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Research Article

△-Convergence Problems for Asymptotically Nonexpansive Mappings in CAT(0) Spaces

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New △-convergence theorems of iterative sequences for asymptotically nonexpansive mappings in CAT(0) spaces are obtained. Consider an asymptotically nonexpansive self-mapping T of a closed convex subset C of a CAT(0) space X. Consider the iteration process $\{x_n\}$, where $x_0 \in C$ is arbitrary and $x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T^n y_n$ or $x_{n+1} = \alpha_n T^n x_n \oplus (1-\alpha_n) y_n$, $y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n$ for $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. It is shown that under certain appropriate conditions on $\alpha_n, \beta_n, \{x_n\}$ △-converges to a fixed point of T.

1. Introduction and Preliminaries

Let C be a nonempty subset of a metric space (X, d). A mapping $T : C \to C$ is a contraction if there exists $k \in [0,1)$ such that for all $x, y \in C$, we have $d(Tx, Ty) < kd(x, y)$. It is said to be nonexpansive if for all $x, y \in C$, we have $d(Tx, Ty) \leq d(x, y)$. T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [1, \infty)$ with $k_n \to 1$ such that $d(T^n x, T^n y) \leq k_n d(x, y)$ for all integers $n \geq 1$ and all $x, y \in C$. Clearly, every contraction mapping is nonexpansive and every nonexpansive mapping is asymptotically nonexpansive with sequence $k_n = 1$, for all $n \geq 1$. There are, however, asymptotically nonexpansive mappings which are not nonexpansive (see, e.g., [1]). As a generalization of the class of nonexpansive mappings, the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] in 1972 and has been studied by several authors (see, e.g., [3–5]). Goebel and Kirk proved that if C is a nonempty closed convex and bounded subset of a uniformly convex Banach space (more general than a Hilbert space, i.e., CAT(0) space), then every asymptotically nonexpansive self-mapping of C has a fixed point. The weak and strong convergence problems to fixed points of nonexpansive and asymptotically nonexpansive mappings have been studied by many authors.

We will denote by $F(T)$ the set of fixed points of T. In 1967, Halpern [6] introduced an explicit iterative scheme for a nonexpansive mapping T on a subset C of a Hilbert space by taking any point $u, x_1 \in C$ and defined the iterative sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + (1-\alpha_n) T x_n, \quad \text{for } n \geq 1, \tag{1}$$

where $\alpha_n \in [0,1]$. He pointed out that under certain appropriate conditions on $\alpha_n, \{x_n\}$ converges strongly to a fixed point of T. In 1994, Tan and Xu [7] introduced the following iterative scheme for asymptotically nonexpansive mapping on uniformly convex Banach space:

$$\begin{gathered} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T^n y_n, \quad n \geq 0, \\ y_n = \gamma_n x_n + (1-\gamma_n) T^n x_n, \quad n \geq 0, \end{gathered} \tag{2}$$

where $\{\alpha_n\}, \{\gamma_n\} \subseteq (0,1)$. They proved that under certain appropriate conditions on $\alpha_n, \gamma_n, \{x_n\}$ converges weakly to a fixed point of T.

In 2012, we [8] studied the viscosity approximation methods for nonexpansive mappings on CAT(0) space. For a contraction f on C, consider the iteration process $\{x_n\}$, where $x_0 \in C$ is arbitrary and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) T x_n, \tag{3}$$

for $n \geq 1$, where $\{\alpha_n\} \subset (0,1)$. We proved that under certain appropriate conditions on α_n, $\{x_n\}$ converges strongly to a fixed point of T which solves some variational inequality.

The purpose of this paper is to study the iterative scheme defined as follows: consider an asymptotically nonexpansive self-mapping T of a closed convex subset C of a CAT(0) space X with coefficient k_n. consider the iteration process $\{x_n\}$, where $x_0 \in C$ is arbitrary and

$$\begin{aligned} x_{n+1} &= \alpha_n x_n \oplus (1-\alpha_n) T^n y_n, \\ y_n &= \beta_n x_n \oplus (1-\beta_n) T^n x_n, \end{aligned} \tag{4}$$

or

$$\begin{aligned} x_{n+1} &= \alpha_n T^n x_n \oplus (1-\alpha_n) y_n, \\ y_n &= \beta_n x_n \oplus (1-\beta_n) T^n x_n, \end{aligned} \tag{5}$$

for $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. We show that $\{x_n\}$ $\triangle$-converges to a fixed point of T under certain appropriate conditions on α_n, β_n, and k_n.

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

Lemma 1. *Let X be a CAT(0) space. Then, one has the following:*

(i) *(see [9, Lemma 2.4]) for each $x, y, z \in X$ and $t \in [0,1]$, one has*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z), \tag{6}$$

(ii) *(see [10]) for each $x, y, z \in X$ and $t, s \in [0,1]$ one has*

$$d((1-t)x \oplus ty, (1-s)x \oplus sy) \leq |t-s|\, d(x,y), \tag{7}$$

(iii) *(see [5, Lemma 3]) for each $x, y, z \in X$ and $t \in [0,1]$, one has*

$$d((1-t)z \oplus tx, (1-t)z \oplus ty) \leq td(x,y), \tag{8}$$

(iv) *(see [9]) for each $x, y, z \in X$ and $t \in [0,1]$, one has*

$$\begin{aligned} &d^2((1-t)x \oplus ty, z) \\ &\quad \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y). \end{aligned} \tag{9}$$

Let X be a complete CAT(0) space and let $\{x_n\}$ be a bounded sequence in a complete X and for $x \in X$ set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n). \tag{10}$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}, \tag{11}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}. \tag{12}$$

It is known (see, e.g., [11, Proposition 7]) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

A sequence $\{x_n\}$ in X is said to *$\triangle$-converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_n x_n = x$ and call x the $\triangle$-limit of $\{x_n\}$.

Lemma 2. *Assume that X is a CAT(0) space. Then, one has the following:*

(i) *(see [12]) every bounded sequence in X has a $\triangle$-convergent subsequence;*

(ii) *(see [13]) if K is a closed convex subset of X and $T : K \to X$ is an asymptotically nonexpansive mapping, then the conditions $\{x_n\}$ $\triangle$-converge to x and $d(x_n, T(x_n)) \to 0$, imply $x \in K$ and $x \in F(T)$.*

Lemma 3 (see [14, 15]). *Let $\{a_n\}, \{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1+b_n)a_n + c_n, \quad \forall n \geq n_0, \tag{13}$$

where n_0 is some nonnegative integer, $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$. Then the limit $\lim_{n\to\infty} a_n$ exists.

2. $\triangle$-Convergence of the Iteration Sequences

In this section, we will study the $\triangle$-convergence of the iteration sequence for asymptotically nonexpansive mappings in CAT(0) spaces.

Suppose that X be a CAT(0) space, C a closed convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping with coefficient k_n. Firstly, we consider the iteration process:

$$\begin{aligned} &x_0 \in C, \\ &x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T^n y_n, \quad n \geq 0, \\ &y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n, \quad n \geq 0, \end{aligned} \tag{14}$$

where $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$ and k_n satisfy the following.

(i) There exist positive integers n_0, n_1, and $\delta > 0$, $0 < b < \min\{1, 1/L\}$, where $L = \sup_n k_n$, such that

$$\begin{aligned} &0 < \delta < \alpha_n < 1-\delta, \quad n \geq n_0, \\ &0 < 1-\beta_n < b, \quad n \geq n_1, \end{aligned} \tag{15}$$

(ii) Consider $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$.

We will prove that $\{x_n\}$ $\triangle$-converges to a fixed point of T.

Lemma 4. *Let X be a CAT(0) space, C a closed convex subset of X, $T : C \to C$ an asymptotically nonexpansive mapping with coefficient k_n, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. If $F(T) \neq \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)T^n y_n$, $y_n = \beta_n x_n \oplus (1-\beta_n)T^n x_n$, $n \geq 0$. Then the limit $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.*

Proof. Taking $p \in F(T)$, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n x_n \oplus (1-\alpha_n) T^n y_n, p) \\ &\le \alpha_n d(x_n, p) + (1-\alpha_n) d(T^n y_n, p) \\ &\le \alpha_n d(x_n, p) + (1-\alpha_n) k_n d(y_n, p) \\ &\le \alpha_n d(x_n, p) \\ &\quad + (1-\alpha_n) k_n \{\beta_n d(x_n, p) \\ &\qquad + (1-\beta_n) d(T^n x_n, p)\} \\ &\le \alpha_n d(x_n, p) \\ &\quad + (1-\alpha_n) k_n \{\beta_n d(x_n, p) \\ &\qquad + (1-\beta_n) k_n d(x_n, p)\} \\ &= \{1 + (1-\alpha_n)(k_n - 1) \\ &\quad \times [k_n(1-\beta_n) + 1]\} d(x_n, p) \\ &\le \left\{1 + \left(k_n^2 - 1\right)\right\} d(x_n, p). \end{aligned} \tag{16}$$

By Lemma 3, we can get that $\lim_{n\to\infty} d(x_n, p)$ exists. □

Remark 5. The above lemma implies that $\{x_n\}$ is bounded and so is the sequence $\{Tx_n\}$. Moreover, let $L = \sup_n k_n$, then we have

$$\begin{aligned} d(T^n x_n, p) &\le k_n d(x_n, p) \le L d(x_n, p), \\ d(y_n, p) &\le \beta_n d(x_n, p) + (1-\beta_n) d(T^n x_n, p) \\ &\le L d(x_n, p) \\ d(T^n y_n, p) &\le k_n d(y_n, p) \le L^2 d(x_n, p). \end{aligned} \tag{17}$$

It follows that the sequences $\{T^n x_n\}$, $\{y_n\}$, $\{T^n y_n\}$ are bounded.

Proposition 6. *Let X be a CAT(0) space, C a closed convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping with coefficient k_n. If $F(T) \ne \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T^n y_n$, $y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n$, $n \ge 0$. Then under the hypotheses (i) and (ii), one can get that $\lim_{n\to\infty} d(x_n, T^n y_n) = 0$.*

Proof. By the assumption, $F(T)$ is nonempty. Take $p \in F(T)$, by Lemma 1(iv), we have

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\alpha_n x_n \oplus (1-\alpha_n) T^n y_n, p) \\ &\le \alpha_n d^2(x_n, p) + (1-\alpha_n) d^2(T^n y_n, p) \\ &\quad - \alpha_n (1-\alpha_n) d^2(x_n, T^n y_n) \\ &\le d^2(x_n, p) + (1-\alpha_n)\left\{d^2(T^n y_n, p) \right. \\ &\qquad \left. - d^2(y_n, p)\right\} \\ &\quad + (1-\alpha_n)\left\{d^2(y_n, p) - d^2(x_n, p)\right\} \\ &\quad - \alpha_n(1-\alpha_n) d^2(x_n, T^n y_n), \end{aligned}$$

$$\begin{aligned} d^2(y_n, p) - d^2(x_n, p) &= d^2(\beta_n x_n \oplus (1-\beta_n) T^n x_n, p) - d^2(x_n, p) \\ &\le \beta_n d^2(x_n, p) + (1-\beta_n) d^2(T^n x_n, p) \\ &\quad - \beta_n(1-\beta_n) d^2(x_n, T^n x_n) - d^2(x_n, p) \\ &\le \beta_n d^2(x_n, p) + (1-\beta_n) d^2(T^n x_n, p) \\ &\quad - d^2(x_n, p), \end{aligned} \tag{18}$$

which implies that

$$\begin{aligned} d^2(y_n, p) - d^2(x_n, p) &\le (1-\beta_n)\left[d^2(T^n x_n, p) - d^2(x_n, p)\right] \\ &\le (1-\beta_n)\left(k_n^2 - 1\right) d^2(x_n, p). \end{aligned} \tag{19}$$

Therefore, we have

$$\begin{aligned} d^2(x_{n+1}, p) &\le d^2(x_n, p) + (1-\alpha_n)\left(k_n^2 - 1\right) d^2(y_n, p) \\ &\quad + (1-\alpha_n)(1-\beta_n)\left(k_n^2 - 1\right) d^2(x_n, p) \\ &\quad - \alpha_n(1-\alpha_n) d^2(x_n, T^n y_n). \end{aligned} \tag{20}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded and $0 < \delta < \alpha_n < 1 - \delta$ for all $n \ge n_0$. we have

$$\begin{aligned} \delta^2 d^2(x_n, T^n y_n) &\le d^2(x_n, p) - d^2(x_{n+1}, p) \\ &\quad + (1-\alpha_n)\left(k_n^2 - 1\right) d^2(y_n, p) \\ &\quad + (1-\alpha_n)(1-\beta_n)\left(k_n^2 - 1\right) d^2(x_n, p). \end{aligned} \tag{21}$$

By the conditions (i) and (ii), we have

$$\sum_{n=1}^{\infty} \delta^2 d^2(x_n, T^n y_n) < \infty, \tag{22}$$

which implies that

$$\lim_{n\to\infty} d^2(x_n, T^n y_n) = 0. \tag{23}$$

□

Theorem 7. *Let X be a CAT(0) space, C a closed convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping with coefficient k_n. If $F(T) \ne \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T^n y_n$, $y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n$, $n \ge 0$. Then under the hypotheses (i) and (ii), one can get that $\{x_n\}$ $\triangle$-converges to a fix point of T.*

Proof. We first show that $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$. Indeed

$$\begin{aligned} d(x_n, y_n) &= d(x_n, \beta_n x_n \oplus (1-\beta_n) T^n x_n) \\ &\le (1-\beta_n) d(x_n, T^n x_n) \\ &\le (1-\beta_n)\{d(x_n, T^n y_n) + d(T^n y_n, T^n x_n)\} \\ &\le (1-\beta_n)\{d(x_n, T^n y_n) + L d(y_n, x_n)\}; \end{aligned} \tag{24}$$

it follows that

$$[1 - L(1-\beta_n)]\, d(x_n, y_n) \le (1-\beta_n) d(x_n, T^n y_n). \tag{25}$$

By the conditions (i) and (ii) and Proposition 6, we get $\lim_{n\to\infty} d(x_n, y_n) = 0$.

And then,

$$\begin{aligned} d(x_n, T^n x_n) &\le d(x_n, T^n y_n) + d(T^n y_n, T^n x_n) \\ &\le d(x_n, T^n y_n) + L d(y_n, x_n). \end{aligned} \tag{26}$$

By Proposition 6, we get that $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$.

We claim that $\lim_{n\to\infty} d(x_n, T x_n) = 0$. Indeed we have

$$\begin{aligned} d(y_n, T^n x_n) &= d(\beta_n x_n \oplus (1-\beta_n) T^n x_n, T^n x_n) \\ &\le \beta_n d(x_n, T^n x_n) \longrightarrow 0. \\ d(x_{n+1}, x_n) &= d(\alpha_n x_n \oplus (1-\alpha_n) T^n y_n, x_n) \\ &\le (1-\alpha_n) d(x_n, T^n y_n) \longrightarrow 0. \\ d\left(x_{n-1}, T^{n-1} x_n\right) &\le d\left(x_{n-1}, T^{n-1} x_{n-1}\right) \\ &\quad + d\left(T^{n-1} x_{n-1}, T^{n-1} x_n\right) \\ &\le d\left(x_{n-1}, T^{n-1} x_{n-1}\right) + L d(x_{n-1}, x_n) \longrightarrow 0. \\ d\left(x_n, T^{n-1} x_n\right) &\le d\left(\alpha_{n-1} x_{n-1} \oplus (1-\alpha_{n-1}) T^{n-1} y_{n-1}, T^{n-1} x_n\right) \\ &\le \alpha_{n-1} d\left(x_{n-1}, T^{n-1} x_n\right) \\ &\quad + (1-\alpha_{n-1}) d\left(T^{n-1} y_{n-1}, T^{n-1} x_n\right) \\ &\le \alpha_{n-1} d\left(x_{n-1}, T^{n-1} x_n\right) \\ &\quad + (1-\alpha_{n-1}) L d(y_{n-1}, x_n) \\ &\le \alpha_{n-1} d\left(x_{n-1}, T^{n-1} x_n\right) \\ &\quad + (1-\alpha_{n-1}) L [d(y_{n-1}, x_{n-1}) \\ &\quad + d(x_{n-1}, x_n)] \longrightarrow 0. \end{aligned} \tag{27}$$

Thus,

$$\begin{aligned} d(x_n, T x_n) &\le d(x_n, T^n x_n) + d(T^n x_n, T x_n) \\ &\le d(x_n, T^n x_n) + L d\left(T^{n-1} x_n, x_n\right) \longrightarrow 0. \end{aligned} \tag{28}$$

Since $\{x_n\}$ is bounded, we may assume that $\{x_n\}$ $\triangle$-converges to a point $\hat{x}$. By Lemma 2, we have $\hat{x} \in F(T)$. □

Next we will consider another iteration process:

$$\begin{aligned} &x_0 \in C, \\ &x_{n+1} = \alpha_n T^n x_n \oplus (1-\alpha_n) y_n, \quad n \ge 0, \\ &y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n, \quad n \ge 0, \end{aligned} \tag{29}$$

where $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$, and k_n satisfy the following

(H1) There exist positive integers n_0 and $\delta > 0$, such that

$$\begin{aligned} &0 < \delta < \alpha_n < 1 - \delta, \quad n \ge n_0; \\ &1 - \beta_n \longrightarrow 0; \end{aligned} \tag{30}$$

(H2) $\sum_{n=1}^{\infty} (k_n - 1) < \infty$.

We will prove that $\{x_n\}$ also $\triangle$-converges to a fixed point of T.

Lemma 8. *Let X be a CAT(0) space, C a closed convex subset of X, $T : C \to C$ an asymptotically nonexpansive mapping with coefficient k_n, and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. If $F(T) \ne \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n T^n x_n \oplus (1-\alpha_n) y_n$, $y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n$, $n \ge 0$. Then the limit $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.*

Proof. Taking $p \in F(T)$, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n T^n x_n \oplus (1-\alpha_n) y_n, p) \\ &\le \alpha_n k_n d(x_n, p) + (1-\alpha_n) d(y_n, p) \\ &\le \alpha_n k_n d(x_n, p) \\ &\quad + (1-\alpha_n)\{\beta_n d(x_n, p) + (1-\beta_n) d(T^n x_n, p)\} \\ &\le \alpha_n k_n d(x_n, p) \\ &\quad + (1-\alpha_n)\{\beta_n d(x_n, p) + (1-\beta_n) k_n d(x_n, p)\} \\ &= \{1 + (k_n - 1)[1 - (1-\alpha_n)\beta_n]\} d(x_n, p). \end{aligned} \tag{31}$$

By Lemma 3, we can get that $\lim_{n\to\infty} d(x_n, p)$ exists. □

Next, we will prove $\lim_{n\to\infty} d(T^n x_n, y_n) = 0$.

Proposition 9. *Let X be a CAT(0) space, C a closed convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping with coefficient k_n. If $F(T) \ne \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n T^n x_n \oplus (1-\alpha_n) y_n$, $y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n$, $n \ge 0$. Then under the hypotheses (H1) and (H2), one can get that $\lim_{n\to\infty} d(T^n x_n, y_n) = 0$.*

Proof. By the assumption, $F(T)$ is nonempty. Take $p \in F(T)$, let $L = \sup_n k_n$, then we have

$$\begin{gathered} d(T^n x_n, p) \le k_n d(x_n, p) \le L d(x_n, p), \\ d(y_n, p) \le \beta_n d(x_n, p) + (1-\beta_n) d(T^n x_n, p) \\ \le L d(x_n, p) \\ d(T^n y_n, p) \le k_n d(y_n, p) \le L^2 d(x_n, p). \end{gathered} \tag{32}$$

It follows that the sequences $\{x_n\}, \{T^n x_n\}, \{y_n\}, \{T^n y_n\}$ are bounded.

By Lemma 1, we have

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\alpha_n T^n x_n \oplus (1-\alpha_n) y_n, p) \\ &\le \alpha_n k_n^2 d^2(x_n, p) + (1-\alpha_n) d^2(y_n, p) \\ &\quad - \alpha_n (1-\alpha_n) d^2(T^n x_n, y_n) \\ &\le d^2(x_n, p) + (1-\alpha_n)\left\{d^2(y_n, p) - d^2(x_n, p)\right\} \\ &\quad + \alpha_n \left(k_n^2 - 1\right) d^2(x_n, p) \\ &\quad - \alpha_n (1-\alpha_n) d^2(T^n x_n, y_n). \end{aligned} \tag{33}$$

Similar to the proof of Proposition 6, we can get

$$d^2(y_n, p) - d^2(x_n, p) \le (1-\beta_n)\left(k_n^2 - 1\right) d^2(x_n, p). \tag{34}$$

Therefore, we have

$$\begin{aligned} d^2(x_{n+1}, p) \le d^2(x_n, p) &+ (1-\alpha_n)(1-\beta_n) \\ &\times \left(k_n^2 - 1\right) d^2(x_n, p) \\ &+ \alpha_n \left(k_n^2 - 1\right) d^2(x_n, p) \\ &- \alpha_n (1-\alpha_n) d^2(T^n x_n, y_n). \end{aligned} \tag{35}$$

Since $\{x_n\}, \{y_n\}$ are bounded and $0 < \delta < \alpha_n < 1 - \delta$ for all $n \ge n_0$. we have

$$\begin{aligned} \delta^2 d^2(T^n x_n, y_n) \le d^2(x_n, p) &- d^2(x_{n+1}, p) \\ &+ (1-\alpha_n)(1-\beta_n)\left(k_n^2 - 1\right) d^2(x_n, p) \\ &+ \alpha_n \left(k_n^2 - 1\right) d^2(x_n, p). \end{aligned} \tag{36}$$

By the conditions (H1) and (H2), we have $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and

$$\sum_{n=1}^{\infty} \delta^2 d^2(T^n x_n, y_n) < \infty, \tag{37}$$

which implies that

$$\lim_{n\to\infty} d^2(T^n x_n, y_n) = 0. \tag{38}$$

□

Theorem 10. *Let X be a CAT(0) space, C a closed convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping with coefficient k_n. If $F(T) \ne \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n T^n x_n \oplus (1-\alpha_n) y_n$, $y_n = \beta_n x_n \oplus (1-\beta_n) T^n x_n$, $n \ge 0$. Then under the hypotheses (H1) and (H2), one can get that $\{x_n\}$ $\triangle$-converges to a fix point of T.*

Proof. We first show that $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$. Indeed, by Lemma 1, and $\beta_n \to 1$, we can get

$$\begin{aligned} d(x_n, y_n) &= d(x_n, \beta_n x_n \oplus (1-\beta_n) T^n x_n) \\ &\le (1-\beta_n) d(x_n, T^n x_n) \longrightarrow 0. \end{aligned} \tag{39}$$

And then,

$$d(x_n, T^n x_n) \le d(x_n, y_n) + d(y_n, T^n x_n). \tag{40}$$

By Proposition 9, we obtain that $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$.

We claim that $\lim_{n\to\infty} d(x_n, T x_n) = 0$. Indeed we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\alpha_n T^n x_n \oplus (1-\alpha_n) y_n, x_n) \\ &\le \alpha_n d(T^n x_n, x_n) + (1-\alpha_n) d(x_n, y_n) \longrightarrow 0. \\ d\left(x_n, T^{n-1} x_n\right) &\le d\left(\alpha_{n-1} T^{n-1} x_{n-1} \oplus (1-\alpha_{n-1}) y_{n-1}, T^{n-1} x_n\right) \\ &\le \alpha_{n-1} d\left(T^{n-1} x_{n-1}, T^{n-1} x_n\right) \\ &\quad + (1-\alpha_{n-1}) d\left(y_{n-1}, T^{n-1} x_n\right) \\ &\le \alpha_{n-1} k_{n-1} d(x_{n-1}, x_n) \\ &\quad + (1-\alpha_{n-1}) \left[d\left(y_{n-1}, T^{n-1} x_{n-1}\right) \right. \\ &\quad \left. + d\left(T^{n-1} x_{n-1}, T^{n-1} x_n\right)\right] \\ &\le \alpha_{n-1} k_{n-1} d(x_{n-1}, x_n) \\ &\quad + (1-\alpha_{n-1}) \left[d\left(y_{n-1}, T^{n-1} x_{n-1}\right) \right. \\ &\quad \left. + k_{n-1} d(x_{n-1}, x_n)\right] \longrightarrow 0. \end{aligned} \tag{41}$$

Thus,

$$\begin{aligned} d(x_n, T x_n) &\le d(x_n, T^n x_n) + d(T^n x_n, T x_n) \\ &\le d(x_n, T^n x_n) + L d\left(T^{n-1} x_n, x_n\right) \longrightarrow 0. \end{aligned} \tag{42}$$

Since $\{x_n\}$ is bounded, we may assume that $\{x_n\}$ $\triangle$-converges to a point $\hat{x}$. By Lemma 2, we have $\hat{x} \in F(T)$. □

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