h(C)]^\beta = \sigma_\infty \cap \ell_\infty. \tag{32}$$

Proof. Consider the equation

$$\begin{aligned}\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} a_k\left(\sum_{j=k}^{\infty}\frac{y_j}{j}\right) \\ &= \sum_{k=0}^{n}\left(\sum_{j=k}^{\infty} a_j\right) y_k = (Ey)_n \quad (n \in \mathbb{N}),\end{aligned} \tag{33}$$

where $E = (e_{nk})$ is defined by

$$e_n^{(k)} = \begin{cases}\sum_{j=k}^{\infty}\frac{a_j}{j} & (0 \le k \le n), \\ 0 & (k > n),\end{cases} \quad (n, k \in \mathbb{N}). \tag{34}$$

Thus, we deduce from (33) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in h(C)$ if and only if $Ey \in c$ whenever $y = (y_k) \in h$. Therefore, we derive the consequence from Lemma 9 (ii) that $[h(C)]^\beta = \sigma_\infty \cap \ell_\infty$. □

Theorem 12. *One has the following:*

$$[h(C)]^\gamma = \sigma_\infty \cap \ell_\infty. \tag{35}$$

Proof. This is obtained in the similar way used in the proof of Theorem 11. □

4. Matrix Transformations

Let us suppose throughout that the sequences $x = (x_k)$ and $y = (y_k)$ are connected with (7), and let the A-transform of the sequence $x = (x_k)$ be $r = (r_n)$, and let the B-transform of the sequence $y = (y_k)$ be $s = (s_n)$; that is,

$$\begin{aligned}r_n &= (Ax)_n = \sum_k a_{nk}x_k \quad (n \in \mathbb{N}), \\ s_n &= (By)_n = \sum_k b_{nk}y_k \quad (n \in \mathbb{N}).\end{aligned} \tag{36}$$

It is clear here that the method B is applied to the C-transform of the sequence $x = (x_k)$, while the method A is directly applied to the terms of the sequence $x = (x_k)$. So the methods A and B are essentially different.

Following Şengönül and Başar [7], we give some knowledge about the dual summability methods of the new type. Let us assume the existence of the matrix product BC. We will say in this situation that the methods A and B in (36) are the dual of the new type if $r = (r_n)$ is reduced to $s = (s_n)$ (or $s = (s_n)$ becomes $r = (r_n)$) under the application of the formal summation by parts. This leads us to the fact that BC exists and is equal to A and $(BC)x = B(Cx)$ formally holds, if one side exists. This statement is equivalent to the relation

$$a_{nk} = \sum_{j=k}^{\infty}\frac{1}{j+1}b_{nj} \; \left(\text{or } b_{nk} = (k+1)\Delta a_{nk}, \text{ where } \Delta a_{nk} = a_{nk} - a_{n,k+1}\right) \quad (n \in \mathbb{N}). \tag{37}$$

Now, we may give the following theorem.

Theorem 13. *Let $A = (a_{nk})$ and $B = (b_{nk})$ be the dual matrices of the new type, and μ be any given sequence space. Then, let $A \in (h(C) : \mu)$ if and only if $B \in (h : \mu)$ and*

$$\{(n+1)a_{nk}\}_{n\in\mathbb{N}} \in c_0 \tag{38}$$

for every fixed $k \in \mathbb{N}$.

Proof. Suppose that $A = (a_{nk})$ and $B = (b_{nk})$ are dual matrices of the new type; that is to say that (37) holds; let μ be any given sequence space, and take account that the spaces $h(C)$ and h are linearly isomorphic.

Let $A \in (h(C) : \mu)$, and take any $y \in h$. Then, BC exists, and $(a_{nk})_{k\in\mathbb{N}} \in d_2 \cap d_3 \cap cs$, which yields that $(a_{nk})_{k\in\mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. Hence, By exists, and, thus, letting $m \to \infty$ in the equality

$$\sum_{k=0}^{m} b_{nk} y_k = \sum_{k=0}^{m} \sum_{j=k}^{m} \frac{1}{j+1} b_{nj} x_k \quad (n, m \in \mathbb{N}), \tag{39}$$

we have by (37) that $By = Ax$, which leads us to the consequence that $B \in (h : \mu)$.

Conversely, let $B \in (h : \mu)$, and (38) hold, and take any $x \in h(C)$. Then, we have $(b_{nk})_{k\in\mathbb{N}} \in \ell_1$, which gives together with (38) that $(a_{nk})_{k\in\mathbb{N}} \in [h(C)]^{\beta}$ for each $n \in \mathbb{N}$. Hence, Ax exists. Therefore, we obtain from the equality

$$\begin{aligned}\sum_{k=0}^{m} a_{nk} x_k &= \sum_{k=0}^{m-1} (k+1)\, \Delta_{nk} y_k + (m+1)\, a_{nm} y_m \\ &= \sum_{k=0}^{m} b_{nk} y_k \quad (n, m \in \mathbb{N})\end{aligned} \tag{40}$$

as $m \to \infty$ that $Ax = By$, and this shows that $A \in (h(C) : \mu)$. This completes the proof. □

By the changing roles of the spaces $h(c)$ and μ in Theorem 13, we have the following.

Theorem 14. *Suppose that the elements of the infinite matrices $F = (f_{nk})$ and $G = (g_{nk})$ are connected with the relation*

$$g_{nk} = \sum_{j=0}^{n} \frac{f_{jk}}{n+1} \tag{41}$$

for all $k, m \in \mathbb{N}$, and let μ be any given sequence space. Then, $F \in (\mu : h(C))$ if and only if $G \in (\mu : h)$.

Proof. Let $x = (x_k) \in \mu$, and consider the following equality with (41):

$$\frac{1}{i+1} \sum_{j=0}^{i} \sum_{k=0}^{n} f_{jk} x_k = \sum_{k=0}^{n} g_{ik} x_k \quad (i, n \in \mathbb{N}) \tag{42}$$

which yields as $n \to \infty$ that

$$\frac{1}{i+1} \sum_{j=0}^{i} (Fx)_j = (Gx)_i \quad (i \in \mathbb{N}). \tag{43}$$

Therefore, one can easily see by (43) that $Fx \in h(C)$ whenever $x \in \mu$ if and only if $Gx \in c$ whenever $x \in \mu$. □

Corollary 15. *(i) $A = (a_{nk}) \in (h(C) : \ell)$ if and only if* (23) *hold with $\sum_{j=k}^{\infty}(1/(k+1))b_{nj}$ instead of a_{nk}.*

(ii) $A = (a_{nk}) \in (h(C) : c)$ if and only if (24) *and* (25) *hold with $\sum_{j=k}^{\infty}(1/(k+1))b_{nj}$ instead of a_{nk}.*

(iii) $A = (a_{nk}) \in (h(C) : c_0)$ if and only if $\lim_{n\to\infty} a_{nk} = 0$ and (24) *hold with $\sum_{j=k}^{\infty}(1/(k+1))b_{nj}$ instead of a_{nk}.*

(iv) $A = (a_{nk}) \in (h(C) : \ell_\infty)$ if and only if (24) *hold with $\sum_{j=k}^{\infty}(1/(k+1))b_{nj}$ instead of a_{nk}.*

(v) $A = (a_{nk}) \in (h(C) : h)$ if and only if (26) *and* (27) *hold with $\sum_{j=k}^{\infty}(1/(k+1))b_{nj}$ instead of a_{nk}.*

5. Conclusion

Hahn [2] defined the space h and gave some of its general properties. G. Goes and S. Goes [3] studied the functional analytic properties of the space h. The study of the Hahn sequence space was initiated by Chandrasekhara Rao [5] with a certain specific purpose in the Banach space theory. Also Chandrasekhara Rao [5] computed some matrix transformations. Chandrasekhara Rao and Subramanian [8] introduced a new class of sequence spaces called semi-replete spaces. Chandrasekhara Rao and Subramanian [8] defined the semi-Hahn space and proved that the intersection of all the semi-Hahn spaces is the Hahn space. Balasubramanian and Pandiarani [9] defined the new sequence space $h(F)$ called the Hahn sequence space of fuzzy numbers and proved that β- and γ-duals of $h(F)$ is the Cesàro space of the set of all fuzzy bounded sequences.

The sequence space h was defined by Hahn [2], and G. Goes and S. Goes [3] and Chandrasekhara Rao et al. [5, 8, 10] investigated some properties of the space h. In exception of these works, there has not been any work related to the Hahn sequence space. In this paper, the Hahn sequence space h defined by the Cesáro mean worked as follows. In Section 2, the new Hahn sequence space is determined by the Cesáro mean, and some properties of this space are investigated. In Section 3, α-, β-, and γ-duals of the new Hahn sequence space are computed. In Section 4, the matrix classes $(h(C) : \mu)$ and $(\mu : h(C))$ are characterized, where μ is an arbitrary sequence space, and some results of these characterizations are given.

We can define the matrix domain h_A of an arbitrary triangle A, compute its α-, β-, and γ-duals, and characterize the matrix transformations on them into the classical sequence spaces, and almost the convergent sequence space is a new result.

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