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Research Article

On the Stability of Trigonometric Functional Equations in Distributions and Hyperfunctions

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We consider the Hyers-Ulam stability for a class of trigonometric functional equations in the spaces of generalized functions such as Schwartz distributions and Gelfand hyperfunctions.

1. Introduction

Hyers-Ulam stability problems of functional equations go back to 1940 when Ulam proposed the following question [1].

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot,\cdot)$ such that

$$d\left(f\left(xy\right),f\left(x\right)f\left(y\right)\right)\le\epsilon. \tag{1}$$

Then does there exist a group homomorphism h and $\delta_\epsilon > 0$ such that

$$d\left(f\left(x\right),h\left(x\right)\right)\le\delta_\epsilon \tag{2}$$

for all $x\in G_1$?

This problem was solved affirmatively by Hyers [2] under the assumption that G_2 is a Banach space. After the result of Hyers, Aoki [3] and Bourgin [4, 5] treated with this problem; however, there were no other results on this problem until 1978 when Rassias [6] treated again with the inequality of Aoki [3]. Generalizing Hyers' result, he proved that if a mapping $f : X \to Y$ between two Banach spaces satisfies

$$\left\|f\left(x+y\right)-f\left(x\right)-f\left(y\right)\right\|\le\Phi\left(x,y\right),\quad\text{for } x,y\in X \tag{3}$$

with $\Phi(x,y)=\epsilon(\|x\|^p+\|y\|^p)$ ($\epsilon\ge 0$, $0\le p<1$), then there exists a unique additive function $A : X\to Y$ such that $\|f(x)-A(x)\|\le 2\epsilon|x|^p/(2-2^p)$ for all $x\in X$. In 1951 Bourgin [4, 5] stated that if Φ is symmetric in $\|x\|$ and $\|y\|$ with $\sum_{j=1}^{\infty}\Phi(2^jx,2^jx)/2^j<\infty$ for each $x\in X$, then there exists a unique additive function $A : X\to Y$ such that $\|f(x)-A(x)\|\le\sum_{j=1}^{\infty}\Phi(2^jx,2^jx)/2^j$ for all $x\in X$. Unfortunately, there was no use of these results until 1978 when Rassias [7] treated with the inequality of Aoki [3]. Following Rassias' result, a great number of papers on the subject have been published concerning numerous functional equations in various directions [6–10, 10–25]. In 1990 Székelyhidi [24] has developed his idea of using invariant subspaces of functions defined on a group or semigroup in connection with stability questions for the sine and cosine functional equations. We refer the reader to [9, 10, 18, 19, 25] for Hyers-Ulam stability of functional equations of trigonometric type. In this paper, following the method of Székelyhidi [24] we consider a distributional analogue of the Hyers-Ulam stability problem of the trigonometric functional inequalities

$$\begin{aligned}&\left|f\left(x-y\right)-f\left(x\right)g\left(y\right)+g\left(x\right)f\left(y\right)\right|\le\psi\left(y\right),\\&\left|g\left(x-y\right)-g\left(x\right)g\left(y\right)-f\left(x\right)f\left(y\right)\right|\le\psi\left(y\right),\end{aligned} \tag{4}$$

where $f,g:\mathbb{R}^n\to\mathbb{C}$ and $\psi:\mathbb{R}^n\to[0,\infty)$ is a continuous function. As a distributional version of the inequalities (4), we consider the inequalities for the generalized functions $u, v \in \mathcal{G}'(\mathbb{R}^n)$ (resp., $\mathcal{S}'(\mathbb{R}^n)$),

$$\begin{aligned}&\left\|u \circ (x-y) - u_x \otimes v_y + v_x \otimes u_y\right\| \le \psi(y),\\&\left\|v \circ (x-y) - v_x \otimes v_y - u_x \otimes u_y\right\| \le \psi(y),\end{aligned} \tag{5}$$

where $\circ$ and $\otimes$ denote the pullback and the tensor product of generalized functions, respectively, and $\psi : \mathbb{R}^n \to [0,\infty)$ denotes a continuous infraexponential function of order 2 (resp., a function of polynomial growth). For the proof we employ the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernel

$$E_t(x) = (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right), \quad x \in \mathbb{R}^n,\ t > 0. \tag{6}$$

For the first step, convolving $E_t(x)E_s(y)$ in both sides of (5) we convert (5) to the Hyers-Ulam stability problems of *trigonometric-hyperbolic type* functional inequalities, respectively,

$$\begin{aligned}&\left|U(x-y,t+s) - U(x,t)V(y,s) + V(x,t)U(y,s)\right|\\&\quad\le \Psi(y,s),\\&\left|V(x-y,t+s) - V(x,t)V(y,s) - U(x,t)U(y,s)\right|\\&\quad\le \Psi(y,s),\end{aligned} \tag{7}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, where U, V are the Gauss transforms of u, v, respectively, given by

$$U(x,t) = u * E_t(x) = \left\langle u_y, E_t(x-y)\right\rangle, \tag{8}$$

$$V(x,t) = v * E_t(x), \tag{9}$$

which are solutions of the heat equation, and

$$\Psi(y,s) = \int \psi(\eta) E_s(\eta - y)\, d\eta = (\psi * E_s)(y). \tag{10}$$

For the second step, using similar idea of Székelyhidi [24] we prove the Hyers-Ulam stabilities of inequalities (7). For the final step, taking initial values as $t \to 0^+$ for the results we arrive at our results.

2. Generalized Functions

We first introduce the spaces $\mathcal{S}'$ of Schwartz tempered distributions and $\mathcal{G}'$ of Gelfand hyperfunctions (see [26–29] for more details of these spaces). We use the notations: $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of nonnegative integers and $\partial_j = \partial/\partial x_j$.

Definition 1 (see [29]). One denotes by $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{\alpha,\beta} = \sup_x \left|x^\alpha \partial^\beta \varphi(x)\right| < \infty \tag{11}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. The elements of $\mathcal{S}$ are called rapidly decreasing functions, and the elements of the dual space $\mathcal{S}'$ are called tempered distributions.

Definition 2 (see [26]). One denotes by $\mathcal{G}$ or $\mathcal{G}(\mathbb{R}^n)$ the Gelfand space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{h,k} = \sup_{x\in\mathbb{R}^n,\alpha,\beta\in\mathbb{N}_0^n} \frac{\left|x^\alpha \partial^\beta \varphi(x)\right|}{h^{|\alpha|} k^{|\beta|} \alpha!^{1/2} \beta!^{1/2}} < \infty \tag{12}$$

for some $h, k > 0$. One says that $\varphi_j \to 0$ as $j \to \infty$ if $\|\varphi_j\|_{h,k} \to 0$ as $j \to \infty$ for some h, k, and one denotes by $\mathcal{G}'$ the dual space of $\mathcal{G}$ and calls its elements Gelfand hyperfunctions.

It is well known that the following topological inclusions hold:

$$\mathcal{G} \hookrightarrow \mathcal{S}, \qquad \mathcal{S}' \hookrightarrow \mathcal{G}'. \tag{13}$$

It is known that the space $\mathcal{G}(\mathbb{R}^n)$ consists of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ which can be extended to an entire function on $\mathbb{C}^n$ satisfying

$$\left|\varphi(x+iy)\right| \le C \exp\left(-a|x|^2 + b|y|^2\right), \quad x, y \in \mathbb{R}^n \tag{14}$$

for some a, b, and $C > 0$ (see [26]).

By virtue of Theorem 6.12 of [27, p. 134] we have the following.

Definition 3. Let $u_j \in \mathcal{G}'(\mathbb{R}^{n_j})$ for $j = 1, 2$, with $n_1 \ge n_2$, and let $\lambda : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ be a smooth function such that, for each $x \in \mathbb{R}^{n_1}$, the Jacobian matrix $\nabla\lambda(x)$ of λ at x has rank n_2. Then there exists a unique continuous linear map $\lambda^* : \mathcal{G}'(\mathbb{R}^{n_2}) \to \mathcal{G}'(\mathbb{R}^{n_1})$ such that $\lambda^* u = u \circ \lambda$ when u is a continuous function. One calls $\lambda^* u$ the pullback of u by λ which is often denoted by $u \circ \lambda$.

In particular, let $\lambda : \mathbb{R}^{2n} \to \mathbb{R}^n$ be defined by $\lambda(x, y) = x - y$, $x, y \in \mathbb{R}^n$. Then in view of the proof of Theorem 6.12 of [27, p. 134] we have

$$\langle u \circ \lambda, \varphi(x,y)\rangle = \left\langle u, \int \varphi(x-y, y)\, dy\right\rangle. \tag{15}$$

Definition 4. Let $u_x \in \mathcal{G}'(\mathbb{R}^{n_1})$, $u_y \in \mathcal{G}'(\mathbb{R}^{n_2})$. Then the tensor product $u_x \otimes u_y$ of u_x and u_y, defined by

$$\langle u_x \otimes u_y, \varphi(x,y)\rangle = \langle u_x, \langle u_y, \varphi(x,y)\rangle\rangle \tag{16}$$

for $\varphi(x,y) \in \mathcal{G}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, belongs to $\mathcal{G}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

For more details of pullback and tensor product of distributions we refer the reader to Chapter V-VI of [27].

3. Main Theorems

Let f be a Lebesgue measurable function on $\mathbb{R}^n$. Then f is said to be an *infraexponential function of order 2 (resp.,* *a function of polynomial growth)* if for every $\epsilon > 0$ there exists $C_\epsilon > 0$ (resp., there exist positive constants C, N, and d) such that

$$\left|f(x)\right| \le C_\epsilon e^{\epsilon|x|^2} \quad \left[\text{resp. } \le C|x|^N + d\right] \tag{17}$$

for all $x \in \mathbb{R}^n$. It is easy to see that every infraexponential function f of order 2 (resp., every function of polynomial growth) defines an element of $\mathcal{G}'(\mathbb{R}^n)$ (resp., $\mathcal{S}'(\mathbb{R}^n)$) via the correspondence

$$\langle f, \varphi\rangle = \int f(x)\varphi(x)\,dx \tag{18}$$

for $\varphi \in \mathcal{G}(\mathbb{R}^n)$ (resp. $\mathcal{S}(\mathbb{R}^n)$).

Let $u, v \in \mathcal{G}'(\mathbb{R}^n)$ (resp., $\mathcal{S}'(\mathbb{R}^n)$). We prove the stability of the following functional inequalities:

$$\left\|u \circ (x-y) - u_x \otimes v_y + v_x \otimes u_y\right\| \le \psi(y), \tag{19}$$

$$\left\|v \circ (x-y) - v_x \otimes v_y - u_x \otimes u_y\right\| \le \psi(y), \tag{20}$$

where $\circ$ and $\otimes$ denote the pullback and the tensor product of generalized functions, respectively, $\psi : \mathbb{R}^n \to [0,\infty)$ denotes a continuous infraexponential functional of order 2 (resp. a continuous function of polynomial growth) with $\psi(0) = 0$, and $\|\cdot\| \le \psi$ means that $|\langle\cdot,\varphi\rangle| \le \|\psi\varphi\|_{L^1}$ for all $\varphi \in \mathcal{G}(\mathbb{R}^n)$ (resp., $\mathcal{S}(\mathbb{R}^n)$).

In view of (14) it is easy to see that the n-dimensional heat kernel

$$E_t(x) = (4\pi t)^{-n/2}\exp\left(-\frac{|x|^2}{4t}\right), \quad t > 0, \tag{21}$$

belongs to the Gelfand space $\mathcal{G}(\mathbb{R}^n)$ for each $t > 0$. Thus the convolution $(u * E_t)(x) := \langle u_y, E_t(x-y)\rangle$ is well defined for all $u \in \mathcal{G}'(\mathbb{R}^n)$. It is well known that $U(x,t) = (u * E_t)(x)$ is a smooth solution of the heat equation $(\partial/\partial_t - \Delta)U = 0$ in $\{(x,t) : x \in \mathbb{R}^n, t > 0\}$ and $(u * E_t)(x) \to u$ as $t \to 0^+$ in the sense of generalized functions that is, for every $\varphi \in \mathcal{G}(\mathbb{R}^n)$,

$$\langle u, \varphi\rangle = \lim_{t\to 0^+}\int (u * E_t)(x)\varphi(x)\,dx. \tag{22}$$

We call $(u * E_t)(x)$ the Gauss transform of u.

A function A from a semigroup $\langle S, +\rangle$ to the field $\mathbb{C}$ of complex numbers is said to be *an additive function* provided that $A(x+y) = A(x) + A(y)$, and $m : S \to \mathbb{C}$ is said to be *an exponential function* provided that $m(x+y) = m(x)m(y)$.

For the proof of stabilities of (19) and (20) we need the following.

Lemma 5 (see [15]). *Let S be a semigroup and $\mathbb{C}$ the field of complex numbers. Assume that $f, g : S \to \mathbb{C}$ satisfy the inequality; for each $y \in S$ there exists a positive constant M_y such that*

$$\left|f(x+y) - f(x)g(y)\right| \le M_y \tag{23}$$

for all $x \in S$. Then either f is a bounded function or g is an exponential function.

Proof. Suppose that g is not exponential. Then there are $y, z \in S$ such that $g(y+z) \ne g(y)g(z)$. Now we have

$$\begin{aligned} f(x+y+z) &- f(x+y)g(z) \\ &= (f(x+y+z) - f(x)g(y+z)) \\ &\quad - g(z)(f(x+y) - f(x)g(y)) \\ &\quad + f(x)(g(y+z) - g(y)g(z)), \end{aligned} \tag{24}$$

and hence

$$\begin{aligned} f(x) = {} & (g(y+z) - g(y)g(z))^{-1} \\ & \times ((f(x+y+z) - f(x+y)g(z)) \\ & \quad - (f(x+y+z) - f(x)g(y+z)) \\ & \quad + g(z)(f(x+y) - f(x)g(y))). \end{aligned} \tag{25}$$

In view of (23) the right hand side of (25) is bounded as a function of x. Consequently, f is bounded. □

Lemma 6 (see [30, p. 122]). *Let $f(x,t)$ be a solution of the heat equation. Then $f(x,t)$ satisfies*

$$\left|f(x,t)\right| \le M, \quad x \in \mathbb{R}^n,\ t \in (0,1) \tag{26}$$

for some $M > 0$, if and only if

$$f(x,t) = (f_0 * E_t)(x) = \int f_0(y)E_t(x-y)\,dy \tag{27}$$

for some bounded measurable function f_0 defined in $\mathbb{R}^n$. In particular, $f(x,t) \to f_0(x)$ in $\mathcal{G}'(\mathbb{R}^n)$ as $t \to 0^+$.

We discuss the solutions of the corresponding trigonometric functional equations

$$u \circ (x-y) - u_x \otimes v_y + v_x \otimes u_y = 0, \tag{28}$$

$$v \circ (x-y) - v_x \otimes v_y - u_x \otimes u_y = 0, \tag{29}$$

in the space $\mathcal{G}'$ of Gelfand hyperfunctions. As a consequence of the results [8, 31, 32] we have the following.

Lemma 7. *The solutions $u, v \in \mathcal{G}'(\mathbb{R}^n)$ of (28) and (29) are equal, respectively, to the continuous solutions $f, g : \mathbb{R}^n \to \mathbb{C}$ of corresponding classical functional equations*

$$f(x-y) - f(x)g(y) + g(x)f(y) = 0, \tag{30}$$

$$g(x-y) - g(x)g(y) - f(x)f(y) = 0. \tag{31}$$

The continuous solutions (f, g) of the functional equation (30) are given by one of the following:

(i) *$f = 0$ and g is arbitrary,*

(ii) *$f(x) = c_1 \cdot x$, $g(x) = 1 + c_2 \cdot x$ for some $c_1, c_2 \in \mathbb{C}^n$,*

(iii) *$f(x) = \lambda_1 \sin(c\cdot x)$ and $g(x) = \cos(c\cdot x) + \lambda_2\sin(c\cdot x)$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$, $c \in \mathbb{C}^n$.*

Also, the continuous solutions (f, g) of the functional equation (31) *are given by one of the following:*

(i) $g(x) = \lambda$ *and* $f(x) = \pm\sqrt{\lambda - \lambda^2}$ *for some* $\lambda \in \mathbb{C}$,

(ii) $g(x) = \cos(c \cdot x)$ *and* $f(x) = \sin(c \cdot x)$ *for some* $c \in \mathbb{C}^n$.

For the proof of the stability of (19) we need the followings.

Lemma 8. *Let G be an Abelian group and let $U, V : G \times (0, \infty) \to \mathbb{C}$ satisfy the inequality; there exists a nonnegative function $\Psi : G \times (0, \infty) \to \mathbb{R}$ such that*

$$\begin{aligned} &\left|U(x-y, t+s) - U(x,t)V(y,s) + V(x,t)U(y,s)\right| \\ &\quad \leq \Psi(y,s) \end{aligned} \tag{32}$$

for all $x, y \in G$, $t, s > 0$. Then either there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both are zero, and $M > 0$ such that

$$\left|\lambda_1 U(x,t) - \lambda_2 V(x,t)\right| \leq M, \tag{33}$$

or else

$$U(x-y, t+s) - U(x,t)V(y,s) + V(x,t)U(y,s) = 0 \tag{34}$$

for all $x, y \in G$, $t, s > 0$.

Proof. Suppose that inequality (33) holds only when $\lambda_1 = \lambda_2 = 0$. Let

$$\begin{aligned} K(x,y,t,s) = {} & U(x+y, t+s) - U(x,t)V(-y,s) \\ & + V(x,t)U(-y,s), \end{aligned} \tag{35}$$

and choose y_1 and s_1 satisfying $U(-y_1, s_1) \neq 0$. Now it can be easily calculated that

$$\begin{aligned} V(x,t) = {} & \lambda_0 U(x,t) + \lambda_1 U(x+y_1, t+s_1) \\ & - \lambda_1 K(x, y_1, t, s_1), \end{aligned} \tag{36}$$

where $\lambda_0 = V(-y_1, s_1)/U(-y_1, s_1)$ and $\lambda_1 = -1/U(-y_1, s_1)$. By (35) we have

$$\begin{aligned} U(x + (y+z), t + (s+r)) = {} & U(x,t)V(-y-z, s+r) \\ & - V(x,t)U(-y-z, s+r) \\ & + K(x, y+z, t, s+r). \end{aligned} \tag{37}$$

Also by (35) and (36) we have

$$\begin{aligned} & U((x+y)+z, (t+s)+r) \\ & = U(x+y, t+s)V(-z,r) - V(x+y, t+s)U(-z,r) \\ & \quad + K(x+y, z, t+s, r) \\ & = (U(x,t)V(-y,s) - V(x,t)U(-y,s) \\ & \quad + K(x,y,t,s))V(-z,r) \\ & \quad - (\lambda_0 U(x+y, t+s) + \lambda_1 U(x+y+y_1, t+s+s_1) \\ & \quad - \lambda_1 K(x+y, y_1, t+s, s_1))U(-z,r) \\ & \quad + K(x+y, z, t+s, r) \\ & = (U(x,t)V(-y,s) - V(x,t)U(-y,s) \\ & \quad + K(x,y,t,s))V(-z,r) \\ & \quad - \lambda_0 (U(x,t)V(-y,s) - V(x,t)U(-y,s) \\ & \quad + K(x,y,t,s))U(-z,r) \\ & \quad - \lambda_1 (U(x,t)V(-y-y_1, s+s_1) \\ & \quad - V(x,t)U(-y-y_1, s+s_1) \\ & \quad + K(x, y+y_1, t, s+s_1))U(-z,r) \\ & \quad + \lambda_1 K(x+y, y_1, t+s, s_1)U(-z,r) \\ & \quad + K(x+y, z, t+s, r). \end{aligned} \tag{38}$$

From (37) and (38) we have

$$\begin{aligned} & (V(-y,s)V(-z,r) - \lambda_0 V(-y,s)U(-z,r) \\ & \quad - \lambda_1 V(-y-y_1, s+s_1)U(-z,r) \\ & \quad - V(-y-z, s+r))U(x,t) \\ & + (-U(-y,s)V(-z,r) + \lambda_0 U(-y,s)U(-z,r) \\ & \quad + \lambda_1 U(-y-y_1, s+s_1)U(-z,r) \\ & \quad + U(-y-z, s+r))V(x,t) \\ & = -K(x,y,t,s)V(-z,r) + \lambda_0 K(x,y,t,s)U(-z,r) \\ & \quad + \lambda_1 K(x, y+y_1, t, s+s_1)U(-z,r) \\ & \quad - \lambda_1 K(x+y, y_1, t+s, s_1)U(-z,r) \\ & \quad - K(x+y, z, t+s, r) + K(x, y+z, t, s+r). \end{aligned} \tag{39}$$

Since $K(x,y,t,s)$ is bounded by $\Psi(-y,s)$, if we fix y, z, r, and s, the right hand side of (39) is bounded by a constant M, where

$$\begin{aligned} M = {} & \Psi(-y,s)\,|V(-z,r)| + \Psi(-y,s)\,|\lambda_0 U(-z,r)| \\ & + \Psi(-y-y_1,s+s_1)\,|\lambda_1 U(-z,r)| \\ & + \Psi(-y_1,s_1)\,|\lambda_1 U(-z,r)| + \Psi(-z,r) \\ & + \Psi(-y-z,r+s). \end{aligned} \tag{40}$$

So by our assumption, the left hand side of (39) vanishes, so is the right hand side. Thus we have

$$\begin{aligned} & (-\lambda_0 K(x,y,t,s) - \lambda_1 K(x,y+y_1,t,s+s_1) \\ & \quad + \lambda_1 K(x+y,y_1,t+s,s_1))\,U(-z,r) \\ & + K(x,y,t,s)\,V(-z,r) = K(x,y+z,t,s+r) \\ & - K(x+y,z,t+s,r). \end{aligned} \tag{41}$$

Now by the definition of K we have

$$\begin{aligned} & K(x+y,z,t+s,r) - K(x,y+z,t,s+r) \\ & = U(x+y+z,t+s+r) - U(x+y,t+s)\,V(-z,r) \\ & \quad + V(x+y,t+s)\,U(-z,r) - U(x+y+z,t+s+r) \\ & \quad + U(x,t)\,V(-y-z,s+r) - V(x,t)\,U(-y-z,s+r) \\ & = U(-y-z-x,s+r+t) - U(-y-z,s+r)\,V(x,t) \\ & \quad + V(-y-z,s+r)\,U(x,t) - U(-z-x-y,r+t+s) \\ & \quad + U(-z,r)\,V(x+y,t+s) - V(-z,r)\,U(x+y,t+s) \\ & = K(-y-z,-x,s+r,t) - K(-z,-x-y,r,t+s). \end{aligned} \tag{42}$$

Hence the left hand side of (41) is bounded by $\Psi(x,t)+\Psi(x+y,t+s)$. So if we fix x, y, t, and s in (41), the left hand side of (41) is a bounded function of z and r. Thus $K(x,y,t,s) \equiv 0$ by our assumption. This completes the proof. □

In the following lemma we assume that $\Psi : \mathbb{R}^n \times (0,\infty) \to [0,\infty)$ is a continuous function such that

$$\psi(x) := \lim_{t\to 0^+} \Psi(x,t) \tag{43}$$

exists and satisfies the conditions $\psi(0) = 0$ and

$$\Phi_1(x) := \sum_{k=0}^{\infty} 2^{-k}\psi(-2^k x) < \infty \tag{44}$$

or

$$\Phi_2(x) := \sum_{k=1}^{\infty} 2^{k}\psi(-2^{-k} x) < \infty. \tag{45}$$

Lemma 9. *Let $U, V : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be continuous functions satisfying*

$$\begin{aligned} & |U(x-y,t+s) - U(x,t)\,V(y,s) + V(x,t)\,U(y,s)| \\ & \quad \le \Psi(y,s) \end{aligned} \tag{46}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, and there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both are zero, and $M > 0$ such that

$$|\lambda_1 U(x,t) - \lambda_2 V(x,t)| \le M. \tag{47}$$

Then (U,V) satisfies one of the followings:

(i) *$U = 0$, V is arbitrary,*

(ii) *U and V are bounded functions,*

(iii) *$V(x,t) = \lambda U(x,t) + e^{ic\cdot x - bt}$ for some $\lambda \in \mathbb{C}^n$, $c(\neq 0) \in \mathbb{R}^n$, and $b \in \mathbb{C}$, and $f(x) := \lim_{t\to 0^+} U(x,t)$ is a continuous function; in particular, there exists $\delta : (0,\infty) \to [0,\infty)$ with $\delta(t) \to 0$ as $t \to 0^+$ such that*

$$|U(x,t) - f(x)\,e^{-bt}| \le \delta(t) \tag{48}$$

for all $x \in \mathbb{R}^n$, $t > 0$, and satisfies the condition; there exists $d \ge 0$ satisfying

$$|f(x)| \le \psi(-x) + d \tag{49}$$

for all $x \in \mathbb{R}^n$,

(iv) *$V(x,t) = \lambda U(x,t) + e^{-bt}$ for some $\lambda \in \mathbb{C}^n$, $b \in \mathbb{C}$, and $f(x) := \lim_{t\to 0^+} U(x,t)$ is a continuous function; in particular, there exists $\delta : (0,\infty) \to [0,\infty)$ with $\delta(t) \to 0$ as $t \to 0^+$ such that*

$$|U(x,t) - f(x)\,e^{-bt}| \le \delta(t) \tag{50}$$

for all $x \in \mathbb{R}^n$, $t > 0$, and satisfies one of the following conditions; there exists $a_1 \in \mathbb{C}^n$ such that

$$|f(x) - a_1 \cdot x| \le \Phi_1(x) \tag{51}$$

for all $x \in \mathbb{R}^n$, or there exists $a_2 \in \mathbb{C}^n$ such that

$$|f(x) - a_2 \cdot x| \le \Phi_2(x) \tag{52}$$

for all $x \in \mathbb{R}^n$.

Proof. If $U = 0$, V is arbitrary which is case (i). If U is a nontrivial bounded function, in view of (46) V is also bounded which gives case (ii). If U is unbounded, it follows from (47) that $\lambda_2 \neq 0$ and

$$V(x,t) = \lambda U(x,t) + R(x,t) \tag{53}$$

for some $\lambda \in \mathbb{C}$ and a bounded function R. Putting (53) in (46) we have

$$\begin{aligned} & |U(x-y,t+s) - U(x,t)\,R(y,s) + R(x,t)\,U(y,s)| \\ & \quad \le \Psi(y,s) \end{aligned} \tag{54}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Replacing y by $-y$ and using the triangle inequality, we have, for some $C > 0$,

$$\begin{aligned} &\left|U(x+y, t+s) - U(x,t) R(-y,s)\right| \\ &\quad \le C\left|U(-y,s)\right| + \Psi(-y,s) \end{aligned} \tag{55}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. By Lemma 5, $R(-y,s)$ is an exponential function. If $R = 0$, putting $y = 0, s \to 0^+$ in (54) we have

$$|U(x,t)| \le \psi(0) = 0. \tag{56}$$

Thus we have $R \ne 0$ since U is unbounded. Given the continuity of U and V we have

$$R(x,t) = e^{ic\cdot x - bt} \tag{57}$$

for some $c \in \mathbb{R}^n$, $b \in \mathbb{C}$ with $\Re b \ge 0$. Putting $y = 0$ and $s = 1$ in (54), dividing $R(0,1)$, and using the triangle inequality we have

$$|U(x,t)| \le |R(0,1)|^{-1} \left(|U(x,t+1)| + C|U(0,1)| + \Psi(0,1)\right) \tag{58}$$

for all $x \in \mathbb{R}^n$, $t > 0$.

From (58) and the continuity of U it is easy to see that

$$\limsup_{t\to 0^+} U(x,t) := f(x) \tag{59}$$

exists. Putting $x = y = 0$ and replacing s and t by $t/2$ in (54) we have

$$|U(0,t)| \le \Psi\left(0, \frac{t}{2}\right) \tag{60}$$

for all $t > 0$.

Fixing x, putting $y = 0$ letting $t \to 0^+$ so that $U(x,t) \to f(x)$ in (54), and using the triangle inequality and (60) we have

$$\left|U(x,s) - f(x) e^{-bs}\right| \le \Psi\left(0, \frac{s}{2}\right) + \Psi(0,s) := \delta(s) \tag{61}$$

for all $x \in \mathbb{R}^n$, $s > 0$. Letting $s \to 0^+$ in (61) we have

$$\lim_{s\to 0^+} U(x,s) = f(x) \tag{62}$$

for all $x \in \mathbb{R}^n$. From (61) the continuity of f can be checked by a usual calculus. Letting $t \to 0^+$ in (60) we see that $f(0) = 0$. Letting $t, s \to 0^+$ in (54) we have

$$\left|f(x-y) - f(x)e^{ic\cdot y} + e^{ic\cdot x} f(y)\right| \le \psi(y) \tag{63}$$

for all $x, y \in \mathbb{R}^n$. Putting $x = 0$ in (63) and replacing y by $-y$ we have

$$\left|f(-y) + f(y)\right| \le \psi(-y) \tag{64}$$

for all $y \in \mathbb{R}^n$.

Replacing y by $-y$ and using (64) and the triangle inequality we have

$$\left|f(x+y) - f(x)e^{-ic\cdot y} - e^{ic\cdot x} f(y)\right| \le 2\psi(-y) \tag{65}$$

for all $x, y \in \mathbb{R}^n$. Now we divide (65) into two cases: $c = 0$ and $c \ne 0$. First we consider the case $c \ne 0$. Replacing x by y and y by x in (65) we have

$$\left|f(x+y) - f(y)e^{-ic\cdot x} - e^{ic\cdot y} f(x)\right| \le 2\psi(-x) \tag{66}$$

for all $x, y \in \mathbb{R}^n$. From (65) and (66), using the triangle inequality and dividing $|e^{ic\cdot y} - e^{-ic\cdot y}|$ we have

$$\left|f(x)\right| \le \frac{2\left(\psi(-x) + \psi(-y) + \left|f(y)\right|\right)}{\left|e^{ic\cdot y} - e^{-ic\cdot y}\right|} \tag{67}$$

for all $x, y \in \mathbb{R}^n$ such that $c \cdot y \ne 0$. Choosing $y_0 \in \mathbb{R}^n$ so that $c \cdot y_0 = \pi/2$ and putting $y = y_0$ in (67) we have

$$\left|f(x)\right| \le \psi(-x) + d, \tag{68}$$

where $d = \psi(\pi/2) + |f(\pi/2)|$, which gives (iii). Now we consider the case $c = 0$. It follows from (65) that

$$\left|f(x+y) - f(x) - f(y)\right| \le 2\psi(-y) \tag{69}$$

for all $x, y \in \mathbb{R}^n$. By the well-known results in [3], there exists a unique additive function $A_1(x)$ given by

$$A_1(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{70}$$

such that

$$\left|f(x) - A_1(x)\right| \le \Phi_1(x) \tag{71}$$

if $\Phi_1(x) := \sum_{k=0}^{\infty} 2^{-k}\psi(-2^k x) < \infty$, and there exists a unique additive function $A_2(x)$ given by

$$A_2(x) = \lim_{n\to\infty} 2^n f(2^{-n} x) \tag{72}$$

such that

$$\left|f(x) - A_2(x)\right| \le \Phi_2(x) \tag{73}$$

if $\Phi_2(x) := \sum_{k=0}^{\infty} 2^k \psi(-2^{-k} x) < \infty$. Now by the continuity of U and inequality (61), it is easy to see that f is continuous. In view of (70) and (72), $A_j(x)$, $j = 1, 2$, are Lebesgue measurable functions. Thus there exist $a_1, a_2 \in \mathbb{C}^n$ such that $A_1(x) = a_1 \cdot x$ and $A_2(x) = a_2 \cdot x$ for all $x \in \mathbb{R}^n$, which gives (iv). This completes the proof. □

In the following we assume that ψ satisfies (44) or (45).

Theorem 10. *Let $u, v \in \mathcal{G}'$ satisfy* (19). *Then (u, v) satisfies one of the followings:*

(i) $u = 0$, *and v is arbitrary,*

(ii) *u and v are bounded measurable functions,*

(iii) $v(x) = \lambda u(x) + e^{ic\cdot x}$ *for some* $\lambda \in \mathbb{C}$, $c(\neq 0) \in \mathbb{R}^n$, *where* u *is a continuous function satisfying the condition; there exists* $d \geq 0$

$$|u(x)| \leq \psi(-x) + d \tag{74}$$

for all $x \in \mathbb{R}^n$,

(iv) $v(x) = \lambda u(x) + 1$ *for some* $\lambda \in \mathbb{C}$, *where* u *is a continuous function satisfying one of the following conditions; there exists* $a_1 \in \mathbb{C}^n$ *such that*

$$|u(x) - a_1 \cdot x| \leq \Phi_1(x) \tag{75}$$

for all $x \in \mathbb{R}^n$, *or there exists* $a_2 \in \mathbb{C}^n$ *such that*

$$|u(x) - a_2 \cdot x| \leq \Phi_2(x) \tag{76}$$

for all $x \in \mathbb{R}^n$,

(v) $u = \lambda \sin(c \cdot x)$, $v = \cos(c \cdot x) + \lambda \sin(c \cdot x)$, *for some* $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$.

Proof. Convolving in (19) the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels in both sides of inequality (19) we have

$$\begin{aligned}
&[u \circ (\xi - \eta) * (E_t(\xi) E_s(\eta))](x, y) \\
&= \left\langle u_\xi, \int E_t(x - \xi - \eta) E_s(y - \eta) \, d\eta \right\rangle \\
&= \left\langle u_\xi, (E_t * E_s)(x - y - \xi) \right\rangle \\
&= \left\langle u_\xi, E_{t+s}(x - y - \xi) \right\rangle \\
&= U(x - y, t + s).
\end{aligned} \tag{77}$$

Similarly we have

$$\begin{aligned}
[(u \otimes v) * (E_t(\xi) E_s(\eta))](x, y) &= U(x, t) V(y, s), \\
[(v \otimes u) * (E_t(\xi) E_s(\eta))](x, y) &= V(x, t) U(y, s),
\end{aligned} \tag{78}$$

where U, V are the Gauss transforms of u, v, respectively. Thus we have the following inequality:

$$\begin{aligned}
&|U(x - y, t + s) - U(x, t) V(y, s) + V(x, t) U(y, s)| \\
&\leq \Psi(y, s)
\end{aligned} \tag{79}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, where

$$\begin{aligned}
\Psi(y, s) &= \int \psi(\eta) E_t(x - \xi) E_s(y - \eta) \, d\xi \, d\eta \\
&= \int \psi(\eta) E_s(\eta - y) \, d\eta = (\psi * E_s)(y).
\end{aligned} \tag{80}$$

By Lemma 8 there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both are zero, and $M > 0$ such that

$$|\lambda_1 U(x, t) - \lambda_2 V(x, t)| \leq M, \tag{81}$$

or else U, V satisfy

$$U(x - y, t + s) - U(x, t) V(y, s) + V(x, t) U(y, s) = 0 \tag{82}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Assume that (81) holds. Applying Lemma 9, case (i) follows from (i) of Lemma 9. Using (ii) of Lemma 9, it follows from Lemma 7 the initial values u, v of $U(x, t), V(x, t)$ as $t \to 0^+$ are bounded measurable functions, respectively, which gives (ii). For case (iii), it follows from (50) that, for all $\varphi \in \mathcal{G}(\mathbb{R}^n)$,

$$\begin{aligned}
&|\langle u, \varphi \rangle - \langle f, \varphi \rangle| \\
&\quad = \left| \lim_{t \to 0^+} \int U(x, t) \varphi(x) \, dx - \int f(x) \varphi(x) \, dx \right| \\
&\quad = \left| \lim_{t \to 0^+} \int \left(U(x, t) - f(x) e^{-bt} \right) \varphi(x) \, dx \right| \\
&\quad \leq \lim_{t \to 0^+} \int \left| U(x, t) - f(x) e^{-bt} \right| |\varphi(x)| \, dx \\
&\quad \leq \lim_{t \to 0^+} \delta(t) \int |\varphi(x)| \, dx = 0.
\end{aligned} \tag{83}$$

Thus we have $u = f$ in $\mathcal{G}'(\mathbb{R}^n)$. Letting $t \to 0^+$ in (iii) of Lemma 9 we get case (iii). Finally we assume that (82) holds. Letting $t, s \to 0^+$ in (82) we have

$$u \circ (x - y) - u_x \otimes v_y + v_x \otimes u_y = 0. \tag{84}$$

By Lemma 6 the solutions of (84) satisfy (i), (iv), or (v). This completes the proof. □

Let $\psi(x) = \epsilon |x|^p$, $p > 0$. Then ψ satisfies the conditions assumed in Theorem 10. In view of (44) and (45) we have

$$\Phi_1(x) = \frac{2\epsilon |x|^p}{2 - 2^p} \tag{85}$$

if $0 < p < 1$, and

$$\Phi_2(x) = \frac{2\epsilon |x|^p}{2^p - 2} \tag{86}$$

if $p > 1$. Thus as a direct consequence of Theorem 10 we have the following.

Corollary 11. *Let* $0 < p < 1$ *or* $p > 1$. *Suppose that* $u, v \in \mathcal{G}'$ *satisfy*

$$\left\| u \circ (x - y) - u_x \otimes v_y + v_x \otimes u_y \right\| \leq \epsilon |y|^p. \tag{87}$$

Then (u, v) *satisfies one of the followings:*

(i) $u = 0$, *and* v *is arbitrary,*

(ii) u *and* v *are bounded measurable functions,*

(iii) $v(x) = \lambda u(x) + e^{ic\cdot x}$ *for some* $\lambda \in \mathbb{C}$, $c(\neq 0) \in \mathbb{R}^n$, *where* u *is a continuous function satisfying the condition; there exists* $d \geq 0$

$$|u(x)| \leq \epsilon |x|^p + d \tag{88}$$

for all $x \in \mathbb{R}^n$,

(iv) $v(x) = \lambda u(x) + 1$ *for some* $\lambda \in \mathbb{C}$, *where* u *is a continuous function satisfying the conditions; there exists* $a \in \mathbb{C}^n$ *such that*

$$|u(x) - a \cdot x| \leq \frac{2\epsilon |x|^p}{|2^p - 2|} \tag{89}$$

for all $x \in \mathbb{R}^n$,

(v) $u = \lambda \sin(c \cdot x)$, $v = \cos(c \cdot x) + \lambda \sin(c \cdot x)$, *for some* $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$.

Now we prove the stability of (20). For the proof we need the following.

Lemma 12. *Let* $U, V : G \times (0, \infty) \to \mathbb{C}$ *satisfy the inequality; there exists a* $\Psi : G \times (0, \infty) \to [0, \infty)$ *such that*

$$\begin{aligned} &\left|V(x-y, t+s) - V(x,t) V(y,s) - U(x,t) U(y,s)\right| \\ &\quad \leq \Psi(y,s) \end{aligned} \tag{90}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. *Then either there exist* λ_1, $\lambda_2 \in \mathbb{C}$, *not both are zero, and* $M > 0$ *such that*

$$\left|\lambda_1 U(x,t) - \lambda_2 V(x,t)\right| \leq M, \tag{91}$$

or else

$$V(x-y, t+s) - V(x,t) V(y,s) - U(x,t) U(y,s) = 0 \tag{92}$$

for all $x, y \in G$, $t, s > 0$.

Proof. As in Lemma 9, suppose that $\lambda_1 U(x,t) - \lambda_2 V(x,t)$ is bounded only when $\lambda_1 = \lambda_2 = 0$, and let

$$\begin{aligned} L(x, y, t, s) = {} & V(x+y, t+s) - V(x,t) V(-y,s) \\ & - U(x,t) U(-y,s). \end{aligned} \tag{93}$$

Since we may assume that U is nonconstant, we can choose y_1 and s_1 satisfying $U(-y_1, s_1) \neq 0$. Now it can be easily got that

$$\begin{aligned} U(x,t) = {} & \lambda_0 V(x,t) + \lambda_1 V(x + y_1, t + s_1) \\ & - \lambda_1 L(x, y_1, t, s_1), \end{aligned} \tag{94}$$

where $\lambda_0 = -V(-y_1, s_1)/U(-y_1, s_1)$ and $\lambda_1 = 1/U(-y_1, s_1)$. From the definition of L and the use of (94), we have the following two equations:

$$\begin{aligned} & V((x+y)+z, (t+s)+r) \\ & \quad = V(x+y, t+s) V(-z, r) + U(x+y, t+s) U(-z, r) \\ & \qquad + L(x+y, z, t+s, r) \\ & \quad = (V(x,t) V(-y,s) + U(x,t) U(-y,s) \\ & \qquad + L(x,y,t,s)) V(-z,r) \\ & \qquad + (\lambda_0 V(x+y, t+s) + \lambda_1 V(x+y+y_1, t+s+s_1) \\ & \qquad - \lambda_1 L(x+y, y_1, t+s, s_1)) U(-z,r) \\ & \qquad + L(x+y, z, t+s, r) \\ & \quad = (V(x,t) V(-y,s) + U(x,t) U(-y,s) \\ & \qquad + L(x,y,t,s)) V(-z,r) \\ & \qquad + \lambda_0 (V(x,t) V(-y,s) + U(x,t) U(-y,s) \\ & \qquad + L(x,y,t,s)) U(-z,r) \\ & \qquad + \lambda_1 (V(x,t) V(-y-y_1, s+s_1) \\ & \qquad + U(x,t) U(-y-y_1, s+s_1) \\ & \qquad + L(x, y+y_1, t, s+s_1)) U(-z,r) \\ & \qquad - \lambda_1 L(x+y, y_1, t+s, s_1) U(-z,r) \\ & \qquad + L(x+y, z, t+s, r), \end{aligned} \tag{95}$$

$$\begin{aligned} & V(x+(y+z), t+(s+r)) \\ & \quad = V(x,t) V(-y-z, s+r) + U(x,t) U(-y-z, s+r) \\ & \qquad + L(x, y+z, t, s+r). \end{aligned} \tag{96}$$

By equating (95) and (96), we have

$$\begin{aligned} & V(x,t) (V(-y,s) V(-z,r) + \lambda_0 V(-y,s) U(-z,r) \\ & \qquad + \lambda_1 V(-y-y_1, s+s_1) U(-z,r) \\ & \qquad - V(-y-z, s+r)) \\ & \quad + U(x,t) (U(-y,s) V(-z,r) + \lambda_0 U(-y,s) U(-z,r) \\ & \qquad + \lambda_1 U(-y-y_1, s+s_1) U(-z,r) \\ & \qquad - U(-y-z, s+r)) \end{aligned}$$

$$\begin{aligned}
&= -L(x,y,t,s)V(-z,r) - \lambda_0 L(x,y,t,s)U(-z,r) \\
&\quad - \lambda_1 L(x,y+y_1,t,s+s_1)U(-z,r) \\
&\quad + \lambda_1 L(x+y,y_1,t+s,s_1)U(-z,r) \\
&\quad - L(x+y,z,t+s,r) + L(x,y+z,t,s+r).
\end{aligned} \tag{97}$$

In (97), when y, s, z, and r are fixed, the right hand side is bounded; so by our assumption we have

$$\begin{aligned}
&L(x,y,t,s)V(-z,r) \\
&\quad + (\lambda_0 L(x,y,t,s) + \lambda_1 L(x,y+y_1,t,s+s_1) \\
&\quad -\lambda_1 L(x+y,y_1,t+s,s_1))U(-z,r) \\
&= L(x,y+z,t,s+r) - L(x+y,z,t+s,r).
\end{aligned} \tag{98}$$

Here, we have

$$\begin{aligned}
&L(x,y+z,t,s+r) - L(x+y,z,t+s,r) \\
&= V(x+y+z,t+s+r) - V(x,t)V(-y-z,s+r) \\
&\quad - U(x,t)U(-y-z,s+r) - V(x+y+z,t+s+r) \\
&\quad + V(x+y,t+s)V(-z,r) + U(x+y,t+s)U(-z,r) \\
&= L(-y-z,-x,s+r,t) - L(-z,-x-y,r,t+s) \\
&\leq \Psi(x,t) + \Psi(x+y,t+s).
\end{aligned} \tag{99}$$

Considering (98) as a function of z and r for all fixed x, y, t, and s again, we have $L(x,y,t,s) \equiv 0$. This completes the proof. □

In the following lemma we assume that $\Psi : \mathbb{R}^n \times (0,\infty) \to [0,\infty)$ is a continuous function such that

$$\psi(x) := \lim_{t\to 0^+} \Psi(x,t) \tag{100}$$

exists and satisfies the condition $\psi(0) = 0$.

Lemma 13. *Let $U, V : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be continuous functions satisfying*

$$\begin{aligned}
&|V(x-y,t+s) - V(x,t)V(y,s) - U(x,t)U(y,s)| \\
&\quad \leq \Psi(y,s)
\end{aligned} \tag{101}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, and there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both zero, and $M > 0$ such that

$$|\lambda_1 U(x,t) - \lambda_2 V(x,t)| \leq M. \tag{102}$$

Then (U, V) satisfies one of the followings:

(i) *U and V are bounded functions in $\mathbb{R}^n \times (0,1)$,*

(ii) *$\pm iU(x,t) = V(x,t) - e^{ia\cdot x - bt}$ for some $a \in \mathbb{R}^n$, $b \in \mathbb{C}$, and $g(x) := \lim_{t\to 0^+} V(x,t)$ is a continuous function; in particular, there exists $\delta : (0,\infty) \to [0,\infty)$ with $\delta(t) \to 0$ as $t \to 0^+$ such that*

$$\left|V(x,t) - g(x)e^{-bt}\right| \leq \delta(t) \tag{103}$$

for all $x \in \mathbb{R}^n$, $t > 0$, and g satisfies

$$|g(x) - \cos(a\cdot x)| \leq \frac{1}{2}\psi(x) \tag{104}$$

for all $x \in \mathbb{R}^n$.

Proof. If U is bounded, then in view of inequality (100), for each y, s, $V(x+y,t+s) - V(x,t)V(-y,s)$ is also bounded. It follows from Lemma 5 that V is (101). If V is bounded, case (i) follows. If V is a nonzero exponential function, then by the continuity of V we have

$$V(x,t) = e^{c\cdot x + bt} \tag{105}$$

for some $c \in \mathbb{C}^n$, $b \in \mathbb{C}$. Putting (105) in (101) and using the triangle inequality we have for some $d \geq 0$

$$\left|e^{c\cdot x}e^{b(t+s)}\left(e^{-c\cdot y} - e^{c\cdot y}\right)\right| \leq \Psi(y,s) + d \tag{106}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. In view of (106) it is easy to see that $c = ia$, $a \in \mathbb{R}^n$. Thus $V(x,t)$ is bounded on $\mathbb{R}^n \times (0,1)$. If U is unbounded; then in view of (101) V is also unbounded, hence $\lambda_1\lambda_2 \neq 0$ and

$$U(x,t) = \lambda V(x,t) + R(x,t) \tag{107}$$

for some $\lambda \neq 0$ and a bounded function R. Putting (107) in (101), replacing y by $-y$, and using the triangle inequality we have

$$\begin{aligned}
&\left|V(x+y,t+s) - V(x,t)\left(\left(\lambda^2+1\right)V(-y,s) + \lambda R(-y,s)\right)\right| \\
&\quad \leq |(\lambda V(-y,s) + R(-y,s))R(x,t)| + \Psi(-y,s).
\end{aligned} \tag{108}$$

From Lemma 5 we have

$$\left(\lambda^2+1\right)V(y,s) + \lambda R(y,s) = m(y,s) \tag{109}$$

for some exponential function m. From (107) and (109), m is continuous, and we have

$$m(x,t) = e^{c\cdot x + bt} \tag{110}$$

for some $c \in \mathbb{C}^n$, $b \in \mathbb{C}$. If $\lambda^2 \neq -1$, we have

$$U = \frac{\lambda m + R}{\lambda^2+1}, \qquad V = \frac{m - \lambda R}{\lambda^2+1}. \tag{111}$$

Putting (111) in (101), multiplying $|\lambda^2+1|$ in the result, and using the triangle inequality we have, for some $d \geq 0$,

$$|m(x,t)(m(-y,s) - m(y,s))| \leq \left|\lambda^2+1\right|\Psi(y,s) + d \tag{112}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Since m is unbounded, we have

$$m(y, s) = m(-y, s) \tag{113}$$

for all $y \in \mathbb{R}$ and $s > 0$. Thus it follows that $m(x, t) = e^{bt}$ and that U, V are bounded in $\mathbb{R}^n \times (0, 1)$. If $\lambda^2 = -1$, we have

$$U = \pm i (V - m), \tag{114}$$

where m is a bounded exponential function. Putting (114) in (101) we have

$$\begin{aligned} &\left|V(x - y, t + s) - V(x, t) m(y, s) - V(y, s) m(x, t)\right. \\ &\quad \left. + m(x, t) m(y, s)\right| \le \Psi(y, s) \end{aligned} \tag{115}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Since m is a bounded continuous function, we have

$$m(x, t) = e^{ia\cdot x - bt} \tag{116}$$

for some $a \in \mathbb{R}^n$, $b \in \mathbb{C}$ with $\Re b \ge 0$.

Similarly as in the proof of Lemma 9, by (101) and the continuity of V, it is easy to see that

$$\limsup_{t \to 0^+} V(x, t) := g(x) \tag{117}$$

exists. Putting $x = y = 0$ in (115), multiplying $|e^{bt}|$ in both sides of the result, and using the triangle inequality we have

$$\left|V(0, s) - e^{-bs}\right| \le \left|e^{bt}\right| \left(\left|V(0, t + s) - V(0, t) e^{-bs}\right| + \Psi(0, s)\right) \tag{118}$$

for all $t, s > 0$. Letting $s \to 0^+$ in (118) we have

$$\lim_{t \to 0^+} V(0, t) = 1. \tag{119}$$

Putting $y = 0$, fixing x, letting $t \to 0^+$ in (115) so that $V(x, t) \to g(x)$, and using the triangle inequality we have

$$\left|V(x, s) - g(x) e^{-bs}\right| \le \left|V(0, s) - e^{-bs}\right| + \Psi(0, s) \tag{120}$$

for all $x \in \mathbb{R}^n$, $s > 0$. Letting $s \to 0^+$ in (120) we have

$$\lim_{s \to 0^+} V(x, s) = g(x) \tag{121}$$

for all $x \in \mathbb{R}^n$. The continuity of g follows from (120). Letting $t, s \to 0^+$ in (115) we have

$$\left|g(x - y) - g(x) e^{ia\cdot y} - g(y) e^{ia\cdot x} + e^{ia\cdot(x+y)}\right| \le \psi(y) \tag{122}$$

for all $x, y \in \mathbb{R}^n$. Replacing y by x in (122) and dividing the result by $2e^{ia\cdot x}$ we have

$$\left|g(x) - \cos(a \cdot x)\right| \le \frac{1}{2}\psi(x). \tag{123}$$

From (114), (116), (120) and (123) we get (ii). This completes the proof. □

Theorem 14. *Let $u, v \in \mathcal{G}'$ satisfy (20). Then (u, v) satisfies one of the followings:*

(i) *u and v are bounded measurable functions,*

(ii) *$v(x) = \cos(a \cdot x) + r(x)$, $\pm u(x) = \sin(a \cdot x) + ir(x)$ for some $a \in \mathbb{R}^n$, where $r(x)$ is a continuous function satisfying*

$$|r(x)| \le \frac{1}{2}\psi(x) \tag{124}$$

for all $x \in \mathbb{R}^n$,

(iii) *$v(x) = \cos(c \cdot x)$ and $u(x) = \sin(c \cdot x)$ for some $c \in \mathbb{C}^n$.*

Proof. Similarly as in the proof of Theorem 10 convolving in (20) the tensor product $E_t(x)E_s(y)$ we obtain the inequality

$$\begin{aligned} &\left|V(x - y, t + s) - V(x, t) V(y, s) - U(x, t) U(y, s)\right| \\ &\quad \le \Psi(y, s) \end{aligned} \tag{125}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, where U, V are the Gauss transforms of u, v, respectively, and

$$\begin{aligned} \Psi(y, s) &= \int \psi(\eta) E_t(x - \xi) E_s(y - \eta) \, d\xi \, d\eta \\ &= \int \psi(\eta) E_s(\eta - y) \, d\eta = (\psi * E_s)(y). \end{aligned} \tag{126}$$

By Lemma 12 there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both zero, and $M > 0$ such that

$$\left|\lambda_1 U(x, t) - \lambda_2 V(x, t)\right| \le M, \tag{127}$$

or else U, V satisfy

$$V(x - y, t + s) - V(x, t) V(y, s) - U(x, t) U(y, s) = 0 \tag{128}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$.

Firstly we assume that (127) holds. Letting $t \to 0^+$ in (i) of Lemma 13, by Lemma 6, the initial values u, v of $U(x, t), V(x, t)$ as $t \to 0^+$ are bounded measurable functions, respectively, which gives case (i). Using the same approach of the proof of case (iii) of Theorem 10, we have $v = g$ in $\mathcal{G}'$. It follows from (104) that

$$v(x) = \cos(a \cdot x) + r(x), \tag{129}$$

where $r(x)$ is a continuous function satisfying

$$|r(x)| \le \frac{1}{2}\psi(x) \tag{130}$$

for all $x \in \mathbb{R}^n$. Letting $t \to 0^+$ in (ii) of Lemma 13 we have

$$\pm iu(x) = v(x) - e^{ia\cdot x}. \tag{131}$$

Putting (129) in (131) we have

$$\pm u(x) = \sin(a \cdot x) + ir(x). \tag{132}$$

Secondly we assume that (128) holds. Letting $t, s \to 0^+$ in (127) we have

$$v \circ (x - y) - v_x \otimes v_y - u_x \otimes u_y = 0. \quad (133)$$

By Lemma 7 the solution of (133) satisfies (i) or (iii). This completes the proof. □

Every infraexponential function f of order 2 defines an element of $\mathcal{G}'(\mathbb{R}^n)$ via the correspondence

$$\langle f, \varphi \rangle = \int f(x)\varphi(x)\,dx \quad (134)$$

for $\varphi \in \mathcal{G}$. Thus as a direct consequence of Corollary 11 and Theorem 14 we have the followings.

Corollary 15. *Let $0 < p < 1$ or $p > 1$. Suppose that f, g are infraexponential functions of order 2 satisfying the inequality*

$$|f(x - y) - f(x)g(y) + g(x)f(y)| \le \epsilon |x|^p \quad (135)$$

for almost every $(x, y) \in \mathbb{R}^{2n}$. Then (f, g) satisfies one of the following:

(i) *$f(x) = 0$, almost everywhere $x \in \mathbb{R}^n$, and g is arbitrary,*

(ii) *f and g are bounded in almost everywhere,*

(iii) *$f(x) = f_0(x)$, $g(x) = \lambda f_0(x) + e^{ic\cdot x}$ for almost everywhere $x \in \mathbb{R}^n$, where $\lambda \in \mathbb{C}$, $c(\neq 0) \in \mathbb{R}^n$, and f_0 is a continuous function satisfying the condition; there exists $d \ge 0$*

$$|f_0(x)| \le \epsilon |x|^p + d \quad (136)$$

for all $x \in \mathbb{R}^n$,

(iv) *$f(x) = f_0(x)$, $g(x) = \lambda f_0(x) + 1$ for a.e. $x \in \mathbb{R}^n$, where $\lambda \in \mathbb{C}$ and f_0 is a continuous function satisfying the condition; there exists $a \in \mathbb{C}^n$ such that*

$$|f_0(x) - a \cdot x| \le \frac{2\epsilon |x|^p}{|2^p - 2|} \quad (137)$$

for all $x \in \mathbb{R}^n$,

(v) *$f(x) = \lambda \sin(c \cdot x)$, $g(x) = \cos(c \cdot x) + \lambda \sin(c \cdot x)$ for a.e. $x \in \mathbb{R}^n$, where $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$.*

Corollary 16. *Suppose that f, g are infraexponential functions of order 2 satisfying the inequality*

$$|g(x - y) - g(x)g(y) - f(x)f(y)| \le \epsilon |y|^p \quad (138)$$

for almost every $(x, y) \in \mathbb{R}^{2n}$. Then (f, g) satisfies one of the followings:

(i) *f and g are bounded in almost everywhere,*

(ii) *there exists $a \in \mathbb{R}^n$ such that*

$$|g(x) - \cos(a \cdot x)| \le \frac{1}{2}\epsilon |x|^p, \quad (139)$$

$$|f(x) \pm \sin(a \cdot x)| \le \frac{1}{2}\epsilon |x|^p \quad (140)$$

for almost every $x \in \mathbb{R}^n$,

(iii) *$g(x) = \cos(c \cdot x)$ and $f(x) = \sin(c \cdot x)$ for a.e. $x \in \mathbb{R}^n$, where $c \in \mathbb{C}^n$.*

Remark 17. Taking the growth of $u = e^{c\cdot x}$ as $|x| \to \infty$ into account, $u \in \mathcal{S}'(\mathbb{R}^n)$ only when $c = ia$ for some $a \in \mathbb{R}^n$. Thus Theorems 10 and 14 are reduced to the following:

Corollary 18. *Let $u, v \in \mathcal{S}'$ satisfy* (19). *Then (u, v) satisfies one of the followings*:

(i) *$u = 0$, and v is arbitrary,*

(ii) *u and v are bounded measurable functions,*

(iii) *$v(x) = \lambda u(x) + e^{ic\cdot x}$ for some $\lambda \in \mathbb{C}$, $c(\neq 0) \in \mathbb{R}^n$, where u is a continuous function satisfying the condition; there exists $d \ge 0$*

$$|u(x)| \le \psi(-x) + d \quad (141)$$

for all $x \in \mathbb{R}^n$,

(iv) *$v(x) = \lambda u(x) + 1$ for some $\lambda \in \mathbb{C}$, where u is a continuous function satisfying one of the following conditions; there exists $a_1 \in \mathbb{C}^n$ such that*

$$|u(x) - a_1 \cdot x| \le \Phi_1(x) \quad (142)$$

for all $x \in \mathbb{R}^n$, or there exists $a_2 \in \mathbb{C}^n$ such that

$$|u(x) - a_2 \cdot x| \le \Phi_2(x) \quad (143)$$

for all $x \in \mathbb{R}^n$.

Corollary 19. *Let $u, v \in \mathcal{S}'$ satisfy* (20). *Then (u, v) satisfies one of the followings:*

(i) *u and v are bounded measurable functions,*

(ii) *$v(x) = \cos(a \cdot x) + r(x)$, $\pm u(x) = \sin(a \cdot x) + ir(x)$ for some $a \in \mathbb{R}^n$, where $r(x)$ is a continuous function satisfying*

$$|r(x)| \le \frac{1}{2}\psi(x) \quad (144)$$

for all $x \in \mathbb{R}^n$.

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