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Research Article

Some Remarks on the Extended Hartley-Hilbert and Fourier-Hilbert Transforms of Boehmians

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We obtain generalizations of Hartley-Hilbert and Fourier-Hilbert transforms on classes of distributions having compact support. Furthermore, we also study extension to certain space of Lebesgue integrable Boehmians. New characterizing theorems are also established in an adequate performance.

1. Introduction

The classical theory of integral transforms and their applications have been studied for a long time, and they are applied in many fields of mathematics. Later, after [1], the extension of classical integral transformations to generalized functions has comprised an active area of research. Several integral transforms are extended to various spaces of generalized functions, distributions [2], ultradistributions, Boehmians [3, 4], and many more.

In recent years, many papers are devoted to those integral transforms which permit a factorization identity (of Fourier convolution type) such as Fourier transform, Mellin transform, Laplace transform, and few others that have a lot of attraction, the reason that the theory of integral transforms, generally speaking, became an object of study of integral transforms of Boehmian spaces.

The Hartley transform is an integral transformation that maps a real-valued temporal or spacial function into a real-valued frequency function via the kernel

$$k(\upsilon; x) = \operatorname{cas}(\upsilon x). \tag{1}$$

This novel symmetrical formulation of the traditional Fourier transform, attributed to Hartley 1942, leads to a parallelism that exists between a function of the original variable and that of its transform. In any case, signal and systems analysis and design in the frequency domain using the Hartley transform may be deserving an increased awareness due to the necessity of the existence of a fast algorithm that can substantially lessen the computational burden when compared to the classical complex-valued fast Fourier transform.

The Hartley transform of a function $f(x)$ can be expressed as either [5]

$$\mathscr{A}(\upsilon) =: \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \operatorname{cas}(\upsilon x)\, dx \tag{2}$$

or

$$\mathscr{A}(f) =: \int_{-\infty}^{\infty} f(x) \operatorname{cas}(2\pi f x)\, dx, \tag{3}$$

where the angular or radian frequency variable υ is related to the frequency variable f by $\upsilon = 2\pi f$ and

$$\mathscr{A}(f) = \sqrt{2\pi}\mathscr{A}(2\pi f) = \sqrt{2\pi}\mathscr{A}(\upsilon). \tag{4}$$

The integral kernel, known as cosine-sine function, is defined as

$$\operatorname{cas}(\upsilon x) = \cos \upsilon x + \sin(\upsilon x). \tag{5}$$

Inverse Hartley transform may be defined as either

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathscr{A}(\upsilon) \operatorname{cas}(\upsilon x)\, d\upsilon \tag{6}$$

or

$$f(x)=\int_{-\infty}^{\infty}\mathscr{A}(f)\operatorname{cas}(2\pi fx)\,df. \tag{7}$$

The theory of convolutions of integral transforms has been developed for a long time and is applied in many fields of mathematics. Historically, the convolution product [2]

$$(f*g)(y)=\int_{-\infty}^{\infty}f(x)\,g(x-y)\,dy \tag{8}$$

has a relationship with the Fourier transform with the factorization property

$$\mathscr{F}(f*g)(y)=\mathscr{F}(f)(y)\,\mathscr{F}(g)(y). \tag{9}$$

The more complicated convolution theorem of Hartley transforms, compared to that of Fourier transforms, is that

$$\mathscr{A}(f*g)(y)=\frac{1}{2}\mathscr{G}(\mathscr{A}f\times\mathscr{A}g)(y), \tag{10}$$

where

$$\begin{aligned}\mathscr{G}(f*g)(y)&=f(y)\,g(y)+f(y)\,g(-y)\\&\quad+f(y)\,g(y)-f(-y)\,g(-y).\end{aligned} \tag{11}$$

Some properties of Hartley transforms can be listed as follows.

(i) Linearity: if f and g are real functions then

$$\mathscr{A}(af+bg)(y)=a\mathscr{A}(f)(y)+b\mathscr{A}(g)(y),\quad a,b\in\mathscr{R}. \tag{12}$$

(ii) Scaling: if f is a real function then

$$\int_{-\infty}^{\infty}f(\alpha\varsigma)\operatorname{cas}(2\pi y\varsigma)\,d\varsigma=\frac{1}{a}(\mathscr{A}f)\left(\frac{y}{a}\right). \tag{13}$$

2. Distributional Hartley-Hilbert Transform of Compact Support

The Hilbert transform via the Hartley transform is defined by [6, 7]

$$\mathscr{B}^{\mathscr{A}}(y)=-\frac{1}{\pi}\int_{0}^{\infty}\left(\mathscr{A}^{o}(x)\cos(xy)+\mathscr{A}^{e}(x)\sin(xy)\right)dx, \tag{14}$$

where

$$\begin{aligned}\mathscr{A}^{o}(x)&=\frac{\mathscr{A}(x)-\mathscr{A}(-x)}{2},\\\mathscr{A}^{e}(x)&=\frac{\mathscr{A}(x)+\mathscr{A}(-x)}{2}\end{aligned} \tag{15}$$

are the respective odd and even components of (2).

We denote, $\mathscr{C}(\mathscr{R})$, $\mathscr{C}(\mathscr{R})=\mathscr{C}$, the space of smooth functions and $\mathscr{C}'(\mathscr{R})$, $\mathscr{C}'(\mathscr{R})=\mathscr{C}'$, the strong dual of $\mathscr{C}$ of distributions of compact support over $\mathscr{R}$.

Following is the convolution theorem of $\mathscr{B}^{\mathscr{A}}$.

Theorem 1 (Convolution Theorem). *Let f and $g\in\mathscr{C}$ then*

$$\mathscr{B}^{\mathscr{A}}(f*g)(y)=\int_{0}^{\infty}\left(k_1(x)\cos(yx)+k_2(x)\sin(yx)\right)dx, \tag{16}$$

where

$$\begin{aligned}k_1(x)&=\mathscr{A}^{e}f(x)\,\mathscr{A}^{o}g(x)+\mathscr{A}^{o}f(x)\,\mathscr{A}^{e}g(x),\\k_2(x)&=\mathscr{A}^{e}f(x)\,\mathscr{A}^{e}g(x)-\mathscr{A}^{o}f(x)\,\mathscr{A}^{o}g(x).\end{aligned} \tag{17}$$

Proof. To prove this theorem it is sufficient to establish that

$$k_1(x)=\mathscr{A}^{o}(f*g)(x), \tag{18}$$

$$k_2(x)=\mathscr{A}^{e}(f*g)(x). \tag{19}$$

Therefore, we have

$$\begin{aligned}&\mathscr{A}^{o}(f*g)(x)\\&\quad=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(\gamma)\,g(y-\gamma)\,dr\right)\operatorname{cas}(xy)\,dy\\&\quad=\int_{-\infty}^{\infty}f(\gamma)\int_{-\infty}^{\infty}g(y-\gamma)\operatorname{cas}(xy)\,dy\,dr.\end{aligned} \tag{20}$$

The substitution $y-\gamma=z$ and using of (1) together with Fubini theorem imply

$$\begin{aligned}&\mathscr{A}^{o}(f*g)(x)\\&\quad=\int_{-\infty}^{\infty}f(\gamma)\int_{-\infty}^{\infty}g(z)\left(\cos(x(z+\gamma))+\sin(x(z+\gamma))\right)\\&\qquad\times dz\,dr.\end{aligned} \tag{21}$$

By invoking the formulae

$$\begin{aligned}\cos(x(z+\gamma))&=\cos(xz)\cos(x\gamma)-\sin(xz)\sin(x\gamma),\\\sin(x(z+\gamma))&=\sin(xz)\cos(x\gamma)+\cos(xz)\sin(x\gamma),\end{aligned} \tag{22}$$

then (18) follows from simple computation. Proof of (19) has a similar technique. Hence, the theorem is completely proved. □

It is of interest to know that $\cos(xy)$ and $\sin(xy)$ are members of $\mathscr{C}$ and, therefore, $\mathscr{A}^{o}f,\mathscr{A}^{e}f\in\mathscr{C}'$. This leads to the following statement.

Definition 2. Let $f\in\mathscr{C}'$ then we define the distributional Hartley-Hilbert transform of f as

$$\widehat{\mathscr{B}^{\mathscr{A}}}f(y)=\left\langle\mathscr{A}^{o}f(x),\cos(xy)\right\rangle+\left\langle\mathscr{A}^{e}f(x),\sin(xy)\right\rangle. \tag{23}$$

The extended transform $\widehat{\mathscr{B}^{\mathscr{A}}}f$ is clearly well defined for each $f\in\mathscr{C}'$.

Theorem 3. *The distributional Hartley-Hilbert transform* $\widehat{\mathscr{B}^{\mathscr{A}}} f$ *is linear.*

Proof. Let $f, g \in \mathscr{C}'$ then their components $\mathscr{A}^e f, \mathscr{A}^o f, \mathscr{A}^e g, \mathscr{A}^o g \in \mathscr{C}'$. Hence,

$$\widehat{\mathscr{B}^{\mathscr{A}}}(f+g)(y) = \langle \mathscr{A}^o (f+g)(x), \cos(xy)\rangle + \langle \mathscr{A}^e (f+g)(x), \sin(xy)\rangle. \quad (24)$$

By factoring and rearranging components we get that

$$\widehat{\mathscr{B}^{\mathscr{A}}}(f+g)(y) = \widehat{\mathscr{B}^{\mathscr{A}}} f(y) + \widehat{\mathscr{B}^{\mathscr{A}}} g(y). \quad (25)$$

Furthermore,

$$\widehat{\mathscr{B}^{\mathscr{A}}}(kf)(y) = \langle k\mathscr{A}^o f(x), \cos(xy)\rangle + \langle k\mathscr{A}^e f(x), \sin(xy)\rangle. \quad (26)$$

Hence,

$$\widehat{\mathscr{B}^{\mathscr{A}}}(kf)(y) = k\widehat{\mathscr{B}^{\mathscr{A}}} f(y). \quad (27)$$

This completes the proof of the theorem. □

Theorem 4. *Let* $f \in \mathscr{C}'$ *then* $\widehat{\mathscr{B}^{\mathscr{A}}} f$ *is a continuous mapping on* $\mathscr{C}'$.

Proof. Let $f_n, f \in \mathscr{C}'$, $n \in \mathscr{N}$ and $f_n \to f$ as $n \to \infty$. Then,

$$\begin{aligned}\widehat{\mathscr{B}^{\mathscr{A}}} f_n(y) &= \langle \mathscr{A}^o f_n(x), \cos(xy)\rangle + \langle \mathscr{A}^e f_n(x), \sin(xy)\rangle \\ &\longrightarrow \langle \mathscr{A}^o f(x), \cos(xy)\rangle + \langle \mathscr{A}^e f(x), \sin(xy)\rangle \\ &= \widehat{\mathscr{B}^{\mathscr{A}}} f(y) \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (28)$$

Hence we have the following theorem. □

Theorem 5. *The mapping* $\widehat{\mathscr{B}^{\mathscr{A}}} f$ *is one-to-one.*

Proof. Let $f, g \in \mathscr{C}'$ and that $\widehat{\mathscr{B}^{\mathscr{A}}} f = \widehat{\mathscr{B}^{\mathscr{A}}} g$ then, using (23) we get

$$\langle \mathscr{A}^o f(x), \cos xy\rangle + \langle \mathscr{A}^e f(x), \sin xy\rangle = \langle \mathscr{A}^o g(x), \cos xy\rangle + \langle \mathscr{A}^e g(x), \sin xy\rangle. \quad (29)$$

Basic properties of inner product implies

$$\langle \mathscr{A}^o f(x) - \mathscr{A}^o g(x), \cos(xy)\rangle + \langle \mathscr{A}^e f(x) - \mathscr{A}^e g(x), \sin(xy)\rangle = 0. \quad (30)$$

Hence,

$$\mathscr{A}^o f(x) = \mathscr{A}^o g(x), \qquad \mathscr{A}^e f(x) = \mathscr{A}^e g(x). \quad (31)$$

Therefore,

$$\begin{aligned}\mathscr{A} f(x) &= \mathscr{A}^o f(x) + \mathscr{A}^e f(x) \\ &= \mathscr{A}^o g(x) + \mathscr{A}^e g(x) = \mathscr{A} g(x)\end{aligned} \quad (32)$$

for all x. This completes the proof of the theorem. □

Theorem 6. *Let* $f \in \mathscr{C}'$ *then* f *is analytic and*

$$\mathscr{D}_y^k \widehat{\mathscr{B}^{\mathscr{A}}} f(y) = \left\langle \mathscr{A}^o f(x), \mathscr{D}_y^k \cos(xy)\right\rangle + \left\langle \mathscr{A}^e f(x), \mathscr{D}_y^k \sin(xy)\right\rangle. \quad (33)$$

Proof of this theorem is analogous to that of the previous theorem and is thus avoided.

Denote by δ the dirac delta function then it is easy to see that

$$\mathscr{A}^e \delta(y) = 1, \qquad \mathscr{A}^e \delta(y) = 0. \quad (34)$$

3. Lebesgue Space of Boehmians for Hartley-Hilbert Transforms

The original construction of Boehmians produce a concrete space of generalized functions. Since the space of Boehmians was introduced, many spaces of Boehmians were defined. In references, we list selected papers introducing different spaces of Boehmians. One of the main motivations for introducing different spaces of Boehmians was the generalization of integral transforms. The idea requires a proper choice of a space of functions for which a given integral transform is well defined, a choice of a class of delta sequences that is transformed by that integral transform to a well-behaved class of approximate identities, and finally a convolution product that behaves well under the transform. If these conditions are met, the transform has usually an extension to the constructed space of Boehmians and the extension has desirable properties. For general construction of Boehmians, see [8–12].

Let $\mathscr{D}$ be the space of test functions of bounded support. By delta sequence, we mean a subset of $\mathscr{D}$ of sequences (δ_n) such that

$$\begin{gathered}\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1, \\ \|\delta_n\| = \int_{-\infty}^{\infty} |\delta_n(x)|\,dx < \mathscr{M}, \quad 0 < \mathscr{M} \in \mathscr{R}, \\ \lim_{n\to\infty} \int_{|x|>\varepsilon} |\delta_n(x)|\,dx = 0 \quad \text{for each } \varepsilon > 0,\end{gathered} \quad (35)$$

where $\varepsilon(\delta_n)(x) = \{x \in \mathscr{R} : \delta_n(x) \neq 0\}$.

The set of all such delta sequences is usually denoted as Δ. Each element in Δ corresponds to the dirac delta function δ, for large values of n.

Proposition 7. *Let* $(\delta_n) \in \Delta$ *then*

$$\begin{aligned}\mathscr{A}^e \delta_n(y) &= \int_{-\infty}^{\infty} \delta_n(x) \cos(xy)\,dx \longrightarrow 1 \quad \textit{as } n \longrightarrow \infty, \\ \mathscr{A}^o \delta_n(y) &= \int_{-\infty}^{\infty} \delta_n(x) \sin(xy)\,dx \longrightarrow 0 \quad \textit{as } n \longrightarrow \infty.\end{aligned} \quad (36)$$

Let $\mathscr{L}^1(\mathscr{R})$, $\mathscr{L}^1(\mathscr{R}) = \mathscr{L}^1$, be the space of complex valued Lebesgue integrable functions. From Proposition 7 we establish the following theorem.

Theorem 8. *Let $f \in \mathscr{L}^1$ then $\mathscr{B}^{\mathscr{A}}(f * \delta_n)(y) \to \mathscr{B}^{\mathscr{A}} f(y)$ as $n \to \infty$.*

Proof. For $f \in \mathscr{L}^1$, $(\delta_n) \in \Delta$, then using of (14) implies

$$\mathscr{B}^{\mathscr{A}}(f * \delta_n)(y) = \int_{-\infty}^{\infty} (\mathscr{A}^o (f * \delta_n)(x) \cos xy + \mathscr{A}^e (f * \delta_n)(x) \sin xy)\, dx. \tag{37}$$

Since

$$(f * \delta_n)(\zeta) = \int_{-\infty}^{\infty} f(t)\, \delta_n(\zeta - t)\, dt \longrightarrow f(\zeta) \tag{38}$$

as $n \to \infty$, we see that

$$\begin{aligned}\mathscr{A}^o (f * \delta_n)(x) &= \int_{-\infty}^{\infty} (f * \delta_n)(\zeta) \sin x\zeta d\zeta \\ &= \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} \delta_n(\zeta - t) \sin(x\zeta)\, d\zeta dt \\ &\longrightarrow \int_{-\infty}^{\infty} f(t) \sin(xt)\, dt \\ &= \mathscr{A}^o(f)(x).\end{aligned} \tag{39}$$

Similarly,

$$\mathscr{A}^e (f * \delta_n)(x) \longrightarrow \mathscr{A}^e(f)(x) \quad \text{as } n \longrightarrow \infty. \tag{40}$$

Therefore, invoking the above equations in (37), we get $\mathscr{B}^{\mathscr{A}}(f * \delta_n)(y) \to \mathscr{B}^{\mathscr{A}} f(y)$ as $n \to \infty$. Hence the theorem. □

Denote by $\rho_{\mathscr{L}^1}$ the space of integrable Boehmians, then $\rho_{\mathscr{L}^1}$ is a convolution algebra when multiplication by scalar, addition, and convolution is defined as [9]

$$\begin{gathered} k\left[\frac{f_n}{\delta_n}\right] = \left[\frac{kf_n}{\delta_n}\right], \\ \left[\frac{f_n}{\delta_n}\right] + \left[\frac{g_n}{\gamma_n}\right] = \left[\frac{f_n * \gamma_n + g_n * \delta_n}{\delta_n * \gamma_n}\right], \\ \left[\frac{f_n}{\delta_n}\right] * \left[\frac{g_n}{\gamma_n}\right] = \left[\frac{f_n * g_n}{\delta_n * \gamma_n}\right]. \end{gathered} \tag{41}$$

Each function $f \in \mathscr{L}^1$ is identified with the Boehmian $[f * \delta_n/\delta_n]$. Also, $[f_n/\delta_n] * \delta_n = f_n \in \mathscr{L}^1$, for every $n \in \mathscr{N}$. Since $[\delta_n/\delta_n]$ corresponds to Dirac delta distribution δ, the kth-derivative of each $\rho \in \rho_{\mathscr{L}^1}$ is defined as

$$\mathscr{D}^k \rho = \rho * \mathscr{D}^k \delta. \tag{42}$$

The integral of a Boehmian $\rho = [f_n/\delta_n] \in \rho_{\mathscr{L}^1}$ is defined as [11]

$$\int_{-\infty}^{\infty} \rho(x)\, dx = \int_{-\infty}^{\infty} f_1(x)\, dx. \tag{43}$$

It is of great interest to observe the following example.

Example 9. Every infinitely smooth function $f(x) \in \mathscr{L}^1$ such that $\mathscr{D}^k f(x) \notin \mathscr{L}^1$ is integrable as Boehmian but not integrable as function.

The following has importance in the sense of analysis.

Theorem 10. *Let $[f_n/\delta_n] \in \rho_{\mathscr{L}^1}$, then the sequence*

$$\mathscr{B}^{\mathscr{A}}(f_n)(y) = \int_{-\infty}^{\infty} (\mathscr{A}^o f_n(x) \cos(xy) + \mathscr{A}^e f_n(x) \sin(xy))\, dx \tag{44}$$

converges uniformly on each compact subset $\mathscr{K}$ of $\mathscr{R}$.

Proof. By aid of the Theorem 8 and the concept of quotient of sequences, we have,

$$\begin{aligned}\mathscr{B}^{\mathscr{A}}(f_n)(y) &= \mathscr{B}^{\mathscr{A}}\left(f_n * \frac{\delta_k}{\delta_k}\right)(y) \\ &= \mathscr{B}^{\mathscr{A}}\left(\frac{f_n * \delta_k}{\delta_k}\right)(y) \\ &= \mathscr{B}^{\mathscr{A}}\left(\frac{f_k}{\delta_k} * \delta_n\right)(y) \\ &\longrightarrow \mathscr{B}^{\mathscr{A}} \frac{f_k}{\delta_k}(y) \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{45}$$

where convergence ranges over compact subsets of $\mathscr{R}$. The theorem is completely proved. □

Let $[f_n/\delta_n] \in \rho_{\mathscr{L}^1}$, then by virtue of Theorem 10 we define the Hartley-Hilbert transform of the Lebesgue Boehmian $[f_n/\delta_n]$ as

$$\widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_n}{\delta_n}\right] = \lim_{n\to\infty} \mathscr{B}^{\mathscr{A}} f_n. \tag{46}$$

on compact subsets of $\mathscr{R}$.

The next objective is to establish that our definition is well defined. Let $[f_n/\delta_n] = [g_n/\gamma_n]$ in $\rho_{\mathscr{L}^1}$, then

$$f_n * \gamma_m = g_m * \delta_n, \quad \text{for every } m, n \in \mathscr{N}. \tag{47}$$

Hence, applying the Hartley-Hilbert transform to both sides of the above equation and using the concept of quotients of sequences imply

$$\mathscr{B}^{\mathscr{A}}(f_n * \gamma_m) = \mathscr{B}^{\mathscr{A}}(g_m * \delta_n) = \mathscr{B}^{\mathscr{A}}(g_n * \delta_m). \tag{48}$$

Thus, in particular, for $n = m$, and considering Theorem 3, we get

$$\lim_{n\to\infty} \mathscr{B}^{\mathscr{A}} f_n = \lim_{n\to\infty} \mathscr{B}^{\mathscr{A}} g_n. \tag{49}$$

Hence,

$$\widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_n}{\delta_n}\right] = \widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{g_n}{\gamma_n}\right]. \tag{50}$$

Definition (46) is therefore well defined.

Theorem 11. *The generalized transform $\widetilde{\mathscr{B}^{\mathscr{A}}}$ is linear.*

Proof. Let $\rho_1 = [f_n/\delta_n]$ and $\rho_2 = [g_n/\gamma_n]$ be arbitrary in $\rho_{\mathscr{L}^1}$ and $\alpha \in \mathbb{C}$ then $\rho_1 + \rho_2 = [(f_n * \gamma_n + g_n * \delta_n)/\delta_n * \gamma_n]$. Hence, employing (46) yields

$$\widetilde{\mathscr{B}^{\mathscr{A}}}(\rho_1 + \rho_2) = \lim_{n\to\infty}\left(\mathscr{B}^{\mathscr{A}}(f_n * \gamma_n) + \mathscr{B}^{\mathscr{A}}(g_n * \delta_n)\right). \tag{51}$$

By Theorem 8, we get

$$\widetilde{\mathscr{B}^{\mathscr{A}}}(\rho_1 + \rho_2) = \lim_{n\to\infty}\mathscr{B}^{\mathscr{A}} f_n + \lim_{n\to\infty}\mathscr{B}^{\mathscr{A}} g_n. \tag{52}$$

Hence,

$$\widetilde{\mathscr{B}^{\mathscr{A}}}(\rho_1 + \rho_2) = \widetilde{\mathscr{B}^{\mathscr{A}}}\rho_1 + \widetilde{\mathscr{B}^{\mathscr{A}}}\rho_2. \tag{53}$$

Also, for each complex number α, we have

$$\begin{aligned}\widetilde{\mathscr{B}^{\mathscr{A}}}(\alpha\rho_1) &= \widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{\alpha f_n}{\delta_n}\right]\\ &= \alpha\lim_{n\to\infty}\mathscr{B}^{\mathscr{A}} f_n\\ &= \alpha\widetilde{\mathscr{B}^{\mathscr{A}}}\rho_1.\end{aligned} \tag{54}$$

Hence we have the following theorem. □

Theorem 12. *Let $\rho \in \rho_{\mathscr{L}^1}$ and $(\epsilon_n) \in \Delta$, then*

$$\widetilde{\mathscr{B}^{\mathscr{A}}}(\rho * \epsilon_n) = \widetilde{\mathscr{B}^{\mathscr{A}}}\rho = \widetilde{\mathscr{B}^{\mathscr{A}}}(\epsilon_n * \rho). \tag{55}$$

Proof. Let $\rho = [f_n/\delta_n] \in \rho_{\mathscr{L}^1}$, then $\widetilde{\mathscr{B}^{\mathscr{A}}}(\rho * \epsilon_n) = \widetilde{\mathscr{B}^{\mathscr{A}}}[f_n * \epsilon_n/\delta_n] = \lim_{n\to\infty}\mathscr{B}^{\mathscr{A}}(f_n * \epsilon_n)$.

Hence, $\widetilde{\mathscr{B}^{\mathscr{A}}}(\rho * \epsilon_n) = \lim_{n\to\infty}\mathscr{B}^{\mathscr{A}} f_n = \widetilde{\mathscr{B}^{\mathscr{A}}}\rho$.

Similarly, we proceed for $\widetilde{\mathscr{B}^{\mathscr{A}}}\rho = \widetilde{\mathscr{B}^{\mathscr{A}}}(\epsilon_n * \rho)$.

This completes the theorem. The following theorem is obvious. □

Theorem 13. *If $\widetilde{\mathscr{B}^{\mathscr{A}}}\rho_1 = 0$, then $\rho_1 = 0$.*

Theorem 14. *The Hartley-Hilbert transform $\widetilde{\mathscr{B}^{\mathscr{A}}}$ is continuous with respect to the δ-convergence.*

Proof. Let $\rho_n \xrightarrow{\delta} \rho$ in $\rho_{\mathscr{L}^1}$ as $n \to \infty$, then we show that $\widetilde{\mathscr{B}^{\mathscr{A}}}\rho_n \xrightarrow{\delta} \widetilde{\mathscr{B}^{\mathscr{A}}}\rho$ as $n \to \infty$. Using ([11, Theorem 2.6]), we find $f_{n,k}$, $f_k \in \mathscr{L}^1$, $(\delta_k) \in \Delta$ such that $[f_{n,k}/\delta_k] = \rho_n$, $[f_k/\delta_k] = \rho$ and $f_{n,k} \to f_k$ as $n \to \infty$, $k \in \mathscr{N}$.

Applying the Hartley-Hilbert transform for both sides implies $\mathscr{B}^{\mathscr{A}} f_{n,k} \to \mathscr{B}^{\mathscr{A}} f_k$ in the space of continuous functions. Therefore, considering limits we get

$$\widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_{n,k}}{\delta_k}\right] \longrightarrow \widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_k}{\delta_k}\right]. \tag{56}$$

This completes the proof of the theorem. □

Theorem 15. *The Hartley-Hilbert transform $\widetilde{\mathscr{B}^{\mathscr{A}}}$ is continuous with respect to the Δ-convergence.*

Proof. Let $\rho_n \xrightarrow{\Delta} \rho$ as $n \to \infty$ in $\rho_{\mathscr{L}^1}$, then there is $f_n \in \mathscr{L}^1$ and $\delta_n \in \Delta$ such that

$$(\rho_n - \rho) * \delta_n = \left[\frac{f_n * \delta_n}{\delta_k}\right], \quad f_n \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{57}$$

Thus by the aid of Theorem 3 and the hypothesis of the theorem we have

$$\begin{aligned}\widetilde{\mathscr{B}^{\mathscr{A}}}((\rho_n - \rho) * \delta_n) &= \widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_n * \delta_n}{\delta_k}\right]\\ &\longrightarrow \mathscr{B}^{\mathscr{A}}(f_n * \delta_n) \quad \text{as } n \longrightarrow \infty\\ &\longrightarrow \mathscr{B}^{\mathscr{A}} f_n \quad \text{as } n \longrightarrow \infty\\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{58}$$

Therefore, $\widetilde{\mathscr{B}^{\mathscr{A}}}(\rho_n - \rho) \to 0$ as $n \to \infty$. Thus, $\widetilde{\mathscr{B}^{\mathscr{A}}}\rho_n \xrightarrow{\Delta} \widetilde{\mathscr{B}^{\mathscr{A}}}\rho$ as $n \to \infty$.

This completes the proof. □

Lemma 16. *Let $[f_n/\delta_n] \in \rho_{\mathscr{L}^1}$, and δ has the usual meaning of* (34) *then*

$$\widetilde{\mathscr{B}^{\mathscr{A}}}\left(\left[\frac{f_n}{\delta_n}\right] * \delta\right) = \widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_n}{\delta_n}\right]. \tag{59}$$

Proof. Let $\rho = [f_n/\delta_n] \in \rho_{\mathscr{L}^1}$, then

$$\begin{aligned}\widetilde{\mathscr{B}^{\mathscr{A}}}\left(\left[\frac{f_n}{\delta_n}\right] * \delta\right) &= \widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_n * \delta}{\delta_n}\right]\\ &= \lim_{n\to\infty}\mathscr{B}^{\mathscr{A}}(f_n * \delta)\\ &= \lim_{n\to\infty}\mathscr{B}^{\mathscr{A}} f_n.\end{aligned} \tag{60}$$

Hence,

$$\widetilde{\mathscr{B}^{\mathscr{A}}}\left(\left[\frac{f_n}{\delta_n}\right] * \delta\right) = \widetilde{\mathscr{B}^{\mathscr{A}}}\left[\frac{f_n}{\delta_n}\right]. \tag{61}$$

□

Theorem 17. *The Hartley-Hilbert transform $\widetilde{\mathscr{B}^{\mathscr{A}}}$ is one-to-one.*

Proof. Let $\widetilde{\mathscr{B}^{\mathscr{A}}}[f_n/\delta_n] = \widetilde{\mathscr{B}^{\mathscr{A}}}[g_n/\gamma_n]$, then by the aid of (46), we get

$$\lim_{n\to\infty}\mathscr{B}^{\mathscr{A}} f_n = \lim_{n\to\infty}\mathscr{B}^{\mathscr{A}} g_n. \tag{62}$$

Hence,

$$\mathscr{B}^{\mathscr{A}}\left(\lim_{n\to\infty} f_n\right) = \mathscr{B}^{\mathscr{A}}\left(\lim_{n\to\infty} g_n\right), \tag{63}$$

that is, $\mathscr{B}^{\mathscr{A}} f = \mathscr{B}^{\mathscr{A}} g$. The fact that $\mathscr{B}^{\mathscr{A}}$ is one-to-one implies $f = g$. Hence we have the following theorem. □

4. A Comparative Study: Fourier-Hilbert Transform

In [6, 7], the Hilbert transform via the Fourier transform of $f(x)$ is defined as

$$\mathscr{B}^{\mathscr{F}}(f)(y) = \frac{1}{\pi}\int_0^{\infty}\left(\mathscr{F}_i(f)(x)\cos(xy) - \mathscr{F}_r(f)(x)\sin(xy)\right)dx, \tag{64}$$

where

$$\mathscr{F}_r(f)(x) = \int_0^{\infty} f(t)\cos(xt)\,dt,$$
$$\mathscr{F}_i(f)(x) = \int_0^{\infty} f(t)\sin(xt)\,dt \tag{65}$$

are, respectively, the real and imaginary components of the Fourier transform of f, which are related by

$$\mathscr{F}(f)(x) = \mathscr{F}_r(f)(x) - i\mathscr{F}_i(f)(x). \tag{66}$$

It is interesting to know that a concrete relationship between $\mathscr{F}_r, \mathscr{A}^e$ and $\mathscr{F}_i, \mathscr{A}^o$ is described as $\mathscr{F}_r(x) = \mathscr{A}^e(x)$ and $\mathscr{F}_i(x) = \mathscr{A}^o(x)$ [2]. Those equations, above, justify the following statements of the next theorems.

Theorem 18. *Let $[f_n/\delta_n] \in \rho_{\mathscr{L}^1}$, then the sequence of Fourier-Hilbert transforms of (f_n) satisfies*

$$\mathscr{B}^{\mathscr{F}}(f_n)(y) = \frac{1}{\pi}\int_0^{\infty}\left(\mathscr{F}_i(f_n)(x)\cos(xy) - \mathscr{F}_r(f_n)(x)\sin(xy)\right)dx \tag{67}$$

and converges uniformly on each compact subset $\mathscr{K}$ of $\mathscr{R}$.

Thus, for $[f_n/\delta_n] \in \rho_{\mathscr{L}^1}$, the Fourier-Hilbert transform of $[f_n/\delta_n]$ is similarly defined by

$$\widetilde{\mathscr{B}^{\mathscr{F}}}\left[\frac{f_n}{\delta_n}\right] = \lim_{n\to\infty}\mathscr{B}^{\mathscr{F}} f_n \tag{68}$$

on compact subsets of $\mathscr{R}$.

The following theorems are stated and their proofs are justified for similar reasons. We prefer to we omit the details.

Theorem 19. *The generalized Fourier-Hilbert transform $\widetilde{\mathscr{B}^{\mathscr{F}}}$ is linear.*

Theorem 20. $\widetilde{\mathscr{B}^{\mathscr{F}}}(\rho * \epsilon_n) = \widetilde{\mathscr{B}^{\mathscr{F}}}\rho = \widetilde{\mathscr{B}^{\mathscr{F}}}(\epsilon * \rho)$, $(\epsilon_n) \in \Delta$.

Theorem 21. *If $\widetilde{\mathscr{B}^{\mathscr{F}}}\rho_1 = 0$, then $\rho_1 = 0$.*

Theorem 22. *If $\rho_n \xrightarrow{\Delta} \rho$ as $n \to \infty$ in $\rho_{\mathscr{L}^1}$, then $\widetilde{\mathscr{B}^{\mathscr{F}}}\rho_n \xrightarrow{\Delta} \widetilde{\mathscr{B}^{\mathscr{F}}}\rho$ as $n \to \infty$ in $\rho_{\mathscr{L}^1}$ on compact subsets.*

Theorem 23. *The Fourier-Hilbert transform $\widetilde{\mathscr{B}^{\mathscr{F}}}$ is continuous with respect to the δ-convergence.*

Theorem 24. *The Fourier-Hilbert transform $\widetilde{\mathscr{B}^{\mathscr{F}}}$ is continuous with respect to the Δ-convergence.*

Proofs of the above theorems are similar to that given for the corresponding ones of Hartley-Hilbert transform.

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