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Research Article

On a Kind of Dirichlet Character Sums

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Let $p \geq 3$ be a prime and let χ denote the Dirichlet character modulo p. For any prime q with $q < p$, define the set $E(q,p) = \{a \mid 1 \leq a, \bar{a} \leq p, a\bar{a} \equiv 1(\mathrm{mod} p) \text{ and } a \equiv \bar{a}(\mathrm{mod} q)\}$. In this paper, we study a kind of mean value of Dirichlet character sums $\sum_{a\leq p\ a\in E(q,p)} \chi(a)$, and use the properties of the Dirichlet L-functions and generalized Kloosterman sums to obtain an interesting estimate.

1. Introduction

Let $k \geq 3$ be an integer and let χ denote the Dirichlet character modulo k, for any real number $x \geq 1$, many scholars have studied the following sums:

$$\sum_{n\leq x} \chi(n), \tag{1}$$

where n are positive integers.

Perhaps one of the most famous results is Pólya's inequality [1]. That is, when χ is the primitive character modulo k, we have

$$\sum_{n\leq x} \chi(n) < k^{1/2} \log k. \tag{2}$$

In fact, the result can be extended to the nonprincipal character χ modulo k [2]. Further details about the estimates of character sums can be found in the literature, for example, [3, 4].

For any fixed integer $H > 0$ and any positive integer $k \geq 3$, define the following set:

$$\begin{aligned} L(H,k) = \{&a \mid 1 \leq a, \bar{a} \leq k-1, \\ &(a,k) = 1, a\bar{a} \equiv 1 (\mathrm{mod} k), |a-\bar{a}| \leq H\}, \end{aligned} \tag{3}$$

let χ denote the Dirichlet character modulo k, define the sums as follows:

$$\sum_{\substack{n\leq k \\ n\in L(H,k)}} \chi(n). \tag{4}$$

Xi and Yi [5] studied the problem for χ the nonprincipal Dirichlet character modulo k, and got

$$\sum_{\substack{n\leq k \\ n\in L(H,k)}} \chi(n) \ll k^{1/2} d(k) \log H, \tag{5}$$

where $0 < H \leq q$ was a constant and $d(k)$ was the divisor function. Before this, Wenpeng [6] got an asymptotic formula for the case that χ was the principal Dirichlet character modulo k.

On the other hand, for each integer a with $1 \leq a \leq k$ and $(a,k) = 1$, we know that there exists one and only one b with $1 \leq b \leq k$ such that $ab \equiv 1(\mathrm{mod} k)$. Let $r_2(k)$ be the number of solutions of the congruent equation $ab \equiv 1(\mathrm{mod}\, k)$ for $1 \leq a, b \leq k$ in which a and b are of opposite parity, this can be expressed as follows:

$$r_2(k) = \sum_{\substack{a=1 \\ a\bar{a}\equiv 1(\mathrm{mod}\ k) \\ 2\dagger(a+\bar{a})}}^{k} 1. \tag{6}$$

Richard [7] asks us to find $r_2(k)$ or at least to say something nontrivial about it. About this problem, a lot of scholars have studied it [8–12]. Now we let m be another integer with $m < k$ and let $r_m(k)$ denote the number of all pairs of integers a, b satisfying $ab \equiv 1(\mathrm{mod} k)$, $1 \leq a, b \leq k$, and $m \dagger (a+b)$. Lu and Yi [13] have obtained the asymptotic formula of generalized D. H. Lehmer problem as follows:

$$r_m(k) = \sum_{\substack{a=1 \\ a\bar{a}\equiv 1(\bmod k) \\ m\dagger(a+\bar{a})}}^{k} 1 = \left(1-\frac{1}{m}\right)\phi(k) + O\left(k^{1/2}\log^2 k\right), \tag{7}$$

where the O constant only depends on m.

In this paper, let p be an odd prime and let q be a fixed prime with $q < p$, define the set $E(q,p)$ for $a(1 \le a \le p)$ such that $a\bar{a} \equiv 1(\bmod p)$ and $a \equiv \bar{a}(\bmod q)$, that is,

$$\begin{aligned} E(q,p) = \{&a \mid 1 \le a, \bar{a} \le p-1, \\ &a\bar{a} \equiv 1 (\bmod p), a \equiv \bar{a} (\bmod q)\}. \end{aligned} \tag{8}$$

As another case of (7), we will consider the mean value of Dirichlet character sums as follows:

$$\sum_{\substack{a\le p-1 \\ a\in E(q,p)}} \chi(a), \tag{9}$$

and get an interesting estimate. That is, we will prove the following theorem.

Theorem 1. *Let p be an odd prime and let q be a fixed prime with $q < p$, and let χ denote the Dirichlet character modulo p. Let $E(q,p)$ denote the following set:*

$$\begin{aligned} E(q,p) = \{&a \mid 1 \le a \le p-1, \\ &a\bar{a} \equiv 1 (\bmod p), a \equiv \bar{a} (\bmod q)\}, \end{aligned} \tag{10}$$

then, for any nonprincipal Dirichlet character χ mod p, we have the following estimate:

$$\sum_{\substack{a\le p-1 \\ a\in E(q,p)}} \chi(a) = O\left(p^{1/2+\epsilon}\right), \tag{11}$$

where the O constant only depends on q.

From this Theorem we can get

$$\begin{aligned} \sum_{\substack{a=1 \\ a\bar{a}\equiv 1(\bmod p) \\ q\dagger(a-\bar{a})}}^{p-1} \chi(a) &= \sum_{\substack{a=1 \\ a\bar{a}\equiv 1(\bmod p)}}^{p-1} \chi(a) - \sum_{\substack{a\le p-1 \\ a\in E(q,p)}} \chi(a) \\ &= O\left(p^{1/2+\epsilon}\right). \end{aligned} \tag{12}$$

For any integer k and fixed integer m such that $(m,k) = 1$, whether or not there exists an estimate for

$$\sum_{\substack{a\le k \\ a\in E(m,k)}} \chi(a) \tag{13}$$

is still an open problem.

2. Some Lemmas

In this section, we will give several lemmas which are necessary in the proof of the theorem.

Lemma 2. *Let Q be an integer, and let χ be a primitive character modulo Q. Then, for any real number u and v with $u < v$, we have*

$$\begin{aligned} \sum_{uQ<n\le vQ} \chi(n) = \tau(\chi) \sum_{0<|h|\le H} \bar{\chi}(h) \frac{e(-hu) - e(-hv)}{2\pi i h} \\ + O\left(1 + \frac{Q\log Q}{H}\right), \end{aligned} \tag{14}$$

where $e(x) = e^{2\pi i x}$ and $\tau(\chi) = \sum_{a=1}^{Q} \chi(a)e(a/Q)$ are Gauss sums.

Especially, let $u = 0$, we have a slight modification

$$\sum_{0<n\le vQ} \chi(n) = \begin{cases} \dfrac{\tau(\chi)}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\sin(2\pi n v)}{n} + O(1), & \text{if } \chi(-1) = 1, \\ \dfrac{\tau(\chi)}{\pi i} \displaystyle\sum_{n=1}^{\infty} \frac{\bar{\chi}(n)(1-\cos(2\pi n v))}{n} + O(1), & \text{if } \chi(-1) = -1. \end{cases} \tag{15}$$

Proof. (See [1]). □

Lemma 3. *Let q be a prime, let Q be an integer with $Q > q$, and let χ be a primitive character modulo Q, then, we have*

$$\sum_{n\le Q/q} \chi(n) = \begin{cases} \dfrac{\tau(\chi)}{(q-1)\pi} \displaystyle\sum_{l=1}^{q-1} \sum_{\chi_2 \bmod q} \left(\chi_2(l)\sin\frac{2\pi l}{q}\right) \times L\left(1, \overline{\chi\chi_2}\right) + O(1), & \chi(-1) = 1, \\ \dfrac{\tau(\chi)}{(q-1)\pi i} \displaystyle\sum_{l=1}^{q-1} \sum_{\chi_2 \bmod q} \chi_2(l)\left(1-\cos\frac{2\pi l}{q}\right) \times L\left(1, \overline{\chi\chi_2}\right) + O(1), & \chi(-1) = -1, \end{cases} \tag{16}$$

where $L(1,\chi)$ are the Dirichlet L-functions corresponding to χ.

Proof. From Lemma 2, we take $u = 0$, and $v = 1/q$ and get

$$\begin{aligned} \sum_{n\le Q/q} \chi(n) = \tau(\chi) \sum_{0<h\le H} \bar{\chi}(h) \frac{1-e(-h/q)}{2\pi i h} \\ + \tau(\chi) \sum_{-H<h\le 0} \bar{\chi}(h) \frac{1-e(-h/q)}{2\pi i h} \\ + O\left(1 + \frac{Q\log Q}{H}\right). \end{aligned} \tag{17}$$

When $\chi(-1)=1$, we have

$$\begin{aligned}\sum_{n\le Q/q}\chi(n)&=\tau(\chi)\sum_{0<h\le H}\overline{\chi}(h)\frac{\sin(2\pi h/q)}{\pi h}\\&\quad+O\left(1+\frac{Q\log Q}{H}\right)\\&=\frac{\tau(\chi)}{\pi}\sum_{l=1}^{q-1}\left(\sin\frac{2\pi l}{q}\right)\sum_{\substack{0<h\le H\\ h\equiv l(\bmod q)}}\frac{\overline{\chi}(h)}{h}\\&\quad+O\left(1+\frac{Q\log Q}{H}\right)\\&=\frac{\tau(\chi)}{(q-1)\pi}\sum_{l=1}^{q-1}\left(\sin\frac{2\pi l}{q}\right)\sum_{0<h\le H}\frac{\overline{\chi}(h)}{h}\\&\quad\times\sum_{\chi_2\bmod q}\overline{\chi}_2(h)\chi_2(l)+O\left(1+\frac{Q\log Q}{H}\right)\\&=\frac{\tau(\chi)}{(q-1)\pi}\sum_{l=1}^{q-1}\sum_{\chi_2\bmod q}\left(\chi_2(l)\sin\frac{2\pi l}{q}\right)\\&\quad\times\sum_{0<h\le H}\frac{\overline{\chi\chi_2}(h)}{h}+O\left(1+\frac{Q\log Q}{H}\right).\end{aligned}\tag{18}$$

Let $H\to\infty$, then, we have

$$\sum_{n\le Q/q}\chi(n)=\frac{\tau(\chi)}{(q-1)\pi}\sum_{l=1}^{q-1}\sum_{\chi_2\bmod q}\left(\chi_2(l)\sin\frac{2\pi l}{q}\right)L\left(1,\overline{\chi\chi_2}\right)+O(1).\tag{19}$$

The case of $\chi(-1)=-1$ can be treated in the same way. This proves Lemma 3. □

Lemma 4. *Let p, q be odd primes and let χ_1, χ_2 be the Dirichlet characters modulo p and q, respectively, such that $(p,q)=1$, denote $\chi=\chi_1\chi_2$, $k=pq$, and χ is the Dirichlet character modulo pq, the famous Gauss sums are defined as follows:*

$$G(n,\chi)=\sum_{a=1}^{k}\chi(a)e\left(\frac{na}{k}\right),\tag{20}$$

where $e(y)=e^{2\pi iy}$. Hence, we have

$$G(n,\chi)=\chi_1(q)\chi_2(p)G(n,\chi_1)G(n,\chi_2).\tag{21}$$

When $n=1$, we denote $\tau(\chi)=G(1,\chi)$; therefore, we have

$$\tau(\chi)=\chi_1(q)\chi_2(p)\tau(\chi_1)\tau(\chi_2).\tag{22}$$

Proof. (See [14]). □

Lemma 5. *Let m, and n be integers and let $q\ge 3$ be prime, let χ denote the Dirichlet character modulo q, the generalized Kloosterman sums are defined by*

$$S_\chi(m,n;q)=\sum_{a\bmod q}\chi(a)e\left(\frac{m\overline{a}+na}{q}\right),\tag{23}$$

where $a\overline{a}\equiv 1(\bmod q)$ and $e(y)=e^{2\pi iy}$.

Then, we have the following estimate:

$$S_\chi(m,n;q)\ll q^{1/2+\epsilon}(m,n,q)^{1/2},\tag{24}$$

where (m,n,q) denotes the gcd of m, n, and q.

Proof. (See [15]). □

Lemma 6. *Let $p\ge 3$ be an odd prime, let χ, χ_1 be a Dirichlet character modulo p, and $\chi\chi_1\ne\chi_0^p$. For any odd prime q with $q<p$, let χ_2,χ_3,χ_4 be any Dirichlet characters with χ_2 mod q, χ_3 mod q and χ_4 mod q, respectively, then, no matter χ is odd character or even character modulo p, we have*

$$\begin{aligned}&\sum_{\substack{\chi_1\ne\chi_p^0\\ \chi_1\chi_2(-1)=1}}\chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1)\\&\quad\times L\left(1,\overline{\chi\chi_1\chi_2\chi_3}\right)L\left(1,\overline{\chi}_1\chi_2\overline{\chi}_4\right)\ll p^{3/2+\epsilon};\end{aligned}\tag{25}$$

$$\begin{aligned}&\sum_{\substack{\chi_1\ne\chi_p^0\\ \chi_1\chi_2(-1)=-1}}\chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1)\\&\quad\times L\left(1,\overline{\chi\chi_1\chi_2\chi_3}\right)L\left(1,\overline{\chi}_1\chi_2\overline{\chi}_4\right)\ll p^{3/2+\epsilon};\end{aligned}\tag{26}$$

$$\begin{aligned}&\sum_{\substack{\chi_1\ne\chi_p^0\\ \chi(-1)=1}}\chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1)\\&\quad\times L\left(1,\overline{\chi\chi_1\chi_3}\right)L\left(1,\overline{\chi_1\chi_4}\right)\ll p^{3/2+\epsilon};\end{aligned}\tag{27}$$

$$\begin{aligned}&\sum_{\substack{\chi_1\ne\chi_p^0\\ \chi(-1)=-1}}\chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1)\\&\quad\times L\left(1,\overline{\chi\chi_1\chi_3}\right)L\left(1,\overline{\chi_1\chi_4}\right)\ll p^{3/2+\epsilon},\end{aligned}\tag{28}$$

where the $\ll$ constant only depends on q.

Proof. For any integer n with $(n,k)=1$ ($k\ge 3$ is any positive integer), we have

$$\begin{aligned}\sum_{\substack{\chi\bmod k\\ \chi(-1)=1}}\chi(n)&=\begin{cases}\frac{1}{2}\phi(k), & n\equiv\pm1(\bmod k),\\ 0, & \text{otherwise.}\end{cases}\\ \sum_{\substack{\chi\bmod k\\ \chi(-1)=-1}}\chi(n)&=\begin{cases}\frac{1}{2}\phi(k), & n\equiv 1(\bmod k),\\ -\frac{1}{2}\phi(k), & n\equiv -1(\bmod k),\\ 0, & \text{otherwise.}\end{cases}\end{aligned}\tag{29}$$

Now let $y > p$ and let $A(y,\chi) = \sum_{p<n\le y}\chi(n)$. Then, from the Pólya-Vinogradov inequality, we obtain

$$A(y,\chi) \ll \sqrt{p}\log p. \tag{30}$$

Hence, from Abel's identity, for any $\mathrm{Re}(s) \ge 1$, we can easily get

$$\begin{aligned} L(s,\chi) &= \sum_{n\le p}\frac{\chi(n)}{n^s} + \int_p^\infty \frac{A(y,\chi)}{y^{s+1}}dy \\ &= \sum_{n\le p}\frac{\chi(n)}{n^s} + O\left(\frac{\log p}{p^{s-1/2}}\right) \\ &= \sum_{n\le p}\frac{\chi(n)}{n^s} + O\left(\frac{\log p}{p^{1/2}}\right). \end{aligned} \tag{31}$$

We will take (25), for example, to prove this lemma. For $(q,p)=1$, from the definition of $\tau(\chi)$ and (31), since χ_1 is not the principle character modulo p and $\chi\chi_1$ is not the principle character modulo p, we have

$$\begin{aligned} &\sum_{\substack{\chi_1\ne\chi_p^0\\ \chi_1\chi_2(-1)=1}} \chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1) \\ &\qquad\times L(1,\overline{\chi\chi_1\chi_2\chi_3})L(1,\overline{\chi}_1\chi_2\overline{\chi}_4) \\ &= \sum_{\substack{\chi_1 \bmod p\\ \chi_1\chi_2(-1)=1}} \chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1) \\ &\qquad\times\left(\sum_{k\le p}\frac{\overline{\chi\chi_1\chi_2\chi_3}(k)}{k} + O\left(\frac{\log pq}{(pq)^{1/2}}\right)\right) \\ &\qquad\times\left(\sum_{n\le p}\frac{\overline{\chi}_1\chi_2\overline{\chi}_4(n)}{n} + O\left(\frac{\log pq}{(pq)^{1/2}}\right)\right) \\ &\qquad-\chi(q)\tau(\chi)\tau(\chi_0^p)L(1,\overline{\chi\chi_2\chi_3})L(1,\chi_2\overline{\chi}_4) \\ &= \sum_{\substack{\chi_1 \bmod p\\ \chi_1\chi_2(-1)=1}} \chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1) \\ &\qquad\times\sum_{k\le p}\frac{\overline{\chi\chi_1\chi_2\chi_3}(k)}{k}\sum_{n\le p}\frac{\overline{\chi}_1\chi_2\overline{\chi}_4(n)}{n} + O\left(p^{3/2}\log^2 p\right) \\ &= \sum_{k\le p}\sum_{n\le p}\frac{\overline{\chi\chi_1\chi_3}(k)\chi_2\overline{\chi}_4(n)\chi(qa)}{kn} \\ &\qquad\times\sum_{a=1}^{p}{}'\sum_{b=1}^{p}{}'\chi(a)e\left(\frac{a+b}{p}\right) \\ &\qquad\times\sum_{\substack{\chi_1 \bmod p\\ \chi_1\chi_2(-1)=1}}\chi_1(q^2ab)\overline{\chi}_1(kn) + O\left(p^{3/2}\log^2 p\right) \\ &= \frac{\phi(p)}{2}\sum_{k\le q}\sum_{n\le p}\frac{\overline{\chi\chi_1\chi_3}(k)\chi_2\overline{\chi}_4(n)\chi(qa)}{kn} \\ &\qquad\times\left(\sum_{a=1}^{p}{}'\sum_{\substack{b=1\\ q^2ab\equiv kn(\bmod p)}}^{p}{}'\chi(a)e\left(\frac{a+b}{p}\right)\right. \\ &\qquad\left.\pm\sum_{a=1}^{p}{}'\sum_{\substack{b=1\\ q^2ab\equiv -kn(\bmod p)}}^{p}{}'\chi(a)e\left(\frac{a+b}{p}\right)\right) \\ &\qquad+O\left(p^{3/2}\log^2 p\right) \\ &= \frac{\phi(p)}{2}\sum_{k\le p}\sum_{n\le p}\frac{\overline{\chi\chi_1\chi_3}(k)\chi_2\overline{\chi}_4(n)\chi(qa)}{kn} \\ &\qquad\times\left(S_\chi\left(1,\overline{q}^2kn;p\right)\pm S_\chi\left(1,-\overline{q}^2kn;p\right)\right) \\ &\qquad+O\left(p^{3/2}\log^2 p\right), \end{aligned} \tag{32}$$

where $abq^2\overline{kn} \equiv 1(\mathrm{mod}p)$. From Lemma 5, we can easily obtain

$$\begin{aligned} &\sum_{\substack{\chi_1\ne\chi_p^0\\ \chi_1\chi_2(-1)=1}} \chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1)L(1,\overline{\chi\chi_1\chi_2\chi_3}) \\ &\qquad\times L(1,\overline{\chi}_1\chi_2\overline{\chi}_4) \ll p^{3/2+\epsilon_1}\log^2 p \ll p^{3/2+\epsilon}, \end{aligned} \tag{33}$$

where the $\ll$ constant only depends on q. Therefore, this completes the proof of Lemma 6. □

Lemma 7. *Let q be a fixed odd prime and let p be a prime with $p>q$, let χ denote the nonprincipal Dirichlet character modulo p, and χ_1 denote the Dirichlet character modulo p, then, we have*

$$\begin{aligned} &\sum_{\substack{\chi_1 \bmod p\\ \chi_1\ne\chi_p^0}}\chi\chi_1(q)\chi_1(q) \\ &\qquad\times\sum_{a\le(p-1)/q}\chi\chi_1(a)\sum_{b\le(p-1)/q}\chi_1(b)\ll p^{3/2+\epsilon}, \end{aligned} \tag{34}$$

where the $\ll$ constant only depends on q.

Proof. For primes p and q, χ_1 and $\chi\chi_1$ are nonprincipal and primitive characters modulo p, hence, from Lemmas 3 and 6, when $\chi(-1) = 1$, we can get

$$\begin{aligned}
&\sum_{\substack{\chi_1 \bmod p\\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q)\chi_1(q) \sum_{a \le (p-1)/q} \chi\chi_1(a) \sum_{b \le (p-1)/q} \chi_1(b) \\
&= \sum_{\substack{\chi_1(-1)=1\\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q)\chi_1(q) \\
&\quad \times \left(\frac{\tau(\chi\chi_1)}{(q-1)\pi} \sum_{u=1}^{q-1} \sum_{\chi_3 \bmod q} \left(\chi_3(u) \sin\frac{2\pi u}{q} \right) \times L\left(1, \overline{\chi\chi_1\chi_3}\right) + O(1) \right) \\
&\quad \times \left(\frac{\tau(\chi_1)}{(q-1)\pi} \sum_{v=1}^{q-1} \sum_{\chi_4 \bmod q} \left(\chi_4(v) \sin\frac{2\pi v}{q} \right) \times L\left(1, \overline{\chi_1\chi_4}\right) + O(1) \right) \\
&\quad + \sum_{\substack{\chi_1(-1)=-1\\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q)\chi_1(q) \\
&\quad \times \left(\frac{\tau(\chi\chi_1)}{(q-1)\pi i} \sum_{u=1}^{q-1} \sum_{\chi_3 \bmod q} \left(\chi_3(u) \left(1 - \cos\frac{2\pi u}{q}\right) \right) \times L\left(1, \overline{\chi\chi_1\chi_3}\right) + O(1) \right) \\
&\quad \times \left(\frac{\tau(\chi_1)}{(q-1)\pi i} \sum_{v=1}^{q-1} \sum_{\chi_4 \bmod q} \left(\chi_4(v) \left(1 - \cos\frac{2\pi v}{q}\right) \right) \times L\left(1, \overline{\chi_1\chi_4}\right) + O(1) \right) \\
&= \frac{1}{(q-1)^2\pi^2} \sum_{u=1}^{q-1} \sum_{v=1}^{q-1} \sum_{\chi_3 \bmod q} \sum_{\chi_4 \bmod q} \chi_3(u)\chi_4(v) \sin\frac{2\pi u}{q} \sin\frac{2\pi v}{q} \\
&\quad \times \sum_{\substack{\chi_1 \neq \chi_p^0\\ \chi_1(-1)=1}} \chi\chi_1(q)\chi_1(m)\tau(\chi\chi_1)\tau(\chi_1) \times L\left(1, \overline{\chi\chi_1\chi_3}\right) L\left(1, \overline{\chi_1\chi_4}\right) \\
&\quad + \frac{-1}{(q-1)^2\pi^2} \times \sum_{u=1}^{q-1} \sum_{v=1}^{q-1} \sum_{\chi_3 \bmod q} \sum_{\chi_4 \bmod m} \chi_3(u)\chi_4(v) \times \left(1 - \cos\frac{2\pi u}{q}\right)\left(1 - \cos\frac{2\pi v}{q}\right) \\
&\quad \times \sum_{\substack{\chi_1 \neq \chi_p^0\\ \chi_1(-1)=-1}} \chi\chi_1(q)\chi_1(q)\tau(\chi\chi_1)\tau(\chi_1) \times L\left(1, \overline{\chi\chi_1\chi_3}\right) L\left(1, \overline{\chi_1\chi_4}\right) + O\left(p^{1/2+\epsilon}\right) \\
&\ll p^{3/2+\epsilon},
\end{aligned} \tag{35}$$

where the $\ll$ constant is only concerned with q.

When $\chi(-1) = -1$, by the similar method, we can also obtain

$$\sum_{\substack{\chi_1 \bmod p\\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q)\chi_1(q) \sum_{a \le (p-1)/q} \chi\chi_1(a) \sum_{b \le (p-1)/q} \chi_1(b) \ll p^{3/2+\epsilon}, \tag{36}$$

where the $\ll$ constant is only concerned with q. Combining (35) and (36), we can obtain Lemma 7. This completes the proof of Lemma 7. ☐

Lemma 8. *Let q be a fixed odd prime and let p be a prime with $p > q$, let χ denote the nonprincipal Dirichlet character modulo p, let χ_1, χ_2 denote the Dirichlet character modulo p, q respectively, then, we have*

$$\sum_{\substack{\chi_1 \bmod p\\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q\\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \le p-1\\ q \nmid a}} \sum_{\substack{b \le p-1\\ q \nmid b}} \chi\chi_1\chi_2(a)\chi_1\overline{\chi}_2(b) \ll p^{3/2+\epsilon}, \tag{37}$$

where the $\ll$ constant is only concerned with q.

Proof. According to the properties of Dirichlet character, we can get

$$\begin{aligned}
&\sum_{\substack{\chi_1 \bmod p\\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q\\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \le p-1\\ q \nmid a}} \sum_{\substack{b \le p-1\\ q \nmid b}} \chi\chi_1\chi_2(a)\chi_1\overline{\chi}_2(b) \\
&= \sum_{\substack{\chi_1 \bmod p\\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q\\ \chi_2 \neq \chi_q^0}} \left(\sum_{a \le p-1} \chi\chi_1\chi_2(a) - \sum_{\substack{a \le p-1\\ q \mid a}} \chi\chi_1\chi_2(a) \right) \\
&\quad \times \left(\sum_{b \le p-1} \chi_1\overline{\chi}_2(b) - \sum_{\substack{b \le p-1\\ q \mid b}} \chi_1\overline{\chi}_2(b) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \left(\sum_{a \le p-1} \chi\chi_1\chi_2(a) - \chi\chi_1\chi_2(q) \times \sum_{a \le (p-1)/q} \chi\chi_1\chi_2(a) \right) \\
&\quad \times \left(\sum_{b \le p-1} \chi_1\overline{\chi}_2(b) - \chi_1\overline{\chi}_2(q) \sum_{b \le (p-1)/q} \chi_1\overline{\chi}_2(b) \right) \\
&= \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{a \le p-1} \chi\chi_1\chi_2(a) \sum_{b \le p-1} \chi_1\overline{\chi}_2(b).
\end{aligned} \tag{38}
$$

For primes p and q, let $\chi' = \chi\chi_1$ be a nonprincipal and primitive character modulo p, χ_2 is also a primitive character modulo q, so $\chi\chi_1\chi_2$ is a primitive character modulo pq; therefore, from Lemmas 3, 4, and 6 and from (38), it is clear that when $\chi(-1) = 1$, we can obtain

$$
\begin{aligned}
&\sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \le p-1 \\ q \nmid a}} \sum_{\substack{b \le p-1 \\ q \nmid b}} \chi\chi_1\chi_2(a)\,\chi_1\overline{\chi}_2(b) \\
&= \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0 \\ \chi_1\chi_2(-1)=1}} \\
&\quad \times \left(\frac{\tau(\chi\chi_1\chi_2)}{(q-1)\pi} \sum_{u=1}^{q-1} \sum_{\chi_3 \bmod q} \left(\chi_3(u) \sin\frac{2\pi u}{q} \right) \times L\left(1, \overline{\chi\chi_1\chi_2\chi_3}\right) + O(1) \right) \\
&\quad \times \left(\frac{\tau(\chi_1\overline{\chi}_2)}{(q-1)\pi} \sum_{v=1}^{q-1} \sum_{\chi_4 \bmod q} \left(\chi_4(v) \sin\frac{2\pi v}{q} \right) \times L\left(1, \overline{\chi}_1\chi_2\overline{\chi}_4\right) + O(1) \right) \\
&\quad + \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0 \\ \chi_1\chi_2(-1)=-1}} \\
&\quad \times \left(\frac{\tau(\chi\chi_1\chi_2)}{(q-1)\pi i} \sum_{u=1}^{q-1} \sum_{\chi_3 \bmod q} \left(\chi_3(u) \left(1 - \cos\frac{2\pi u}{q}\right) \right) \times L\left(1, \overline{\chi\chi_1\chi_2\chi_3}\right) + O(1) \right) \\
&\quad \times \left(\frac{\tau(\chi_1\overline{\chi}_2)}{(q-1)\pi i} \sum_{v=1}^{q-1} \sum_{\chi_4 \bmod q} \left(\chi_4(v) \left(1 - \cos\frac{2\pi v}{q}\right) \right) \times L\left(1, \overline{\chi}_1\chi_2\overline{\chi}_4\right) + O(1) \right) \\
&= \frac{1}{(q-1)^2\pi^2} \sum_{u=1}^{q-1} \sum_{v=1}^{q-1} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\chi_3 \bmod q} \sum_{\chi_4 \bmod q} \\
&\quad \times \chi_2(p)\,\tau(\chi_2)\,\overline{\chi}_2(p)\,\tau(\overline{\chi}_2)\,\chi_3(u)\,\chi_4(v) \sin\frac{2\pi u}{q} \\
&\quad \times \sin\frac{2\pi v}{q} \sum_{\substack{\chi_1 \neq \chi_p^0 \\ \chi_1\chi_2(-1)=1}} \chi\chi_1(q)\,\chi_1(q)\,\tau(\chi\chi_1)\,\tau(\chi_1) \\
&\quad \times L\left(1, \overline{\chi\chi_1\chi_2\chi_3}\right) L\left(1, \overline{\chi}_1\chi_2\overline{\chi}_4\right) \\
&\quad + \frac{-1}{(q-1)^2\pi^2} \sum_{u=1}^{q-1} \sum_{v=1}^{q-1} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\chi_3 \bmod q} \sum_{\chi_4 \bmod q} \\
&\quad \times \chi_2(p)\,\tau(\chi_2)\,\overline{\chi}_2(p)\,\tau(\overline{\chi}_2)\,\chi_3(u)\,\chi_4(v) \\
&\quad \times \left(1 - \cos\frac{2\pi u}{q}\right)\left(1 - \cos\frac{2\pi v}{q}\right) \\
&\quad \times \sum_{\substack{\chi_1 \neq \chi_p^0 \\ \chi_1\chi_2(-1)=-1}} \chi\chi_1(q)\,\chi_1(q)\,\tau(\chi\chi_1)\,\tau(\chi_1) \\
&\quad \times L\left(1, \overline{\chi\chi_1\chi_2\chi_3}\right) L\left(1, \overline{\chi}_1\chi_2\overline{\chi}_4\right) \\
&\ll p^{3/2+\epsilon},
\end{aligned} \tag{39}
$$

where the $\ll$ constant is only concerned with q.

When $\chi(-1) = -1$, in the similar way, we can also obtain

$$
\sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \le p-1 \\ q \nmid a}} \sum_{\substack{b \le p-1 \\ q \nmid b}} \chi\chi_1\chi_2(a)\,\chi_1\overline{\chi}_2(b) \ll p^{3/2+\epsilon}, \tag{40}
$$

where the $\ll$ constant is only concerned with q.

Therefore, from (39) and (40), we can easily get Lemma 8. This completes the proof of Lemma 8. □

3. Proof of Theorem

In this section, we will complete the proof of the theorem. According to the orthogonality relation for character sums, we have

$$
\begin{aligned}
\sum_{\substack{a \le p \\ a \in E(q,p)}} \chi(a) &= \sum_{b=1}^{p-1} \sum_{\substack{a=1 \\ a \equiv b (\bmod q) \\ ab \equiv 1 (\bmod p)}}^{p-1} \chi(a) \\
&= \frac{1}{p-1} \sum_{\chi_1 \bmod p} \sum_{b=1}^{p-1} \sum_{\substack{a=1 \\ a \equiv b (\bmod q)}}^{p-1} \chi\chi_1(a)\,\chi_1(b)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{p-1} \sum_{\substack{a=1 \\ a \equiv b (\bmod q)}}^{p-1} \sum_{b=1}^{p-1} \chi(a) \\
&\quad + \frac{1}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{a=1 \\ a \equiv b (\bmod q)}}^{p-1} \sum_{b=1}^{p-1} \chi\chi_1(a)\chi_1(b) \\
&= \frac{1}{p-1} \sum_{a=1}^{p-1} \chi(a) \left(\sum_{\substack{b=1 \\ b \equiv a (\bmod q)}}^{p-1} 1 \right) \\
&\quad + \frac{1}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{a=1 \\ q \mid a}}^{p-1} \sum_{\substack{b=1 \\ q \mid b}}^{p-1} \chi\chi_1(a)\chi_1(b) \\
&\quad + \frac{1}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{a=1 \\ q \nmid a \\ a \equiv b (\bmod q)}}^{p-1} \sum_{\substack{b=1 \\ q \nmid b}}^{p-1} \chi\chi_1(a)\chi_1(b) \\
&= \frac{1}{p-1} \sum_{a=1}^{p-1} \chi(a) \left(\frac{p}{q} + O(1) \right) \\
&\quad + \frac{1}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q)\chi_1(q) \\
&\quad \times \sum_{a \leq (p-1)/q} \chi\chi_1(a) \sum_{b \leq (p-1)/q} \chi_1(b) \\
&\quad + \frac{1}{(p-1)\phi(q)} \\
&\quad \times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\chi_2 \bmod q} \sum_{\substack{a \leq p-1 \\ q \nmid a}} \sum_{\substack{b \leq p-1 \\ q \nmid b}} \chi\chi_1(a)\chi_1(b) \\
&\qquad \times \chi_2(a)\overline{\chi}_2(b) \\
&= \frac{p}{(p-1)q} \sum_{a=1}^{p-1} \chi(a) + O(1) \\
&\quad + \frac{1}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q)\chi_1(q) \\
&\quad \times \sum_{a \leq (p-1)/q} \chi\chi_1(a) \sum_{b \leq (p-1)/q} \chi_1(b) \\
&\quad + \frac{1}{(p-1)\phi(q)} \\
&\quad \times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{a \leq p-1 \\ m \nmid a}} \sum_{\substack{b \leq p-1 \\ q \nmid b}} \chi\chi_1(a)\chi_1(b) \\
&\quad + \frac{1}{(p-1)\phi(q)} \\
&\quad \times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \leq p-1 \\ q \nmid a}} \sum_{\substack{b \leq p-1 \\ q \nmid b}} \chi\chi_1(a)\chi_1(b) \\
&\qquad \times \chi_2(a)\overline{\chi}_2(b) \\
&= O(1) + \frac{1}{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q)\chi_1(q) \\
&\quad \times \sum_{a \leq (p-1)/q} \chi\chi_1(a) \sum_{b \leq (p-1)/q} \chi_1(b) \\
&\quad + \frac{1}{(p-1)\phi(q)} \\
&\quad \times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{a \leq p-1 \\ q \nmid a}} \sum_{\substack{b \leq p-1 \\ q \nmid b}} \chi\chi_1(a)\chi_1(b) \\
&\quad + \frac{1}{(p-1)\phi(q)} \\
&\quad \times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \leq p-1 \\ q \nmid a}} \sum_{\substack{b \leq p-1 \\ q \nmid b}} \chi\chi_1\chi_2(a)\chi_1\overline{\chi}_2(b).
\end{aligned}
\tag{41}
$$

Note that for any nonprincipal Dirichlet character χ modulo k ($k \geq 3$ is an positive integer), we have $\sum_{n=1}^{k} \chi(n) = 0$, hence, we obtain

$$
\begin{aligned}
&\sum_{\substack{a \leq p-1 \\ q \nmid a}} \sum_{\substack{b \leq p-1 \\ q \nmid b}} \chi\chi_1(a)\chi_1(b) \\
&\quad = \left(\sum_{a \leq p-1} \chi\chi_1(a) - \sum_{\substack{a \leq p-1 \\ q \mid a}} \chi\chi_1(a) \right) \\
&\qquad \times \left(\sum_{b \leq p-1} \chi_1(b) - \sum_{\substack{b \leq p-1 \\ q \mid b}} \chi_1(b) \right) \\
&\quad = \sum_{\substack{a \leq p-1 \\ q \mid a}} \chi\chi_1(a) \sum_{\substack{b \leq p-1 \\ q \mid b}} \chi_1(b) \\
&\quad = \chi\chi_1(q)\chi_1(q) \sum_{a \leq (p-1)/q} \chi\chi_1(a) \sum_{b \leq (p-1)/q} \chi_1(b).
\end{aligned}
\tag{42}
$$

From (41) and (42) and Lemmas 7 and 8, we get

$$
\begin{aligned}
&\sum_{\substack{a \leq p \\ a \in E(q,p)}} \chi(a) \\
&\quad = \frac{1}{p-1} \left(1 + \frac{1}{\phi(q)} \right)
\end{aligned}
$$

$$\times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q)\,\chi_1(q) \sum_{a \le (p-1)/q} \chi\chi_1(a) \sum_{b \le (p-1)/q} \chi_1(b)$$
$$+ \frac{1}{(p-1)\,\phi(q)}$$
$$\times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \le p-1 \\ q \dagger a}} \sum_{\substack{b \le p-1 \\ q \dagger b}} \chi\chi_1\chi_2(a)\,\chi_1\overline{\chi}_2(b) + O(1)$$
$$\ll p^{1/2+\epsilon}, \tag{43}$$

where the $\ll$ constant only depends on q. This completes the proof of Theorem.

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