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Research Article

A New Extension of Serrin's Lower Semicontinuity Theorem

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We present a new extension of Serrin's lower semicontinuity theorem. We prove that the variational functional $\int_\Omega f(x,u,u')dx$ defined on $W^{1,1}_{\mathrm{loc}}(\Omega)$ is lower semicontinuous with respect to the strong convergence in L^1_{loc}, under the assumptions that the integrand $f(x,s,\xi)$ has the locally absolute continuity about the variable x.

1. Introduction and Main Results

The aim of this paper is to give some new sufficient conditions for lower semicontinuity with respect to the strong convergence in L^1_{loc} for integral functionals

$$F(u,\Omega)=\int_\Omega f(x,u(x),Du(x))\,dx, \tag{1}$$

where Ω is an open set of R^n, $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$, defined on $W^{1,1}_{\mathrm{loc}}(\Omega)=\{u: u\in L^1(K), Du\in L^1(K), \text{for all } K\subset\subset\Omega\}$ [1], Du denotes the generalized gradient of u, and the integrand $f(x,s,\xi):\Omega\times R\times R^n\to[0,\infty)$ satisfies the following condition:

(H1) f is continuous in $\Omega\times R\times R^n$, and $f(x,s,\xi)$ is convex in $\xi\in R^n$ for any fixed $(x,s)\in\Omega\times R$.

The integral functional F is called lower semicontinuous in $W^{1,1}_{\mathrm{loc}}(\Omega)$ with respect to the strong convergence in L^1_{loc}, if, for every $u_m, u\in W^{1,1}_{\mathrm{loc}}(\Omega)$, such that $u_m\to u$ in L^1_{loc}, then

$$\liminf_{m\to+\infty}F(u_m,\Omega)\geq F(u,\Omega). \tag{2}$$

It is well known that condition (H1) is not sufficient for lower semicontinuity of the integral F in (1) (see book [2]). In addition to (H1), Serrin [3] proposed some sufficient conditions for lower semicontinuity of the integral F. One of the most known conclusions is the following one.

Theorem 1 (see [3]). *In addition to (H1), if f satisfies one of the following conditions:*

(a) $f(x,s,\xi)\to+\infty$ *when* $|\xi|\to+\infty$, *for all* $(x,s)\in\Omega\times R$,

(b) $f(x,s,\xi)$ *is strictly convex in* $\xi\in R^n$ *for all* $(x,s)\in\Omega\times R$,

(c) *the derivatives* $f_x(x,s,\xi)$, $f_\xi(x,s,\xi)$, *and* $f_{\xi x}(x,s,\xi)$ *exist and are continuous for all* $(x,s,\xi)\in\Omega\times R\times R^n$.

then $F(u,\Omega)$ is lower semicontinuous in $W^{1,1}_{\mathrm{loc}}(\Omega)$ with respect to the strong convergence in L^1_{loc}.

Conditions (a), (b), and (c) quoted above are clearly independent, in the sense that we can find a continuous function f satisfying just one of them but none of the others. Many scholars have weakened the conditions of integrand f and generalized Theorem 1, such as Ambrosio et al. [4], Cicco and Leoni [5], Fonseca and Leoni [6, 7]. In particular Gori et al. [8, 9] proved the following theorems.

Theorem 2 (see [8, 9]). *Let one assume that f satisfies (H1) and that, for every compact set $K\subset\Omega\times R\times R^n$, there exists a constant $L=L(K)$ such that*

$$\left|f_\xi(x_1,s,\xi)-f_\xi(x_2,s,\xi)\right|\leq L\left|x_1-x_2\right|, \quad \forall (x_1,s,\xi),\ (x_2,s,\xi)\in K, \tag{3}$$

and, for every compact set $K_1 \subset \Omega \times R$, there exists a constant $L_1 = L_1(K_1)$ such that

$$\left|f_\xi(x,s,\xi)\right| \le L_1, \quad \forall (x,s) \in K_1, \ \forall \xi \in R^n,$$

$$\left|f_\xi(x,s,\xi_1) - f_\xi(x,s,\xi_2)\right| \le L_1 \left|\xi_1 - \xi_2\right|, \quad \forall (x,s) \in K_1, \quad \forall \xi_1, \xi_2 \in R^n. \tag{4}$$

Then $F(u,\Omega)$ is lower semicontinuous in $W^{1,1}_{\mathrm{loc}}(\Omega)$ with respect to the strong convergence in L^1_{loc}.

Theorem 3 (see [8, 9]). *Let f satisfy (H1) such that, for every open set $\Omega' \times H \times K \subset\subset \Omega \times R \times R^n$, there exists a constant $L = L_{\Omega' \times H \times K}$ such that, for every $x_1, x_2 \in \Omega'$, $s \in H$, and $\xi \in K$,*

$$\left|f(x_1,s,\xi) - f(x_2,s,\xi)\right| \le L\left|x_1 - x_2\right|. \tag{5}$$

Then the functional $F(u,\Omega)$ is lower semicontinuous in $W^{1,1}_{\mathrm{loc}}(\Omega)$ with respect to the L^1_{loc} convergence.

Condition (5) means that f is locally Lipschitz continuous with respect to x, that is, the Lipschitz constant is not uniform for $(s,\xi) \in R \times R^n$. This is an improvement of (c) of Serrin's Theorem 1. Then a question arises, that is whether there are weaker enough conditions more than locally Lipschitz continuity. In this paper, we consider absolute continuity. Obviously, absolute continuity is weaker than Lipschitz continuity. The following theorems show that, in addition to (H1), the locally absolute continuity on f about x is sufficient for the lower semicontinuity of the variational functional.

Theorem 4. *Let $\Omega \subset R$ be an open set; $f(x,s,\xi) : \Omega \times R \times R \to [0,+\infty)$ satisfies the following conditions:*

(H1) *$f(x,s,\xi)$ is continuous on $\Omega \times R \times R$, and, $f(x,s,\xi)$ is convex in $\xi \in R$ for all $(x,s) \in \Omega \times R$;*

(H2) *$f_\xi(x,s,\xi)$ is continuous on $\Omega \times R \times R$, and for every compact set of $\Omega \times R \times R$, $f_\xi(x,s,\xi)$ is absolutely continuous about x;*

(H3) *for every compact set $K_1 \subset \Omega \times R$, there exists a constant $L_1 = L_1(K_1)$, such that*

$$\left|f_\xi\right| \le L_1, \quad \forall (x,s) \in K_1, \ \forall \xi \in R, \tag{6}$$

$$\left|f_\xi(x,s,\xi_1) - f_\xi(x,s,\xi_2)\right| \le L_1\left|\xi_1 - \xi_2\right|, \quad \forall (x,s) \in K_1, \quad \forall \xi_1, \xi_2 \in R. \tag{7}$$

Then the functional $F(u,\Omega) = \int_\Omega f(x,u(x),u'(x))dx$ is lower semicontinuous in $W^{1,1}_{loc}(\Omega)$ with respect to the strong convergence in $L^1_{loc}(\Omega)$.

Theorem 5. *Let $\Omega \subset R$ be an open set; $f(x,s,\xi) : \Omega \times R \times R \to [0,+\infty)$ satisfies (H1) and the following condition:*

(H4) *for every compact set of $\Omega \times R \times R$, $f(x,s,\xi)$ is absolutely continuous about x.*

Then the functional $F(u,\Omega)$ is lower semicontinuous in $W^{1,1}_{loc}(\Omega)$ with respect to the strong convergence in $L^1_{loc}(\Omega)$.

2. Preliminaries

In this section, we will collect some basic facts which will be used in the proofs of Theorems 4 and 5.

It is well know that a real function $f : [a,b] \to R$ is called an absolutely continuous function on $[a,b]$, if, for all $\varepsilon > 0$, $\exists \delta > 0$, such that for any finite disjoint open interval $\{(a_i,b_i)\}_{i=1}^n$ on $[a,b]$, when $\sum_{i=1}^n (b_i - a_i) < \delta$, we have

$$\sum_{i=1}^n \left|f(b_i) - f(a_i)\right| < \varepsilon. \tag{8}$$

Obviously, if $f(x)$ is Lipschitz continuous on $[a,b]$, $f(x)$ is absolutely continuous on $[a,b]$.

One of the main tool, used in the present paper, in order to prove the lower semicontinuity of the functional $F(u,\Omega)$ in (1), is an approximation result for convex functions due to De Giorgi [10].

Lemma 6 (see [10]). *Let $U \subseteq R^d$ be an open set and $f : U \times R^n \to [0,+\infty)$ a continuous function with compact support on U, such that, for every $t \in U$, $f(t,\cdot)$ is convex on R^n. Then there exists a sequence $\{\eta_q\}_{q=1}^\infty \subseteq C_c^\infty(R^n)$, $\eta_q \ge 0$, $\int_{R^n} \eta_q d\rho = 1$, and $\operatorname{supp}(\eta_q) \subseteq B(0,1)$, such that, if we let*

$$a_q(t) = \int_{R^n} f(t,\rho)\left\{(n+1)\eta_q(\rho) + \left\langle \nabla\eta_q(\rho), \rho\right\rangle\right\} d\rho,$$
$$b_q(t) = -\int_{R^n} f(t,\rho)\nabla\eta_q(\rho)\, d\rho, \tag{9}$$

one has

$$f_j(t,\xi) = \max_{1 \le q \le j}\left\{0, a_q(t) + \left\langle b_q(t), \xi\right\rangle\right\}, \quad j \in N, \tag{10}$$

satisfying the following results:

(i) *for every $j \in N$, $f_j : U \times R^n \to [0,+\infty)$ is a continuous function with compact support on U such that, for all $t \in U$, $f_j(t,\cdot)$ is convex on R^n. Moreover, for all $(t,\xi) \in U \times R^n$, $f_j(t,\xi) \le f_{j+1}(t,\xi)$, and*

$$f(t,\xi) = \sup_{j \in N} f_j(t,\xi), \tag{11}$$

(ii) *for every $j \in N$, there exists a constant $M_j > 0$, such that, for all $(t,\xi) \in U \times R^n$,*

$$\left|f_j(t,\xi)\right| \le M_j\left(1 + |\xi|\right), \tag{12}$$

and, for all $t \in U$, and $\xi_1, \xi_2 \in R^n$;

$$\left|f_j(t,\xi_1) - f_j(t,\xi_2)\right| \le M_j\left|\xi_1 - \xi_2\right|. \tag{13}$$

3. Proof of Theorem 4

We will divide four steps to complete the proof of Theorem 4.

Step 1. Let $\{\beta_i(x,s)\}_{i\in N}$ be a sequence of smooth functions satisfying

(1) there exists a compact set $\Omega' \times H \subset\subset \Omega \times R$, such that $\beta_i(x,s)=0$, for all $(x,s) \in (\Omega \setminus \Omega') \times (R \setminus H)$;

(2) for every $i \in N$, $\beta_i(x,s) \le \beta_{i+1}(x,s)$, for all $(x,s) \in \Omega' \times H$;

(3) $\lim_{i\to+\infty}\beta_i(x,s)=1$, for all $(x,s) \in \Omega' \times H$.

Let

$$f_i(x,s,\xi)=\beta_i(x,s)\, f(x,s,\xi), \quad i=1,2,\dots. \tag{14}$$

It is clear that, for each $i \in N$, f_i satisfies all the hypotheses in Theorem 4 and also vanishes if (x,s) is outside $\Omega' \times H$; thus

$$\begin{gathered}\lim_{i\to+\infty} f_i(x,s,\xi)=f(x,s,\xi), \quad \forall (x,s,\xi)\in\Omega'\times H\times R,\\ f_i(x,s,\xi)\le f_{i+1}(x,s,\xi)\le f(x,s,\xi),\\ \forall i\in N, \quad \forall(x,s,\xi)\in\Omega'\times H\times R.\end{gathered} \tag{15}$$

By Levi's Lemma, we have

$$\lim_{i\to+\infty}\int_{\Omega'} f_i(x,s,\xi)\,dx=\int_{\Omega'} f(x,s,\xi)\,dx. \tag{16}$$

Thus, without loss of generality, we can assume that there exists an open set $\Omega' \times H \subset\subset \Omega\times R$, such that

$$f(x,s,\xi)=0, \quad \forall(x,s,\xi)\in\left(\Omega\setminus\Omega'\right)\times(R\setminus H)\times R. \tag{17}$$

Let $u_m, u \in W^{1,1}_{\text{loc}}(\Omega)$ such that $u_m \to u$ in $L^1_{\text{loc}}(\Omega)$. We will prove that

$$\liminf_{m\to+\infty} F(u_m,\Omega)\ge F(u,\Omega). \tag{18}$$

Without loss of generality, we can assume that

$$\liminf_{m\to+\infty} F(u_m,\Omega)=\lim_{m\to+\infty}F(u_m,\Omega)<+\infty. \tag{19}$$

By (17), we have $F(u_m,\Omega)=F(u_m,\Omega')$ and $F(u,\Omega)=F(u,\Omega')$; thus we will only prove the following inequality:

$$\lim_{m\to+\infty}F\left(u_m,\Omega'\right)\ge F\left(u,\Omega'\right). \tag{20}$$

Step 2. Let $\eta_\varepsilon \in C_c^\infty(R)$ be a mollifier, and, for $\epsilon>0$, define

$$\begin{aligned}v_\varepsilon(x)&=\eta_\varepsilon * u(x)\\ &=\int_\Omega \eta_\varepsilon(x-y)\,u(y)\,dy, \quad x\in[\Omega_\varepsilon],\end{aligned} \tag{21}$$

where $[\Omega_\varepsilon] \triangleq \{x\in\Omega : \operatorname{dist}(x,\partial\Omega)>\varepsilon\}$. We have

$$\begin{aligned}[u_\varepsilon(x)]'&=\left[\int_\Omega \eta_\varepsilon(x-y)\,u(y)\,dy\right]_x\\ &=\int_\Omega[\eta_\varepsilon(x-y)]_x u(y)\,dy\\ &=\int_{B(x,\varepsilon)}\eta_\varepsilon(x-y)\,[u(y)]_y dy=\left[u'\right]_\varepsilon(x),\\ &\qquad x\in\Omega_\varepsilon.\end{aligned} \tag{22}$$

In the following, we denote the derivative of u_ε by u'_ε. When $u \in W^{1,1}_{\text{loc}}(\Omega)$, we know $u' \in L^1_{\text{loc}}(\Omega)$. By the properties of convolution, we know $u'_\varepsilon \in C_0^\infty(\Omega)$ and

$$u'_\varepsilon \longrightarrow u' \quad \text{in } L^1_{\text{loc}}(\Omega) \text{ as } \varepsilon\longrightarrow 0^+, \tag{23}$$

That is, for all $\delta>0$, $\exists\epsilon>0$, such that

$$\int_{\Omega'}\left|u'_\varepsilon-u'\right|dx<\delta. \tag{24}$$

Now we estimate the difference for the integrand values on different points:

$$\begin{aligned}&f\left(x,u_m,u'_m\right)-f\left(x,u,u'\right)\\ &\quad=f\left(x,u_m,u'_m\right)-f\left(x,u_m,u'_\varepsilon\right)\\ &\qquad+f\left(x,u_m,u'_\varepsilon\right)-f\left(x,u,u'_\varepsilon\right)\\ &\qquad+f\left(x,u,u'_\varepsilon\right)-f\left(x,u,u'\right).\end{aligned} \tag{25}$$

By the convexity of $f(x,s,\xi)$ with respect to ξ, we have

$$\begin{aligned}&f\left(x,u_m,u'_m\right)-f\left(x,u_m,u'_\varepsilon\right)\\ &\quad\ge f_\xi\left(x,u_m,u'_\varepsilon\right)\cdot\left(u'_m-u'_\varepsilon\right).\\ &\quad=f_\xi\left(x,u_m,u'_\varepsilon\right)\cdot u'_m-f_\xi\left(x,u,u'_\varepsilon\right)\cdot u'\\ &\qquad+f_\xi\left(x,u,u'_\varepsilon\right)\cdot\left(u'-u'_\varepsilon\right)\\ &\qquad+\left[f_\xi\left(x,u,u'_\varepsilon\right)-f_\xi\left(x,u_m,u'_\varepsilon\right)\right]\cdot u'_\varepsilon.\end{aligned} \tag{26}$$

By (25) and (26), we have

$$\begin{aligned}&\int_{\Omega'}\left[f\left(x,u_m,u'_m\right)-f\left(x,u,u'\right)\right]dx\\ &\quad\ge\int_{\Omega'}\left[f_\xi\left(x,u_m,u'_\varepsilon\right)\cdot u'_m-f_\xi\left(x,u,u'_\xi\right)\cdot u'\right]dx\\ &\qquad+\int_{\Omega'}\left[f_\xi\left(x,u,u'_\varepsilon\right)\cdot\left(u'-u'_\varepsilon\right)\right]dx\\ &\qquad+\int_{\Omega'}\left[f_\xi\left(x,u,u'_\varepsilon\right)-f_\xi\left(x,u_m,u'_\varepsilon\right)\right]\cdot u'_\varepsilon dx\\ &\qquad+\int_{\Omega'}\left[f\left(x,u_m,u'_\varepsilon\right)-f\left(x,u,u'_\varepsilon\right)\right]dx\\ &\qquad+\int_{\Omega'}\left[f\left(x,u,u'_\varepsilon\right)-f\left(x,u,u'\right)\right]dx.\end{aligned} \tag{27}$$

Step 3. Now, we estimate the right side of inequality (27).

By (6) and (24), we have

$$\begin{aligned}&\int_{\Omega'}\left[f_\xi\left(x,u,u_\varepsilon'\right)\cdot\left(u'-u_\varepsilon'\right)\right]dx\\&\quad\geq -L_1\int_{\Omega'}\left|u'-u_\varepsilon'\right|dx\geq -L_1\delta.\end{aligned}\tag{28}$$

Thus

$$\lim_{\varepsilon\to 0}\int_{\Omega'}\left[f_\xi\left(x,u,u_\varepsilon'\right)\cdot\left(u'-u_\varepsilon'\right)\right]dx\geq 0.\tag{29}$$

Since $f(x,s,\xi)$ and $f_\xi(x,s,\xi)$ are continuous functions, they are bounded functions on compact subset. By Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned}&\lim_{m\to+\infty}\int_{\Omega'}\left[f_\xi\left(x,u,u_\varepsilon'\right)-f_\xi\left(x,u_m,u_\varepsilon'\right)\right]\cdot u_\varepsilon' dx=0,\\&\lim_{m\to+\infty}\int_{\Omega'}\left[f\left(x,u_m,u_\varepsilon'\right)-f\left(x,u,u_\varepsilon'\right)\right]dx=0.\end{aligned}\tag{30}$$

Now, we will prove

$$\lim_{\varepsilon\to 0}\int_{\Omega'}\left[f\left(x,u,u_\varepsilon'\right)-f\left(x,u,u'\right)\right]dx\geq 0.\tag{31}$$

By Lemma 6, there exists a sequence of nonnegative continuous functions $f_j(x,s,\xi)$ $(j\in N)$, such that $f_j(x,s,\xi)$ is convex on ξ, and, for all $(x,s,\xi)\in\Omega'\times H\times R$,

$$\begin{aligned}&f_j(x,s,\xi)\leq f_{j+1}(x,s,\xi),\\&f(x,s,\xi)=\sup_{j\in\mathrm{N}}f_j(x,s,\xi),\\&\left|f_j(x,s,\xi_1)-f_j(x,s,\xi_2)\right|\leq M_j\left|\xi_1-\xi_2\right|.\end{aligned}\tag{32}$$

By Levi's Lemma, we obtain

$$\begin{aligned}&\lim_{j\to+\infty}\int_{\Omega'}f_j\left(x,u,u_\varepsilon'\right)dx=\int_{\Omega'}f\left(x,u,u_\varepsilon'\right)dx,\\&\lim_{j\to+\infty}\int_{\Omega'}f_j\left(x,u,u'\right)dx=\int_{\Omega'}f\left(x,u,u'\right)dx.\end{aligned}\tag{33}$$

In order to prove (31), we only need to prove

$$\lim_{\varepsilon\to 0}\int_{\Omega'}\left[f_j\left(x,u,u_\varepsilon'\right)-f_j\left(x,u,u'\right)\right]dx\geq 0,\quad \forall j\in N.\tag{34}$$

By (33), we have

$$\begin{aligned}&\int_{\Omega'}\left[f_j\left(x,u,u_\varepsilon'\right)-f_j\left(x,u,u'\right)\right]dx\\&\quad\geq -M_j\int_{\Omega'}\left|u_\varepsilon'-u'\right|dx\geq -M_j\delta.\end{aligned}\tag{35}$$

Thus (31) holds.

Step 4. Now, we need to prove

$$\begin{aligned}\lim_{m\to+\infty}\int_{\Omega'}&\left[f_\xi\left(x,u_m,u_\varepsilon'\right)\cdot u_m'\right.\\&\left.-f_\xi\left(x,u,u_\varepsilon'\right)\cdot u'\right]dx=0.\end{aligned}\tag{36}$$

Let

$$g(x,s)\triangleq f_\xi\left(x,s,u_\varepsilon'\right),\quad x\in\Omega',\tag{37}$$

$$G_m(x)\triangleq\int_{u(x)}^{u_m(x)}g(x,s)\,ds,\quad x\in\Omega'.\tag{38}$$

By triangle inequality and (7), we have

$$\begin{aligned}&\left|f_\xi\left(y_i,s,u_\varepsilon'(y_i)\right)-f_\xi\left(x_i,s,u_\varepsilon'(x_i)\right)\right|\\&\quad\leq\left|f_\xi\left(y_i,s,u_\varepsilon'(y_i)\right)-f_\xi\left(x_i,s,u_\varepsilon'(y_i)\right)\right|\\&\qquad+\left|f_\xi\left(x_i,s,u_\varepsilon'(y_i)\right)-f_\xi\left(x_i,s,u_\varepsilon'(x_i)\right)\right|\\&\quad\leq\left|f_\xi\left(y_i,s,u_\varepsilon'(y_i)\right)-f_\xi\left(x_i,s,u_\varepsilon'(y_i)\right)\right|\\&\qquad+L_1\left|u_\varepsilon'(y_i)-u_\varepsilon'(x_i)\right|.\end{aligned}\tag{39}$$

By (39), condition (H2) and $u_\varepsilon'\in C_0^\infty(\Omega)$, we know that $g(x,s)$ is a locally absolute continuous function about x. So $g(x,s)$ is almost everywhere differentiable; that is, $\partial g/\partial x$ exists almost everywhere. Taking derivatives in both sides of (38), we obtain

$$\begin{aligned}G_m'(x)=&\;g(x,u_m)\cdot u_m'-g(x,u)\cdot u'\\&+\int_{u(x)}^{u_m(x)}\frac{\partial g}{\partial x}ds,\qquad\text{a.e. } x\in\Omega'.\end{aligned}\tag{40}$$

Because $G_m(x)$ vanishes outside Ω', we obtain

$$\int_{\Omega'}G_m'(x)\,dx=0.\tag{41}$$

By (40), we have

$$\begin{aligned}&\left|\int_{\Omega'}\left[f_\xi\left(x,u_m,u_\varepsilon'\right)\cdot u_m'-f_\xi\left(x,u,u_\varepsilon'\right)\cdot u'\right]dx\right|\\&\quad=\left|\int_{\Omega'}\left[g(x,u_m)\cdot u_m'-g(x,u)\cdot u'\right]dx\right|\\&\quad=\left|-\int_{\Omega'}\int_{u(x)}^{u_m(x)}\frac{\partial g}{\partial x}ds\,dx\right|\leq\int_{D_m}\left|\frac{\partial g}{\partial x}\right|dx\,ds,\end{aligned}\tag{42}$$

where

$$\begin{aligned}D_m=\{&(x,s)\in\Omega'\times H\mid\min\{u_m(x),u(x)\}\\&\leq s(x)\leq\max\{u_m(x),u(x)\}\}.\end{aligned}\tag{43}$$

We note

$$\begin{aligned}\left|D_m\right|&=\left|\int_{\Omega'}\int_u^{u_m}ds\,dx\right|\\&\leq\int_{\Omega'}\left|u_m-u\right|dx\longrightarrow 0\quad(m\longrightarrow+\infty).\end{aligned}\tag{44}$$

By Fubini's Theorem, we have

$$\int_{\Omega\times R}\left|\frac{\partial g}{\partial x}\right|dx\,ds=\int_H ds\int_{\Omega'}\left|\frac{\partial g}{\partial x}\right|dx. \tag{45}$$

Since $g(x,s)$ is absolutely continuous about x, $\partial g/\partial x$ is integrable and absolutely integrable with respect to x; that is,

$$\int_{\Omega'}\left|\frac{\partial g}{\partial x}\right|dx<+\infty. \tag{46}$$

By (17) and (46), we obtain

$$\int_{\Omega\times R}\left|\frac{\partial g}{\partial x}\right|dx\,ds<+\infty. \tag{47}$$

Because of the absolute continuity of integral, we have

$$\lim_{m\to+\infty}\int_{D_m}\left|\frac{\partial g}{\partial x}\right|dx\,ds=0. \tag{48}$$

By (42), we obtain

$$\lim_{m\to+\infty}\left|\int_{\Omega'}\left[f_\xi\left(x,u_m,u'_\varepsilon\right)\cdot u'_m-f_\xi\left(x,u,u'_\varepsilon\right)\cdot u'\right]dx\right|=0. \tag{49}$$

Thus we just proved (36). By (29)–(31) and (36), we have

$$\lim_{m\to+\infty}\int_{\Omega'}\left[f\left(x,u_m,u'_m\right)-f\left(x,u,u'\right)\right]dx\geq 0. \tag{50}$$

Thus we deduce that the functional $F(u,\Omega)$ is lower semicontinuous in $W^{1,1}_{\text{loc}}(\Omega)$ with respect to the strong convergence in $L^1_{\text{loc}}(\Omega)$. We complete the proof.

4. Proof of Theorem 5

In order to prove Theorem 5, we will verify all the conditions in Theorem 4 under the assumptions in Theorem 5. Now we will divide three steps to complete the proof of Theorem 5.

Step 1. Similar to the first step of the proof in Theorem 4, without loss of generality, we assume that the integrand $f(x,s,\xi)$ vanishes outside a compact subset of $\Omega\times R$. Thus we assume that there exists an open set $\Omega'\times H\subset\subset\Omega\times R$, such that

$$f(x,s,\xi)\equiv 0,\quad \forall(x,s,\xi)\in\left(\Omega\setminus\Omega'\right)\times(R\setminus H)\times R. \tag{51}$$

Let $u_m,u\in W^{1,1}_{\text{loc}}(\Omega)$, such that $u_m\to u$ in $L^1_{\text{loc}}(\Omega)$; we need to prove

$$\lim_{m\to+\infty}F\left(u_m,\Omega'\right)\geq F\left(u,\Omega'\right). \tag{52}$$

By Lemma 6, there exists a function sequence $\{f_j(x,s,\xi)\}_{j\in N}$, such that, for all $j\in N$, f_j is a continuous function on $\Omega'\times H\subset\subset\Omega\times R$, for all $(x,s)\in\Omega'\times H$, $f_j(x,s,\cdot)$ is convex on R, and, for all $(x,s,\xi)\in\Omega'\times H\times R$,

$$f_j(x,s,\xi)\leq f_{j+1}(x,s,\xi), \tag{53}$$

$$f(x,s,\xi)=\sup_{j\in N}f_j(x,s,\xi), \tag{54}$$

$$\left|f_j(x,s,\xi_1)-f_j(x,s,\xi_2)\right|\leq M_j\left|\xi_1-\xi_2\right|,\quad (x,s)\in\Omega'\times H,\quad \xi_1,\xi_2\in R. \tag{55}$$

Let $\eta_\varepsilon\in C^\infty_c(R)$ $(0<\varepsilon\ll 1)$ be a mollifier, and define the $f_{j,\varepsilon}=f_j*\eta_\varepsilon$; that is,

$$f_{j,\varepsilon}(x,s,\xi)=\int_R f_j(x,s,\xi-z)\,\eta_\varepsilon(z)\,dz. \tag{56}$$

By (55), we have

$$\begin{aligned}\left|f_{j,\varepsilon}(x,s,\xi)-f_j(x,s,\xi)\right|&\leq\int_R\left|f_j(x,s,\xi-z)-f_j(x,s,\xi)\right|\eta_\varepsilon(z)\,dz\\&\leq\int_{\operatorname{supp}\eta_\varepsilon}M_j|z|\cdot\eta_\varepsilon(z)\,dz\leq M_j\cdot\varepsilon.\end{aligned} \tag{57}$$

Choose $\varepsilon=\varepsilon_j=1/jM_j\to 0$. By (57), we have

$$\left|f_{j,\varepsilon_j}(x,s,\xi)-f_j(x,s,\xi)\right|\leq M_j\varepsilon_j=\frac{1}{j}. \tag{58}$$

So

$$\begin{aligned}f_j(x,s,\xi)-\frac{2}{j}&\leq f_{j,\varepsilon_j}(x,s,\xi)-\frac{1}{j}\\&\leq f_j(x,s,\xi)\leq f(x,s,\xi).\end{aligned} \tag{59}$$

By (53), (54), and Levi's Lemma, we have

$$\begin{aligned}\lim_{i\to+\infty}\int_{\Omega'}f_j\left(x,u(x),u'(x)\right)dx\\=\int_{\Omega'}f\left(x,u(x),u'(x)\right)dx.\end{aligned} \tag{60}$$

Let

$$F_j\left(u,\Omega'\right)=\int_{\Omega'}\left[f_{j,\varepsilon_j}\left(x,u(x),u'(x)\right)-\frac{1}{j}\right]dx. \tag{61}$$

By (59)–(61), we have

$$\begin{aligned}\lim_{j\to+\infty}F_j\left(u,\Omega'\right)&=F\left(u,\Omega'\right)\\&=\int_{\Omega'}f\left(x,u(x),u'(x)\right)dx.\end{aligned} \tag{62}$$

Obviously,

$$F_j\left(u,\Omega'\right)\leq F\left(u,\Omega'\right),\quad \forall j\in N. \tag{63}$$

Thus

$$\sup_{j\in N} F_j\left(u,\Omega'\right) = F\left(u,\Omega'\right). \tag{64}$$

Therefore $F(u,\Omega')$, being the supremum of the family of functionals $\{F_j(u,\Omega')\}_{j\in N}$, will be lower semicontinuous if every $\{F_j(u,\Omega')\}$ is lower semicontinuous.

Step 2. In order to prove that, for all $j\in N, F_j(u,\Omega')$ is lower semicontinuous in $W^{1,1}_{\mathrm{loc}}(\Omega)$ with respect to the strong convergence in $L^1_{\mathrm{loc}}(\Omega)$, we will prove that, for all $j\in N$, the integrand $f_{j,\varepsilon_j}(x,u(x),u'(x))$ satisfies all conditions of Theorem 4. Obviously, for all $j\in N$, f_{j,ε_j} satisfies condition (H1).

For all $(x,s)\in\Omega'\times H$ and $\xi_1,\xi_2\in R$, by (55), we have

$$\begin{aligned}&\left|f_{j,\varepsilon_j}(x,s,\xi_1)-f_{j,\varepsilon_j}(x,s,\xi_2)\right|\\&\quad\le\int_R\left|f_j(x,s,\xi_1-z)-f_j(x,s,\xi_2-z)\right|\cdot\eta_{\varepsilon_j}(z)\,dz\\&\quad\le\int_{\operatorname{supp}\eta_\varepsilon}M_j\left|\xi_1-\xi_2\right|\eta_{\varepsilon_j}(z)\,dz\le M_j\left|\xi_1-\xi_2\right|.\end{aligned}\tag{65}$$

Thus

$$\left|\frac{\partial f_{j,\varepsilon_j}}{\partial\xi}\right|\le M_j. \tag{66}$$

So f_{j,ε_j} satisfies (6) in condition (H3) of Theorem 4.

Now, we will prove that f_{j,ε_j} satisfies (7) in condition (H3) of Theorem 4. By $\operatorname{supp}(\eta_{\varepsilon_j})\subseteq B(0,\varepsilon_j)$, we have

$$\begin{aligned}\frac{\partial f_{j,\varepsilon_j}}{\partial\xi}(x,s,\xi)&=\int_R\frac{\partial f_j(x,s,\xi-z)}{\partial\xi}\cdot\eta_{\varepsilon_j}(z)\,dz\\&=-\int_R\frac{\partial f_j(x,s,\xi-z)}{\partial z}\cdot\eta_{\varepsilon_j}(z)\,dz\\&=\int_R f_j(x,s,\xi-z)\frac{\partial\eta_{\varepsilon_j}(z)}{\partial z}dz.\end{aligned}\tag{67}$$

By (55) and (67), we have

$$\begin{aligned}&\left|\frac{\partial f_{j,\varepsilon_j}}{\partial\xi}(x,s,\xi_1)-\frac{\partial f_{j,\varepsilon_j}}{\partial\xi}(x,s,\xi_2)\right|\\&\quad\le\int_R\left|f_j(x,s,\xi_1-z)-f_j(x,s,\xi_2-z)\right|\cdot\left|\frac{\partial\eta_{\varepsilon_j}(z)}{\partial z}\right|dz\\&\quad\le M_j\left|\xi_1-\xi_2\right|\cdot\int_R\left|\frac{\partial\eta_{\varepsilon_j}(z)}{\partial z}\right|dz=L_jM_j\left|\xi_1-\xi_2\right|,\end{aligned}\tag{68}$$

where

$$L_j=\int_R\left|\frac{\partial\eta_{\varepsilon_j}(z)}{\partial z}\right|dz \tag{69}$$

is a constant depending on ε_j. Thus f_{j,ε_j} satisfies (7). So we proved that f_{j,ε_j} satisfies condition (H3).

Step 3. Next we will prove that f_{j,ε_j} satisfies condition (H2).

By condition (H4), for every compact subset $\Omega'\times H\times K$, $f(x,s,\xi)$ is absolutely continuous about x, that is, for all $\varepsilon_0>0$, $\exists\delta>0$ such that for any finite disjoint open interval $\{(x_i,y_i)\}_{i=1}^n$ in Ω', when $\sum_{i=1}^n(y_i-x_i)<\delta$, we have

$$\sum_{i=1}^n\left|f(y_i,s,\xi)-f(x_i,s,\xi)\right|<\varepsilon_0. \tag{70}$$

By Step 1, $\{f_j(x,s,\xi)\}_{j\in N}$ satisfies (53)-(55) and the following property:

$$f_j(x,s,\xi)=\max_{1\le q\le j}\left\{0,a_q(x,s)+b_q(x,s)\xi\right\},\qquad j\in N, \tag{71}$$

where

$$\begin{aligned}a_q(x,s)&=\int_R f(x,s,\rho)\left[2\eta_q(\rho)+\rho\frac{\partial\eta_q(\rho)}{\partial\rho}\right]d\rho,\\b_q(x,s)&=-\int_R f(x,s,\rho)\frac{\partial\eta_q(\rho)}{\partial\rho}d\rho,\end{aligned}\tag{72}$$

And, for all $(x,s,\xi)\in\Omega'\times H\times R$, $\eta_q\in C_c^\infty(R)$ $(q\in N)$ are mollifiers satisfying $\eta_q\ge0$, $\int_R\eta_q(\rho)d\rho=1$, and $\operatorname{supp}(\eta_q)\subseteq B(0,1)$, for all $j\in N$. By (71), without of loss generality, we assume that there exists $l\in\{1,\dots,j\}$, such that

$$f_j(x,s,\xi)=a_l(x,s)+b_l(x,s)\cdot\xi, \tag{73}$$

where a_l, b_l are given by (72). By (70), we obtain

$$\begin{aligned}&\sum_{i=1}^n\left|a_l(y_i,s)-a_l(x_i,s)\right|\\&\quad\le\int_R\sum_{i=1}^n\left|f(y_i,s,\rho)-f(x_i,s,\rho)\right|\\&\qquad\cdot\left[2\eta_l(\rho)+\left|\rho\frac{\partial\eta_l(\rho)}{\partial\rho}\right|\right]d\rho\\&\quad\le\varepsilon_0\int_{B(0,1)}\left[2\eta_l(\rho)+\left|\rho\frac{\partial\eta_l(\rho)}{\partial\rho}\right|\right]d\rho\\&\quad\le(2+A_l)\cdot\varepsilon_0,\end{aligned}\tag{74}$$

where

$$A_l=\int_{B(0,1)}\left|\frac{\partial\eta_l(\rho)}{\partial\rho}\right|d\rho \tag{75}$$

is a constant. Similar to the above proof, we have

$$\begin{aligned}\sum_{i=1}^{n}\left|b_l(y_i,s)-b_l(x_i,s)\right| &\le \int_R\sum_{i=1}^{n}\left|f(y_i,s,\rho)-f(x_i,s,\rho)\right|\cdot\left|\frac{\partial\eta_l(\rho)}{\partial\rho}\right|d\rho \\ &\le \varepsilon_0\int_{B(0,1)}\left|\frac{\partial\eta_l(\rho)}{\partial\rho}\right|d\rho\le A_l\cdot\varepsilon_0.\end{aligned}\tag{76}$$

Thus

$$\begin{aligned}\sum_{i=1}^{n}\left|f_j(y_i,s,\xi)-f_j(x_i,s,\xi)\right| &\le \sum_{i=1}^{n}\left|a_l(y_i,s)-a_l(x_i,s)\right| \\ &\quad+\sum_{i=1}^{n}\left|b_l(y_i,s)-b_l(x_i,s)\right|\cdot|\xi| \\ &\le (2+A_l)\varepsilon_0+A_l\varepsilon_0K_1=(2+A_l+A_lK_1)\varepsilon_0\triangleq\sigma.\end{aligned}\tag{77}$$

Since ξ belongs to a compact set, then $K_1=\sup_\xi\{|\xi|\}<+\infty$. Choose ε_0 sufficient small, so that σ is small enough. Thus $f_j(x,s,\xi)$ is absolutely continuous about x for all $(x,s,\xi)\in A$, which is a compact subset of $\Omega\times R\times R$. By (56) and (77), we have

$$\begin{aligned}\sum_{i=1}^{n}\left|f_{j,\varepsilon_j}(y_i,s,\xi)-f_{j,\varepsilon_j}(x_i,s,\xi)\right| &\le \int_R\sum_{i=1}^{n}\left|f_j(y_i,s,\xi-z)-f_j(x_i,s,\xi-z)\right|\cdot\eta_{\varepsilon_j}(z)\,dz \\ &\le \sigma\cdot\int_{B(0,\varepsilon_j)}\eta_{\varepsilon_j}(z)\,dz=\sigma.\end{aligned}\tag{78}$$

By (67) and (78), we obtain

$$\begin{aligned}\sum_{i=1}^{n}\left|\frac{\partial f_{j,\varepsilon_j}}{\partial\xi}(y_i,s,\xi)-\frac{\partial f_{j,\varepsilon_j}}{\partial\xi}(x_i,s,\xi)\right| &\le \int_R\sum_{i=1}^{n}\left|f_j(y_i,s,\xi-z)-f_j(x_i,s,\xi-z)\right|\cdot\left|\frac{\partial\eta_{\varepsilon_j}(z)}{\partial z}\right|dz \\ &\le \sigma\int_R\left|\frac{\partial\eta_{\varepsilon_j(z)}}{\partial z}\right|dz=L_j\sigma,\end{aligned}\tag{79}$$

where L_j are constants depending on ε_j and given by (69) (for all $j\in N$). By (79), for every compact subset on $\Omega\times R\times R$, $\partial f_{j,\varepsilon_j}/\partial\xi$ is absolutely continuous about x. Thus f_{j,ε_j} satisfies condition (H2).

Now, we have proved f_{j,ε_j} satisfies all conditions in Theorem 4, so $F_j(u,\Omega')$ is lower semicontinuous in $W^{1,1}_{\mathrm{loc}}(\Omega)$ with respect to the strong convergence in $L^1_{\mathrm{loc}}(\Omega)$. Thus $F(u,\Omega)$ has the same lower semicontinuity. This completes the proof of Theorem 5.

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