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Research Article

Solving Integral Representations Problems for the Stationary Schrödinger Equation

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When solutions of the stationary Schrödinger equation in a half-space belong to the weighted Lebesgue classes, we give integral representations of them, which imply known representation theorems of classical harmonic functions in a half-space.

1. Introduction and Results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the sets of all real numbers and of all positive real numbers, respectively. Let $\mathbf{R}^n$ $(n \geq 2)$ denote the n-dimensional Euclidean space with points $x = (x', x_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}_+^{n-1}$, respectively. The boundary and closure of an open set D of $\mathbf{R}^n$ are denoted by ∂D and $\overline{D}$, respectively. The upper half space is the set $H = \{(x', x_n) \in \mathbf{R}^n;\ x_n > 0\}$, whose boundary is ∂H.

We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points $x, y \in \mathbf{R}^n$ as $x = (x', x_n)$, $y = (y', y_n)$, where $y' = (y_1, y_2, \ldots, y_{n-1}) \in \mathbf{R}^{n-1}$ and putting

$$x \cdot y = \sum_{j=1}^{n} x_j y_j, \qquad |x| = \sqrt{x \cdot x},$$
$$\Theta = \frac{x}{|x|}, \qquad \Phi = \frac{y}{|y|}. \tag{1}$$

For $x \in \mathbf{R}^n$ and $r > 0$, let $B(x, r)$ denote the open ball with center at x and radius $r (> 0)$ in $\mathbf{R}^n$. $S_r = \partial B(O, r)$, where O is the origin of $\mathbf{R}^n$. For a set E, $E \subset \mathbf{R}_+ \cup \{0\}$, we denote $\{x \in H; |x| \in E\}$ and $\{x \in \partial H; |x| \in E\}$ by HE and ∂HE, respectively. By Hr we denote $H \cap Sr$. We denote by dS_r the $(n-1)$-dimensional volume elements induced by the Euclidean metric on S_r.

Let $\mathscr{A}_b$ denote the class of nonnegative radial potentials $b(x)$, that is, $0 \leq b(x) = b(|x|)$, $x \in H$, such that $b \in L^a_{\mathrm{loc}}(H)$ with some $a > n/2$ if $n \geq 4$ and with $a = 2$ if $n = 2$ or $n = 3$.

This paper is devoted to the stationary Schrödinger equation:

$$SSE_b h(x) = -\Delta h(x) + b(x) h(x) = 0, \tag{2}$$

where $x \in H$, Δ is the Laplace operator and $b \in \mathscr{A}_b$. Note that solutions of (2) are (classical) harmonic functions in the case $b = 0$. Under these assumptions the operator SSE_b can be extended in the usual way from the space $C_0^\infty(H)$ to an essentially self-adjoint operator on $L^2(H)$ (see [1]). We will denote it by SSE_b as well. This last one has a Green function $G_b(x, y)$. Here $G_b(x, y)$ is positive on H and its inner normal derivative $\partial G_b(x, y)/\partial n(y') \geq 0$. We denote this derivative by $P_b(x, y')$, which is called the Poisson b-kernel with respect to H. If $G(x, y)$ and $P(x, y')$ are denoted by the Green function and Poisson kernel of the Laplace operator in H, respectively, then

$$P(x, y') = -\left.\frac{\partial G(x, y)}{\partial y_n}\right|_{y_n = 0} = \frac{2x_n}{w_n} \frac{1}{|x - y'|^n}, \tag{3}$$

where $x = (x', x_n)$, $y = (y', y_n)$ and w_n is the area of the unit sphere in $\mathbf{R}^n$.

Let g be a continuous function on ∂H. We say that h is a solution of the Dirichlet problem for the stationary Schrödinger operator SSE_b on H with g, if

$$\begin{aligned} &h \in C^2(H) \cap C^0(\overline{H}), \\ &SSE_b h = 0 \quad \text{in } H, \\ &h = g \quad \text{on } \partial H. \end{aligned} \tag{4}$$

Note that h is a solution of the classical Dirichlet problem for the Laplace operator Δ on H with g in the case $b = 0$.

Let Λ be a Laplace-Beltrami operator (spherical part of the Laplace) on the unit sphere. It is known (see, e.g., [2, page 41]) that the eigenvalue problem

$$\begin{aligned} \Lambda\varphi(\Theta) + \tau\varphi(\Theta) &= 0 \quad \Theta \in \mathbf{S}_+^{n-1}, \\ \varphi(\Theta) &= 0 \quad \Theta \in \partial\mathbf{S}_+^{n-1} \end{aligned} \tag{5}$$

has the eigenvalues $\tau_j = j(j+n-2)$, where $j = 0, 1, 2 \ldots$. Corresponding eigenfunctions are denoted by φ_{jv} $(1 \le v \le v_j)$, where v_j is the multiplicity of τ_j. We norm the eigenfunctions in $L^2(\mathbf{S}_+^{n-1})$ and $\varphi_1 = \varphi_{11} > 0$.

Let $P_j(r)$ and $Q_j(r)$ stand, respectively, for the increasing and nonincreasing, as $r \to +\infty$, solutions of the equation:

$$\begin{aligned} &-T''(r) - \frac{n-1}{r}T'(r) \\ &\quad + \left(\frac{\tau_j}{r^2} + b(r)\right)T(r) = 0, \quad 0 < r < \infty, \end{aligned} \tag{6}$$

normalized under the condition $P_j(1) = Q_j(1) = 1$.

We shall also consider the class $\mathscr{B}_b$, consisting of the potentials $b \in \mathscr{A}_b$ such that there exists a finite limit $\lim_{r\to\infty} r^2 b(r) = s \in [0, \infty)$; moreover, $r^{-1}|r^2 b(r) - s| \in L(1, \infty)$. If $b \in \mathscr{B}_b$, then solutions of (2) are continuous (see [3]).

In the rest of paper, we assume that $b \in \mathscr{B}_b$ and we shall suppress this assumption for simplicity. Further, we use the standard notations $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$, and $[d]$ is the integer part of d and $d = [d] + \{d\}$, where d is a positive real number.

Denote

$$\lambda_{j,s}^{\pm} = \frac{2 - n \pm \sqrt{(n-2)^2 + 4(s + \tau_j)}}{2} \qquad (j = 0, 1, 2, 3, \ldots). \tag{7}$$

Remark 1. $\lambda_{j,0}^{+} = j$ in the case $b = 0$, where $j = 0, 1, 2, 3, \ldots$.

It is known (see [4]) that in the case under consideration the solutions to (6) have the asymptotics

$$P_j(r) \sim d_1 r^{\lambda_{j,s}^{+}}, \quad Q_j(r) \sim d_2 r^{\lambda_{j,s}^{-}}, \quad \text{as} \quad r \longrightarrow \infty, \tag{8}$$

where d_1 and d_2 are some positive constants.

If $b \in \mathscr{A}_b$, it is known that the following expansion for the Green function $G_b(x, y)$ (see [5, Chapter 11], [6]):

$$\begin{aligned} G_b(x, y) = \sum_{j=0}^{\infty} \frac{1}{\chi'(1)} P_j(\min(|x|, |y|)) \\ \times Q_j(\max(|x|, |y|)) \left(\sum_{v=1}^{v_j} \varphi_{jv}(\Theta)\varphi_{jv}(\Phi)\right), \end{aligned} \tag{9}$$

where $|x| \neq |y|$ and $\chi'(1) = w(Q_1(r), P_1(r))|_{r=1}$ is its Wronskian. The series converges uniformly if either $|x| \le k|y|$ or $|y| \le k|x|$, where $0 < k < 1$.

For a nonnegative integer m and two points $x, y \in H$, we define a modified Green function:

$$G(b, m)(x, y) = G_b(x, y) - V(b, m)(x, y), \tag{10}$$

where

$$V(b, m)(x, y) = \begin{cases} 0 & \text{if } |y| < 1, \\ \displaystyle\sum_{j=0}^{m} \frac{1}{\chi'(1)} P_j(|x|) Q_j(|y|) \times \left(\sum_{v=1}^{v_j} \varphi_{jv}(\Theta)\varphi_{jv}(\Phi)\right) & \text{if } 1 \le |y| < \infty. \end{cases} \tag{11}$$

If the generalized Poisson kernel $P(b, m)(x, y')$ with respect to H is defined by

$$P(b, m)(x, y') = \frac{\partial G(b, m)(x, y)}{\partial n(y')}, \tag{12}$$

then we have $P(b, 0)(x, y') = P_b(x, y')$ and $P(0, m)(x, y')$ coincides with ones in Finkelstein and Scheinberg [7], Siegel and Talvila [8], Deng [9], Qiao and Deng [10], and Qiao [11] (see [5, Chapter 11]).

Put

$$U(b, m; u)(x) = \int_{\partial H} P(b, m)(x, y') u(y') \, dy', \tag{13}$$

where $u(y')$ is a continuous function on ∂H.

For real numbers $\beta \ge 1$, we denote $\mathscr{C}(\beta, b)$ the class of all measurable functions $f(y)$ satisfying the following inequality:

$$\int_H \frac{|f(y)| \varphi_1}{1 + P_{[\beta]}(|y|) |y|^{n+\{\beta\}}} dy < \infty, \tag{14}$$

and the class $\mathscr{D}(\beta, b)$ consists of all measurable functions $g(y')$ $(y' \in \partial H)$ satisfying

$$\int_{\partial H} \frac{|g(y')| P_1(|y'|) Q_1(|y'|)}{1 + \chi'(|y'|) P_{[\beta]}(|y'|) |y'|^{n+\{\beta\}-1}} \frac{\partial \varphi_1}{\partial n} dy' < \infty. \tag{15}$$

We will also consider the class of all continuous functions $h(y)$, which is the solution of (2), with $h^+(y) \in \mathscr{C}(\beta, b)$ and $h^+(y') \in \mathscr{D}(\beta, b)$, is denoted by $\mathscr{E}(\beta, b)$.

Remark 2. If $b = 0$ and $\beta = \alpha + 1$, then (14) and (15) are equivalent to

$$\int_H \frac{y_n |f(y)|}{1+|y|^{n+\alpha+2}} dy < \infty,$$
$$\int_{\partial H} \frac{|g(y')|}{1+|y'|^{n+\alpha}} dy' < \infty, \tag{16}$$

respectively, from Remark 1 and (14), which yield that $\mathscr{E}(\alpha+1, 0)$ is equivalent to $(CH)_\alpha$ in the notation of [9].

Let us recall the classical case $b = 0$. If $h(x) \geq 0$ is harmonic in H, continuous on $\overline{H}$, and $h \in \mathscr{E}(1,0)$, then there exists a constant $d_3 \geq 0$ such that (see [12])

$$h(x) = d_3 x_n + \int_{\partial H} P(x, y') u(y') dy', \tag{17}$$

where $x = (x', x_n) \in H$.

Deng (see [9]) has constructed a similar representation to (17) for $h \in \mathscr{E}(\alpha+1, 0) (\alpha \geq 0)$, which is the integral with a modified Poisson kernel derived by subtracting some special harmonic polynomials from $P(x, y')$. By virtue of this modified Poisson kernel, Qiao (see [11, 13]) and Qiao and Deng (see [14–18]) have constructed different integral representations for harmonic functions of finite order and infinite order.

Especially, Su (see [6, 19]) recently writes solutions to the half-space Dirichlet problem with respect to the stationary Schrödinger operator SSE_b. Now we state our main results as follows.

Theorem 3. *If $h \in \mathscr{E}(\beta, b)$, then $h \in \mathscr{D}(\beta, b)$.*

Theorem 4. *If $h \in \mathscr{E}(\beta, b)$, m is an integer such that $P_m(|y|) < P_{[\beta]}(|y|)|y|^{\{\beta\}} \leq P_{m+1}(|y|)(|y| \geq 1)$, and then the following properties hold.*

(I) *If $\beta = 1$, then there exists a constant d_4 such that*

$$h(x) = d_4 P_1(|x|) \varphi_1(\Theta) + U(b, 0; h)(x) \tag{18}$$

for $x \in H$.

(II) *If $\beta > 1$, then we have $h(x) = U(b, m; h)(x) + u(x)$, where*

$$u(x) = \sum_{j=0}^{m} \left(\sum_{\nu=1}^{\nu_j} d_{j\nu} \varphi_{j\nu}(\Theta) \right) P_j(|x|) \tag{19}$$

vanishing continuously on ∂H, $x = (x', x_n) \in H$ and $d_{j\nu}$ are constants.

2. Lemmas

The following Lemma plays an important role in our discussions, which is due to Levin and Kheyfits (see [5, page 356]).

Lemma 5. *If $R > 1$ and $h(y)$ is a solution of (2) on a domain containing $H(1, R)$, then*

$$\int_{HR} \frac{\chi'(R)}{P_1(R)} h\varphi_1(\Phi) dS_R + \int_{\partial H(1,R)} h(y') \frac{\partial \varphi_1}{\partial n} W(|y'|) dy'$$
$$+ d_5 + d_6 \frac{Q_1(R)}{P_1(R)} = 0, \tag{20}$$

where

$$W(|y'|) = Q_1(|y'|) - \frac{Q_1(R)}{P_1(R)} P_1(|y'|),$$
$$d_5 = \int_{H1} h\varphi_1(\Phi) Q_1'(r) - Q_1(1) \varphi_1(\Phi) \frac{\partial h}{\partial n} dS_1, \tag{21}$$
$$d_6 = \int_{H1} P_1(1) \varphi_1(\Phi) \frac{\partial h}{\partial n} - h\varphi_1(\Phi) P_1'(1) dS_1.$$

Lemma 6 (see [6, Corollary 1.6]). *If u is a continuous function on ∂H satisfying*

$$\int_{\partial H} \frac{|u(y')|}{1+|y'|^{\lambda^+_{m+1,s}+n-1}} dy' < \infty, \tag{22}$$

then $U(b, m; u)(x)$ is a solution of the Dirichlet problem for SSE_b on H with h and

$$\lim_{|x|\to\infty, x\in H} |x|^{-\lambda^+_{m+1,s}} U(b, m; u)(x) = 0. \tag{23}$$

Lemma 7 (see [6, Lemma 2.1] or [20, Theorem 1]). *If $u(x)$ is a solution of (2) on H satisfying*

$$\lim_{|x|\to\infty,\, x\in H} |x|^{-\lambda^+_{m+1,s}} u^+(x) = 0, \tag{24}$$

then (19) holds.

3. Proof of Theorem 3

We apply the formula (20) to $h = h^+ - h^-$ in $H(1, R)$:

$$m_+(R) + \int_{\partial H(1,R)} h^+(y') W(|y'|) \frac{\partial \varphi_1}{\partial n} dy' + d_5 + \frac{Q_1(R)}{P_1(R)} d_6$$
$$= m_-(R) + \int_{\partial H(1,R)} h^-(y') W(|y'|) \frac{\partial \varphi_1}{\partial n} dy', \tag{25}$$

where

$$m_\pm(R) = \int_{HR} \frac{\chi'(R)}{P_1(R)} h^\pm \varphi_1 dS_R. \tag{26}$$

Since $u \in \mathscr{E}(\beta, b)$, we obtain by (8)

$$\begin{aligned}&\int_1^\infty \frac{m_+(R)\,P_1(R)}{\chi'(R)\,P_{[\beta]}(R)\,R^{n+\{\beta\}}}dR \\ &\quad = \int_{H(1,\infty)} \frac{h^+\varphi_1}{P_{[\beta]}(|y|)\,|y|^{n+\{\beta\}}}dy \\ &\quad \leq 2\int_H \frac{h^+\varphi_1}{1+P_{[\beta]}(|y|)\,|y|^{n+\{\beta\}}}dy < \infty.\end{aligned} \tag{27}$$

From (15), we conclude that

$$\begin{aligned}&\int_1^\infty \frac{P_1(R)}{\chi'(R)\,P_{[\beta]}(R)\,R^{n+\{\beta\}}} \\ &\quad \times \int_{\partial H(1,R)} h^+(y')\,W(|y'|)\,\frac{\partial\varphi_1}{\partial n}dy'dR \\ &\quad = \int_{\partial H(1,\infty)} h^+(y')\,P_1(|y'|) \\ &\qquad \times \int_{|y'|}^\infty \frac{P_1(R)}{\chi'(R)\,P_{[\beta]}(R)\,R^{n+\{\beta\}}} \\ &\qquad \times \left(\frac{Q_1(|y'|)}{P_1(|y'|)} - \frac{W_1(R)}{P_1(R)}\right)dR\frac{\partial\varphi_1}{\partial n}dy' \\ &\quad \leq M\int_{\partial H(1,\infty)} \frac{h^+(y')\,P_1(|y'|)\,Q_1(|y'|)}{\chi'(|y'|)\,P_{[\beta]}(|y'|)\,|y'|^{n+\{\beta\}-1}}\frac{\partial\varphi_1}{\partial n}dy' \\ &\quad \leq M\int_{\partial H} \frac{h^+(y')\,P_1(|y'|)\,Q_1(|y'|)}{1+\chi'(|y'|)\,P_{[\beta]}(|y'|)\,|y'|^{n+\{\beta\}-1}}\frac{\partial\varphi_1}{\partial n}dy' \\ &\quad < \infty.\end{aligned} \tag{28}$$

Combining (25), (27), and (28), we obtain

$$\begin{aligned}&\int_1^\infty \frac{P_1(R)}{\chi'(R)\,P_{[\beta]}(R)\,R^{n+(\{\beta\}/2)}} \\ &\quad \times \int_{\partial H(1,R)} h^-(y')\,W(|y'|)\,\frac{\partial\varphi_1}{\partial n}dy'dR \\ &\quad \leq \int_1^\infty \frac{m_+(R)\,P_1(R)}{\chi'(R)\,P_{[\beta]}(R)\,R^{n+(\{\beta\}/2)}}dR \\ &\quad + \int_1^\infty \frac{P_1(R)}{\chi'(R)\,P_{[\beta]}(R)\,R^{n+(\{\beta\}/2)}} \\ &\quad \times \int_{\partial H(1,R)} h^+(y')\,W(|y'|)\,\frac{\partial\varphi_1}{\partial n}dy'dR \\ &\quad + \int_1^\infty \frac{1}{\chi'(R)\,P_{[\beta]}(R)\,R^{n+(\{\beta\}/2)}} \\ &\qquad \times (P_1(R)\,d_5 + Q_1(R)\,d_6)\,dR \\ &\quad < \infty.\end{aligned} \tag{29}$$

Set

$$\begin{aligned}\mathscr{H}(\beta) &= \lim_{|y'|\to\infty} \frac{\chi'(|y'|)\,P_{[\beta]}(|y'|)\,|y'|^{n+\{\beta\}-1}}{Q_1(|y'|)} \\ &\quad \times \int_{|y'|}^\infty \frac{P_1(R)}{\chi'(R)\,P_{[\beta]}(R)\,R^{n+(\{\beta\}/2)}} \\ &\qquad \times \left(\frac{Q_1(|y'|)}{P_1(|y'|)} - \frac{W_1(R)}{P_1(R)}\right)dR \\ &= \lim_{|y'|\to\infty} |y'|^{\lambda^+_{[\beta],s}+\lambda^+_{1,s}+n+\{\beta\}-2} \\ &\quad \times \int_{|y'|}^\infty \frac{1}{R^{\lambda^+_{[\beta],s}-\lambda^+_{1,s}+(\{\beta\}/2)+1}}\left(\frac{1}{|y'|^{\chi_{1,s}}} - \frac{1}{R^{\chi_{1,s}}}\right)dR,\end{aligned} \tag{30}$$

where $\chi_{1,s} = \lambda^+_{1,s} - \lambda^-_{1,s}$.

By the L'hospital's rule, we have

$$\mathscr{H}(\beta) = \begin{cases} \dfrac{\chi_{1,s}}{\left(\lambda^+_{[\beta],s} - \lambda^+_{1,s}\right)\left(\lambda^+_{[\beta],s} + \lambda^+_{1,s} + n - 2\right)} & \text{if } \{\beta\} = 0, \\ +\infty & \text{if } \{\beta\} \neq 0, \end{cases} \tag{31}$$

which yields that there exists a positive constant M such that, for any $|y'| \geq 1$,

$$\begin{aligned}&\int_{|y'|}^\infty \frac{P_1(R)}{\chi'(R)\,P_{[\beta]}(R)\,R^{n+(\{\beta\}/2)}}W(|y'|)\,dR \\ &\quad \geq \frac{MP_1(|y'|)\,Q_1(|y'|)}{\chi'(|y'|)\,P_{[\beta]}(|y'|)\,|y'|^{n+\{\beta\}-1}}.\end{aligned} \tag{32}$$

Then

$$\begin{aligned}&M\int_{\partial H(1,\infty)} \frac{h^-(y')\,P_1(|y'|)\,Q_1(|y'|)}{\chi'(|y'|)\,P_{[\beta]}(|y'|)\,|y'|^{n+\{\beta\}-1}}\frac{\partial\varphi_1}{\partial n}dy' \\ &\quad \leq \int_{\partial H(1,\infty)} h^- \int_{|y'|}^\infty \frac{P_1(R)}{\chi'(R)\,P_{[\beta]}(R)\,R^{n+(\{\beta\}/2)}} \\ &\qquad \times W(|y'|)\,dR\frac{\partial\varphi_1}{\partial n}dy' \\ &\quad < \infty,\end{aligned} \tag{33}$$

which shows that $h \in \mathscr{D}(\beta, b)$ from $|h| = h^+ + h^-$. Then Theorem 3 is proved.

4. Proof of Theorem 4

To prove (II), notice that $P_m(|y|) < P_{[\beta]}(|y|)|y|^{\{\beta\}} \leq P_{m+1}(|y|)$ ($|y| \geq 1$) and condition (15) is stronger than condition (22) from Theorem 3.

Consider the function $h(x) - U(a, m; h)(x)$. Then it follows from Lemma 6 and Theorem 3 that this is a solution of (2) in H and vanishes continuously on ∂H.

Then

$$0 \leq (h(x) - U(b, m; h)(x))^{+} \leq h^{+}(x) + (U(b, m; h))^{-}(x) \tag{34}$$

for any $P \in H$. Further, (8) gives that

$$\lim_{|x| \to \infty, x \in H} |x|^{-\lambda^{+}_{m+1,s}} h^{+}(x) = 0, \tag{35}$$

which together with (34) and Lemmas 6 and 7 give the result of (II).

If $u \in \mathscr{E}(1, b)$, then $h \in \mathscr{E}(\beta, b)$ for each $\beta > 1$ and there exists a constant d_7 such that

$$h(x) = d_7 P_1(|x|) \varphi_1(\Theta) + U(b, 1; h)(x) \tag{36}$$

for all $x \in H$. So if we take

$$d_4 = d_7 - \int_{\partial H[1,\infty)} P(b, 1)\left(0, y'\right) h\left(y'\right) dy', \tag{37}$$

we see that (18) holds for all $x \in H$.

Thus we complete the proof of Theorem 4.

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