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Research Article

Existence and Monotone Iteration of Positive Pseudosymmetric Solutions for a Third-Order Four-Point BVP with p-Laplacian

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We study the existence and monotone iteration of solutions for a third-order four-point boundary value problem with p-Laplacian. An existence result of positive, concave, and pseudosymmetric solutions and its monotone iterative scheme are established by using the monotone iterative technique. Meanwhile, as an application of our result, an example is given.

1. Introduction

The third-order equations arise in many areas of applied mathematics and physics [1] and thus have been discussed by many authors and many excellent results were obtained; see [1–31] and the references therein. Recently, wide attention has been paid to the third-order boundary value problems with the p-Laplace operator. In fact, the third-order equations involving the p-Laplace operator can be seen as a generalized model for various physical, natural or physiological phenomena such as the flow of a thin film of viscous fluid over a solid surface, the solitary wave solution of the Korteweg-de Vries equation or a thyroid-pituitary interaction [17].

In 2005, Cabada et al. [7] studied the one-dimensional nonlinear third-order ϕ-Laplacian equation

$$-\left(\phi\left(u''(t)\right)\right)' = f(t,u(t)), \quad t \in [a,b] \tag{1}$$

with the boundary conditions

$$u(a) = A, \qquad u''(a) = B, \qquad u''(b) = C, \tag{2}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$. By applying the monotone iterative technique based on suitable antimaximum principles, they obtained the existence of extremal solutions for the problem.

In 2006, using the monotone iterative technique, Zhou and Ma [30] obtained the existence of positive solutions and established a corresponding iterative scheme for the following third-order p-Laplacian problem of the form:

$$\begin{gathered}\left(\phi_p\left(u''(t)\right)\right)' = q(t)\, f(t,u(t)), \quad t \in [0,1], \\ u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u'(\eta) = 0, \\ u''(1) = \sum_{i=1}^{n} \beta_i u''(\theta_i).\end{gathered} \tag{3}$$

In 2007, Wang and Ge [26] considered third-order differential equation

$$\left(\phi\left(u''(t)\right)\right)' + f\left(t,u(t),u'(t),u''(t)\right) = 0, \quad t \in (0,1) \tag{4}$$

subject to the following integral boundary conditions:

$$\begin{gathered}u(0) = 0, \\ u'(0) - k_1 u''(0) = \int_0^1 h_1(u(s))\,ds, \\ u'(1) + k_2 u''(1) = \int_0^1 h_2(u(s))\,ds.\end{gathered} \tag{5}$$

The existence result to the problem is obtained by applying the method of upper and lower solutions and Leray-Schauder degree theory.

In 2009, Sun et al. [24] studied the existence of positive solutions for the following third-order p-Laplacian problem:

$$\left(\phi_p\left(u''(t)\right)\right)' = q(t) f\left(t, u(t), u'(t), u''(t)\right), \quad t \in [0,1],$$
$$u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \quad u'(\eta) = 0, \quad u''(1) = \sum_{i=1}^{n} \beta_i u''(\theta_i). \tag{6}$$

By applying a monotone iterative method, the authors obtained the existence of positive solutions for the problem and established iterative schemes for approximating the solutions.

In 2010, Jin and Lu [17] considered the following third-order p-Laplacian resonant problem of the form:

$$\left(\phi_p\left(x''(t)\right)\right)' = f\left(t, x(t), x'(x), x''(t)\right), \quad t \in (0,1),$$
$$x(0) = 0, \qquad x'(1) = \sum_{i=1}^{m-2} a_i x'(\xi_i), \qquad x''(0) = 0. \tag{7}$$

The authors obtained the existence of solutions for the problem by using Mawhin's continuation theorem.

In 2010, by using the fixed point index method, Yang and Yan [31] established the existence of at least one or at least two positive solutions for the following third-order p-Laplacian problem:

$$\begin{gathered}\left(\phi_p\left(u''(t)\right)\right)' + f(t, u(t)) = 0, \quad t \in (0,1), \\ \alpha u(0) - \beta u'(0) = 0, \qquad \gamma u(1) + \delta u'(1) = 0, \\ u''(0) = 0.\end{gathered} \tag{8}$$

Motivated by the above works and [32, 33], in this paper, we consider the existence and monotone iteration of positive, pseudosymmetric solutions of the following third-order four-point p-Laplacian boundary value problem:

$$\left(\phi_p\left(u''(t)\right)\right)' + q(t) f\left(t, u(t), u'(t)\right) = 0, \quad t \in (0,1) \tag{9}$$

subject to boundary conditions

$$u(0) = 0, \qquad u(1) = u(\eta), \qquad u''\left(\frac{1+\eta}{2}\right) = 0, \tag{10}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, and $\eta \in (0,1)$ be constant. Here $u^*(t)$ is said to be a positive solution of BVP (9), (10) if and only if $u^*(t)$ is the solution of BVP (9), (10) and satisfies $u^*(t) > 0$ for $t \in (0,1]$. BVP (9), (10) can model the static deflection of an elastic beam with linear supports at both endpoints.

To the best of our knowledge, the existence results of the pseudosymmetric solutions for the third-order boundary value problem has not been considered.

This work is organized as follows. In Section 2, some notations and preliminaries are introduced. The main results are discussed in Section 3. As applications of our results, an example is given in the last section.

2. Preliminary

In this section, we give some definitions and lemmas which help to simplify the presentation of our main result.

Definition 1 (see [32]). Let $u \in C[0,1]$, $\eta \in (0,1)$. One says that u is pseudosymmetric about η on $[0,1]$, if u is symmetric on $[\eta,1]$, that is,

$$u(t) = u(1+\eta-t), \quad \forall t \in [\eta, 1]. \tag{11}$$

Definition 2. Let $u \in C[0,1]$, $\eta \in (0,1)$. One says that u is pseudo-antisymmetric about η on $[0,1]$, if u is antisymmetric on $[\eta,1]$, that is,

$$u(t) + u(1+\eta-t) = 0, \quad \forall t \in [\eta, 1]. \tag{12}$$

Let the Banach space $E = C^1[0,1]$ be endowed with the norm

$$\|u\| = \max_{0\le t\le 1}\left(u^2(t) + u'^2(t)\right)^{1/2}, \tag{13}$$

and define the cone $P \subset E$ by

$$\begin{gathered}P = \{u \in E : u \text{ is nonnegative, concave and} \\ \text{pseudosymmetric about } \eta \text{ on } [0,1]\},\end{gathered} \tag{14}$$

and let

$$\overline{P}_a = \{u \in P : \|u\| \le a\}. \tag{15}$$

For convenience, we consider the following.

(H_0) $q(t)$ is a nonnegative continuous function defined on $(0,1)$, $q(t) \not\equiv 0$ on any subinterval of $(0,1)$. In addition, $\int_0^1 q(t)dt < +\infty$ and $q(t)$ is pseudosymmetric about η on $[0,1]$.

(H_1) $f(t,u,v) : [0,1] \times [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and

$$f(t,u,v) \ge 0, \quad \forall (t,u,v) \in \left[0, \frac{1+\eta}{2}\right] \times [0,+\infty) \times \mathbb{R}. \tag{16}$$

(H_2) $f(t,u,v) + f(1+\eta-t, u, -v) = 0$ for all $(t,u,v) \in [\eta,1] \times [0,+\infty) \times \mathbb{R}$.

(H_3) $f(t,0,0) \not\equiv 0$ on $[0,1]$.

Now, we define an operator $T : C^1[0,1] \rightarrow C^1[0,1]$ as follows: for $u \in C^1[0,1]$,

$$(Tu)(t) = \begin{cases} \int_0^t \int_\tau^{(1+\eta)/2} \phi_p^{-1}\left(\int_s^{(1+\eta)/2} q(r) \times f(r, u(r), u'(r))\, \mathrm{d}r\right) \mathrm{d}s\, \mathrm{d}\tau, & 0 \le t \le \dfrac{1+\eta}{2}, \\ \int_0^\eta \int_\tau^{(1+\eta)/2} \phi_p^{-1}\left(\int_s^{(1+\eta)/2} q(r) \times f(r, u(r), u'(r))\, \mathrm{d}r\right) \mathrm{d}s\, \mathrm{d}\tau - \int_t^1 \int_{(1+\eta)/2}^\tau \phi_p^{-1}\left(\int_{(1+\eta)/2}^s q(r) \times f(r, u(r), u'(r))\, \mathrm{d}r\right) \mathrm{d}s\, \mathrm{d}\tau, & \dfrac{1+\eta}{2} \le t \le 1. \end{cases} \tag{17}$$

Obviously under assumptions (H_0) and (H_1), the operator T is well defined and it is easy to verify that BVP (9), (10) has a solution if and only if $T : C^1[0,1] \rightarrow C^1[0,1]$ has a fixed point.

The next lemmas are some properties of the operator T.

Lemma 3. *Assume that* (H_0), (H_1), *and* (H_2) *hold. Then* $TP \subset P$.

Proof. From the definition of T, it is easy to check that Tu is nonnegative on $[0,1]$ and satisfies (10) for all $u \in P$. Furthermore, since

$$(Tu)''(t) = \begin{cases} -\phi_p^{-1}\left(\int_t^{(1+\eta)/2} q(r) f(r, u(r), u'(r))\, \mathrm{d}r\right) \le 0, & 0 \le t \le \dfrac{1+\eta}{2}, \\ \phi_p^{-1}\left(\int_{(1+\eta)/2}^t q(r) f(r, u(r), u'(r))\, \mathrm{d}r\right) \le 0, & \dfrac{1+\eta}{2} \le t \le 1, \end{cases} \tag{18}$$

it follows that Tu is concave on $[0,1]$.

Next we prove that Tu is pseudosymmetric about η on $[0,1]$. In fact, if $t \in [\eta, (1+\eta)/2]$, then $1+\eta-t \in [(1+\eta)/2, 1]$, and it follows that

$$\begin{aligned} &(Tu)(1+\eta-t) \\ &= \int_0^\eta \int_\tau^{(1+\eta)/2} \phi_p^{-1} \times \left(\int_s^{(1+\eta)/2} q(r) f(r, u(r), u'(r))\, \mathrm{d}r\right) \mathrm{d}s\, \mathrm{d}\tau \\ &\quad - \int_{1+\eta-t}^1 \int_{(1+\eta)/2}^\tau \phi_p^{-1} \times \left(\int_{(1+\eta)/2}^s q(r) f(r, u(r), u'(r))\, \mathrm{d}r\right) \mathrm{d}s\, \mathrm{d}\tau \\ &= \int_0^\eta \int_\tau^{(1+\eta)/2} \phi_p^{-1} \times \left(\int_s^{(1+\eta)/2} q(r) f(r, u(r), u'(r))\, \mathrm{d}r\right) \mathrm{d}s\, \mathrm{d}\tau \\ &\quad + \int_1^\eta \int_{(1+\eta)/2}^\tau \phi_p^{-1} \times \left(\int_{(1+\eta)/2}^s q(r) f(r, u(r), u'(r))\, \mathrm{d}r\right) \mathrm{d}s\, \mathrm{d}\tau \\ &\quad + \int_\eta^t \int_{(1+\eta)/2}^\tau \phi_p^{-1} \times \left(\int_{(1+\eta)/2}^s q(r) f(r, u(r), u'(r))\, \mathrm{d}r\right) \mathrm{d}s\, \mathrm{d}\tau \\ &\quad + \int_t^{1+\eta-t} \int_{(1+\eta)/2}^\tau \phi_p^{-1} \times \left(\int_{(1+\eta)/2}^s q(r) f(r, u(r), u'(r))\, \mathrm{d}r\right) \mathrm{d}s\, \mathrm{d}\tau. \end{aligned} \tag{19}$$

Also since u is pseudosymmetric about η on $[0,1]$, that is, $u(t) = u(1+\eta-t)$ for $t \in [\eta, 1]$, then

$$u'(t) = -u'(1+\eta-t), \quad t \in [\eta, 1]. \tag{20}$$

Thus, for all $t \in [\eta, 1]$, from (H_2), we have

$$\begin{aligned} &q(r) f(r, u(r), u'(r)) \\ &\quad = -q(1+\eta-r) f(1+\eta-r, u(r), -u'(r)) \\ &\quad = -q(1+\eta-r) \\ &\qquad \times f(1+\eta-r, u(1+\eta-r), u'(1+\eta-r)). \end{aligned} \tag{21}$$

Hence $q(r)f(r,u(r),u'(r))$ is pseudo-antisymmetric about η on $[0,1]$, and thus $\int_{(1+\eta)/2}^s q(r)f(r,u(r),u'(r))\mathrm{d}r$ is pseudosymmetric about η on $[0,1]$. Furthermore $\phi_p^{-1}(\int_{(1+\eta)/2}^s q(r)f(r,u(r),u'(r))\mathrm{d}r)$ is pseudosymmetric about η on $[0,1]$. Thus the function $\int_{(1+\eta)/2}^\tau \phi_p^{-1} (\int_{(1+\eta)/2}^s q(r)f(r,u(r),u'(r))\mathrm{d}r)\mathrm{d}s$ is pseudo-antisymmetric about η on $[0,1]$, and hence

$$\int_1^\eta \int_{(1+\eta)/2}^\tau \phi_p^{-1}\left(\int_{(1+\eta)/2}^s q(r) f(r, u(r), u'(r))\, \mathrm{d}r\right) \mathrm{d}s\, \mathrm{d}\tau = 0. \tag{22}$$

Using the similar technique, we can get

$$\int_t^{1+\eta-t}\int_{(1+\eta)/2}^{\tau}\phi_p^{-1}\left(\int_{(1+\eta)/2}^{s}q(r)\,f\big(r,u(r),u'(r)\big)\,dr\right)ds\,d\tau = 0. \tag{23}$$

From (19), (22), and (23), it follows that

$$\begin{aligned}(Tu)(1+\eta-t) &= \int_0^{\eta}\int_{\tau}^{(1+\eta)/2}\phi_p^{-1}\\ &\qquad\times\left(\int_s^{(1+\eta)/2}q(r)\,f\big(r,u(r),u'(r)\big)\,dr\right)ds\,d\tau\\ &\quad+\int_{\eta}^{t}\int_{(1+\eta)/2}^{\tau}\phi_p^{-1}\\ &\qquad\times\left(\int_{(1+\eta)/2}^{s}q(r)\,f\big(r,u(r),u'(r)\big)\,dr\right)ds\,d\tau\\ &=\int_0^{t}\int_{(1+\eta)/2}^{\tau}\phi_p^{-1}\\ &\qquad\times\left(\int_{(1+\eta)/2}^{s}q(r)\,f\big(r,u(r),u'(r)\big)\,dr\right)ds\,d\tau\\ &=(Tu)(t),\quad t\in\left[\eta,\frac{1+\eta}{2}\right].\end{aligned}\tag{24}$$

If $t\in[(1+\eta)/2,1]$, then $1+\eta-t\in[\eta,(1+\eta)/2]$. From (24), it follows that

$$\begin{aligned}(Tu)(1+\eta-t)&=(Tu)(1+\eta-(1+\eta-t))\\ &=(Tu)(t),\quad t\in\left[\frac{1+\eta}{2},1\right].\end{aligned}\tag{25}$$

This together with (24) implies that

$$(Tu)(t)=(Tu)(1+\eta-t),\quad t\in[\eta,1].\tag{26}$$

In summary, $Tu\in P$, and then $TP\subset P$. □

The following lemma can be easily verified by a standard argument.

Lemma 4. *Assume that (H_0), (H_1), and (H_2) hold. Then $T: P\to P$ is completely continuous.*

Lemma 5. *Assume that (H_0), (H_1), and (H_2) hold. Suppose also that there exists $a>0$ such that for $0\le t\le(1+\eta)/2$, $0\le u_1\le u_2\le a$, $0\le|v_1|\le|v_2|\le a$,*

$$f(t,u_1,v_1)\le f(t,u_2,v_2).\tag{27}$$

Then for $u_1,u_2\in\overline{P}_a$ with

$$u_1(t)\le u_2(t),\quad |u_1'(t)|\le|u_2'(t)|,\quad t\in[0,1],\tag{28}$$

we have

$$(Tu_1)(t)\le(Tu_2)(t),\quad \left|(Tu_1)'(t)\right|\le\left|(Tu_2)'(t)\right|,\quad t\in[0,1].\tag{29}$$

Proof. First we prove that, for all $t\in[0,(1+\eta)/2]$,

$$(Tu_1)(t)\le(Tu_2)(t),\qquad \left|(Tu_1)'(t)\right|\le\left|(Tu_2)'(t)\right|.\tag{30}$$

From assumptions, we have

$$f\big(r,u_1(r),u_1'(r)\big)\le f\big(r,u_2(r),u_2'(r)\big),\quad r\in\left[0,\frac{1+\eta}{2}\right],\tag{31}$$

and hence

$$\begin{aligned}&\int_s^{(1+\eta)/2}f\big(r,u_1(r),u_1'(r)\big)\,dr\\ &\quad\le\int_s^{(1+\eta)/2}f\big(r,u_2(r),u_2'(r)\big)\,dr,\quad s\in\left[0,\frac{1+\eta}{2}\right].\end{aligned}\tag{32}$$

Since ϕ_p^{-1} is strictly increasing on $\mathbb{R}$, then for all $s\in[0,(1+\eta)/2]$, we have

$$\begin{aligned}&\phi_p^{-1}\left(\int_s^{(1+\eta)/2}f\big(r,u_1(r),u_1'(r)\big)\,dr\right)\\ &\quad\le\phi_p^{-1}\left(\int_s^{(1+\eta)/2}f\big(r,u_2(r),u_2'(r)\big)\,dr\right).\end{aligned}\tag{33}$$

Thus for $t\in[0,(1+\eta)/2]$,

$$\begin{aligned}&(Tu_1)(t)-(Tu_2)(t)\\ &=\int_0^t\int_{\tau}^{(1+\eta)/2}\left[\phi_p^{-1}\left(\int_s^{(1+\eta)/2}q(r)f\big(r,u_1(r),u_1'(r)\big)dr\right)\right.\\ &\qquad\left.-\phi_p^{-1}\left(\int_s^{(1+\eta)/2}q(r)\,f\big(r,u_2(r),u_2'(r)\big)dr\right)\right]ds\,d\tau\\ &\le 0,\\ &\left|(Tu_1)'(t)\right|-\left|(Tu_2)'(t)\right|\\ &=\int_t^{(1+\eta)/2}\left[\phi_p^{-1}\left(\int_s^{(1+\eta)/2}q(r)\,f\big(r,u_1(r),u_1'(r)\big)\,dr\right)\right.\\ &\qquad\left.-\phi_p^{-1}\left(\int_s^{(1+\eta)/2}q(r)f\big(r,u_2(r),u_2'(r)\big)\,dr\right)\right]ds\\ &\le 0.\end{aligned}\tag{34}$$

Therefore, (30) holds for $t\in[0,(1+\eta)/2]$.

Next we prove that (30) holds for $t\in[(1+\eta)/2,1]$. In fact, if $t\in[(1+\eta)/2,1]$, then $1+\eta-t\in[0,(1+\eta)/2]$, and hence from the fact that Tu_1 and Tu_2 are pseudosymmetric about η on $[0,1]$, it follows that, for $t \in [(1+\eta)/2, 1]$,

$$\begin{aligned}(Tu_1)(t) - (Tu_2)(t) &= (Tu_1)(1+\eta-t) \\ &\quad - (Tu_2)(1+\eta-t) \\ &\le 0, \\ \left|(Tu_1)'(t)\right| - \left|(Tu_2)'(t)\right| &= \left|(Tu_1)'(1+\eta-t)\right| \\ &\quad - \left|(Tu_2)'(1+\eta-t)\right| \\ &\le 0.\end{aligned} \tag{35}$$

In summary,

$$(Tu_1)(t) \le (Tu_2)(t), \quad \left|(Tu_1)'(t)\right| \le \left|(Tu_2)'(t)\right|, \quad t \in [0,1]. \tag{36}$$

□

Now, we introduce some notations as follows:

$$\begin{gathered} A_1 = \int_0^{(1+\eta)/2} \phi_p^{-1}\left(\int_s^{(1+\eta)/2} q(r)\,\mathrm{d}r\right)\mathrm{d}s, \\ A_2 = \phi_p^{-1}\left(\int_0^{(1+\eta)/2} q(r)\,\mathrm{d}r\right), \\ A = \max\left\{\sqrt{2}A_1, \sqrt{2}A_2\right\} = \sqrt{2}A_2. \end{gathered} \tag{37}$$

Lemma 6. *Assume that (H_0), (H_1), and (H_2) hold. Suppose also that there exists $a > 0$ such that for $0 \le t \le (1+\eta)/2$, $0 \le u_1 \le u_2 \le a$, $0 \le |v_1| \le |v_2| \le a$,*

$$\begin{gathered} f(t, u_1, v_1) \le f(t, u_2, v_2), \\ \max_{0\le t\le (1+\eta)/2} f(t, a, a) \le \phi_p\left(\frac{a}{A}\right). \end{gathered} \tag{38}$$

Then $T : \overline{P}_a \to \overline{P}_a$.

Proof. Define two functionals on E as follows:

$$\alpha(u) := \max_{0\le t\le 1}|u(t)|, \qquad \beta(u) := \max_{0\le t\le 1}\left|u'(t)\right|. \tag{39}$$

Then

$$\|u\| \le \sqrt{2}\max\{\alpha(u), \beta(u)\}. \tag{40}$$

If $u \in \overline{P}_a$, then

$$\begin{gathered} 0 \le u(t) \le \max_{0\le t\le 1}|u(t)| \le \|u\| \le a, \quad t \in [0,1], \\ 0 \le \left|u'(t)\right| \le \max_{0\le t\le 1}\left|u'(t)\right| \le \|u\| \le a, \quad t \in [0,1]. \end{gathered} \tag{41}$$

From the assumptions, for all $t \in [0, (1+\eta)/2]$,

$$\begin{aligned} 0 \le f\left(t, u(t), u'(t)\right) &\le f(t, a, a) \\ &\le \max_{0\le t\le (1+\eta)/2} f(t, a, a) \le \phi_p\left(\frac{a}{A}\right). \end{aligned} \tag{42}$$

Then,

$$\begin{aligned} &\alpha(Tu) \\ &= \max_{0\le t\le 1}|(Tu)(t)| = (Tu)\left(\frac{1+\eta}{2}\right) \\ &= \int_0^{(1+\eta)/2}\int_\tau^{(1+\eta)/2}\phi_p^{-1} \\ &\qquad \times\left(\int_s^{(1+\eta)/2} q(r) f\left(r, u(r), u'(r)\right)\mathrm{d}r\right)\mathrm{d}s\,\mathrm{d}\tau \\ &\le \int_0^{(1+\eta)/2}\int_\tau^{(1+\eta)/2}\phi_p^{-1} \\ &\qquad \times\left(\int_s^{(1+\eta)/2} q(r)\phi_p\left(\frac{a}{A}\right)\mathrm{d}r\right)\mathrm{d}s\,\mathrm{d}\tau \\ &\le A_1\cdot\frac{a}{A} \le \frac{\sqrt{2}}{2}a, \\ &\beta(Tu) \\ &\quad = \max_{0\le t\le 1}\left|(Tu)'(t)\right| = (Tu)'(0) \\ &\quad = \int_0^{(1+\eta)/2}\phi_p^{-1}\left(\int_s^{(1+\eta)/2} q(r) f\left(r, u(r), u'(r)\right)\mathrm{d}r\right)\mathrm{d}s \\ &\quad \le \int_0^{(1+\eta)/2}\phi_p^{-1}\left(\int_s^{(1+\eta)/2} q(r)\phi_p\left(\frac{a}{A}\right)\mathrm{d}r\right)\mathrm{d}s \\ &\quad \le A_2\cdot\frac{a}{A} \le \frac{\sqrt{2}}{2}a. \end{aligned} \tag{43}$$

So we have

$$\|Tu\| \le \sqrt{2}\max\{\alpha(Tu), \beta(Tu)\} \le \sqrt{2}\cdot\frac{\sqrt{2}}{2}a = a. \tag{44}$$

Thus $T : \overline{P}_a \to \overline{P}_a$. □

3. Main Result

Now we establish existence result of positive, concave, and pseudosymmetric solutions and its monotone iterative scheme for BVP (9), (10).

Theorem 7. *Assume that (H_0), (H_1), (H_2), and (H_3) hold. Suppose also that there exists $a > 0$ such that for $0 \le t \le (1+\eta)/2$, $0 \le u_1 \le u_2 \le a$, $0 \le |v_1| \le |v_2| \le a$,*

$$\begin{gathered} f(t, u_1, v_1) \le f(t, u_2, v_2), \\ \max_{0\le t\le (1+\eta)/2} f(t, a, a) \le \phi_p\left(\frac{a}{A}\right). \end{gathered} \tag{45}$$

Then BVP (9), (10) *has two positive, concave, and pseudosymmetric solutions* w^* *and* v^* *with*

$$\begin{gathered}0<w^*(t)\le\frac{\sqrt{2}}{2}a,\quad t\in(0,1],\\ \lim_{n\to\infty}w_n=\lim_{n\to\infty}T^nw_0=w^*\ \left(in\ C^1\ \text{norm}\right),\\ 0<v^*(t)\le a,\quad t\in(0,1],\\ \lim_{n\to\infty}v_n=\lim_{n\to\infty}T^nv_0=v^*\ \left(in\ C^1\ \text{norm}\right),\end{gathered}\tag{46}$$

where

$$\begin{gathered}w_0(t)=\begin{cases}\frac{\sqrt{2}}{2}at, & 0\le t\le\frac{1+\eta}{2},\\ \frac{\sqrt{2}}{2}a(1+\eta-t), & \frac{1+\eta}{2}\le t\le 1;\end{cases}\\ v_0(t)\equiv 0,\quad 0\le t\le 1,\\ (Tu)(t)=\begin{cases}\int_0^t\int_\tau^{(1+\eta)/2}\phi_p^{-1}\left(\int_s^{(1+\eta)/2}q(r)\times f\left(r,u(r),u'(r)\right)dr\right)ds\,d\tau, & 0\le t\le\frac{1+\eta}{2},\\ \int_0^\eta\int_\tau^{(1+\eta)/2}\phi_p^{-1}\left(\int_s^{(1+\eta)/2}q(r)\times f\left(r,u(r),u'(r)\right)dr\right)ds\,d\tau\\ \quad-\int_t^1\int_{(1+\eta)/2}^\tau\phi_p^{-1}\left(\int_{(1+\eta)/2}^s q(r)\times f\left(r,u(r),u'(r)\right)dr\right)ds\,d\tau, & \frac{1+\eta}{2}\le t\le 1.\end{cases}\end{gathered}\tag{47}$$

Proof. Let

$$\begin{gathered}w_0(t)=\begin{cases}\frac{\sqrt{2}}{2}at, & 0\le t\le\frac{1+\eta}{2},\\ \frac{\sqrt{2}}{2}a(1+\eta-t), & \frac{1+\eta}{2}\le t\le 1,\end{cases}\\ w_1(t)=\begin{cases}\int_0^t\int_\tau^{(1+\eta)/2}\phi_p^{-1}\left(\int_s^{(1+\eta)/2}q(r)\times f\left(r,w_0(r),w_0'(r)\right)dr\right)ds\,d\tau, & 0\le t\le\frac{1+\eta}{2},\\ \int_0^\eta\int_\tau^{(1+\eta)/2}\phi_p^{-1}\left(\int_s^{(1+\eta)/2}q(r)\times f\left(r,w_0(r),w_0'(r)\right)dr\right)ds\,d\tau,\\ \quad-\int_t^1\int_{(1+\eta)/2}^\tau\phi_p^{-1}\left(\int_{(1+\eta)/2}^s q(r)\times f\left(r,w_0(r),w_0'(r)\right)dr\right)ds\,d\tau, & \frac{1+\eta}{2}\le t\le 1.\end{cases}\end{gathered}\tag{48}$$

Then $w_1(t)\in C^1[0,(1+\eta)/2]\cap C^1[(1+\eta)/2,1]$.

Next we prove that

$$\lim_{t\to((1+\eta)/2)^-}w_1(t)=\lim_{t\to((1+\eta)/2)^+}w_1(t),\tag{49}$$

$$\lim_{t\to((1+\eta)/2)^-}w_1'(t)=\lim_{t\to((1+\eta)/2)^+}w_1'(t).\tag{50}$$

In fact, from (H_2) it follows that

$$\begin{aligned}&\lim_{t\to((1+\eta)/2)^+}w_1(t)\\ &=\int_0^\eta\int_\tau^{(1+\eta)/2}\phi_p^{-1}\times\left(\int_s^{(1+\eta)/2}q(r)f\left(r,\frac{\sqrt{2}}{2}ar,\frac{\sqrt{2}}{2}a\right)dr\right)ds\,d\tau\\ &\quad-\int_{(1+\eta)/2}^1\int_{(1+\eta)/2}^\tau\phi_p^{-1}\times\left(\int_{(1+\eta)/2}^s q(r)f\left(r,\frac{\sqrt{2}}{2}a(1+\eta-r),-\frac{\sqrt{2}}{2}a\right)dr\right)ds\,d\tau\\ &=\left(\int_0^{(1+\eta)/2}+\int_{(1+\eta)/2}^\eta\right)\times\int_\tau^{(1+\eta)/2}\phi_p^{-1}\left(\int_s^{(1+\eta)/2}q(r)f\left(r,\frac{\sqrt{2}}{2}ar,\frac{\sqrt{2}}{2}a\right)dr\right)ds\,d\tau\\ &\quad-\int_{(1+\eta)/2}^1\int_{(1+\eta)/2}^\tau\phi_p^{-1}\times\left(\int_{(1+\eta)/2}^s q(r)\times f\left(r,\frac{\sqrt{2}}{2}a(1+\eta-r),-\frac{\sqrt{2}}{2}a\right)dr\right)ds\,d\tau\\ &=\int_0^{(1+\eta)/2}\int_\tau^{(1+\eta)/2}\phi_p^{-1}\times\left(\int_s^{(1+\eta)/2}q(r)f\left(r,\frac{\sqrt{2}}{2}ar,\frac{\sqrt{2}}{2}a\right)dr\right)ds\,d\tau\\ &\quad+\int_{(1+\eta)/2}^\eta\int_\tau^{(1+\eta)/2}\phi_p^{-1}\times\left(\int_s^{(1+\eta)/2}q(r)f\left(r,\frac{\sqrt{2}}{2}ar,\frac{\sqrt{2}}{2}a\right)dr\right)ds\,d\tau\\ &\quad-\int_{(1+\eta)/2}^1\int_{(1+\eta)/2}^\tau\phi_p^{-1}\times\left(\int_{(1+\eta)/2}^s q(r)\times f\left(r,\frac{\sqrt{2}}{2}a(1+\eta-r),-\frac{\sqrt{2}}{2}a\right)dr\right)ds\,d\tau\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{(1+\eta)/2} \int_\tau^{(1+\eta)/2} \phi_p^{-1} \\
&\qquad \times \left(\int_s^{(1+\eta)/2} q(r)\, f\left(r, \frac{\sqrt{2}}{2} ar, \frac{\sqrt{2}}{2} a\right) \mathrm{d}r \right) \mathrm{d}s\, \mathrm{d}\tau \\
&= \lim_{t \to ((1+\eta)/2)^-} w_1(t).
\end{aligned} \tag{51}
$$

Then (49) holds. Equation (50) can be obtained in a similar way. Thus from (49) and (50), it follows that

$$
w_1(t) \in C^1[0,1]. \tag{52}
$$

We note that for $t \in [0, (1+\eta)/2]$,

$$
\begin{aligned}
0 &\le w_1(t) \\
&= \int_0^t \int_\tau^{(1+\eta)/2} \phi_p^{-1} \\
&\qquad \times \left(\int_s^{(1+\eta)/2} q(r)\, f\left(r, \frac{\sqrt{2}}{2} ar, \frac{\sqrt{2}}{2} a\right) \mathrm{d}r \right) \mathrm{d}s\, \mathrm{d}\tau \\
&\le \int_0^t \int_\tau^{(1+\eta)/2} \phi_p^{-1} \left(\int_s^{(1+\eta)/2} q(r)\, \phi_p\left(\frac{a}{A}\right) \mathrm{d}r \right) \mathrm{d}s\, \mathrm{d}\tau \\
&\le \frac{a}{A} \int_0^t \int_\tau^{(1+\eta)/2} \phi_p^{-1} \left(\int_0^{(1+\eta)/2} q(r)\, \mathrm{d}r \right) \mathrm{d}s\, \mathrm{d}\tau \\
&\le \frac{a}{A} A_2 t = w_0(t),
\end{aligned} \tag{53}
$$

and for $t \in [(1+\eta)/2, 1]$,

$$
\begin{aligned}
&w_1(t) \\
&= \int_0^\eta \int_\tau^{(1+\eta)/2} \phi_p^{-1} \\
&\qquad \times \left(\int_s^{(1+\eta)/2} q(r)\, f\left(r, \frac{\sqrt{2}}{2} ar, \frac{\sqrt{2}}{2} a\right) \mathrm{d}r \right) \mathrm{d}s\, \mathrm{d}\tau \\
&\quad - \int_t^1 \int_{(1+\eta)/2}^\tau \phi_p^{-1} \\
&\qquad \times \left(\int_{(1+\eta)/2}^s q(r) \right. \\
&\qquad\qquad \left. \times f\left(r, \frac{\sqrt{2}}{2} a(1+\eta-r), -\frac{\sqrt{2}}{2} a\right) \mathrm{d}r \right) \mathrm{d}s\, \mathrm{d}\tau \\
&\le \int_0^\eta \int_\tau^{(1+\eta)/2} \phi_p^{-1} \left(\int_s^{(1+\eta)/2} q(r)\, \phi_p\left(\frac{a}{A}\right) \mathrm{d}r \right) \mathrm{d}s\, \mathrm{d}\tau \\
&\quad + \int_t^1 \int_{(1+\eta)/2}^\tau \phi_p^{-1} \left(\int_{(1+\eta)/2}^s q(r)\, \phi_p\left(\frac{a}{A}\right) \mathrm{d}r \right) \mathrm{d}s\, \mathrm{d}\tau \\
&\le \frac{a}{A} \int_0^\eta \int_\tau^{(1+\eta)/2} \phi_p^{-1} \left(\int_0^{(1+\eta)/2} q(r)\, \mathrm{d}r \right) \mathrm{d}s\, \mathrm{d}\tau \\
&\quad + \frac{a}{A} \int_t^1 \int_{(1+\eta)/2}^\tau \phi_p^{-1} \left(\int_{(1+\eta)/2}^1 q(r)\, \mathrm{d}r \right) \mathrm{d}s\, \mathrm{d}\tau \\
&\le \frac{a}{A} \int_0^\eta \int_\tau^{(1+\eta)/2} A_2 \mathrm{d}s\, \mathrm{d}\tau \\
&\quad + \frac{a}{A} \int_t^1 \int_{(1+\eta)/2}^\tau \phi_p^{-1} \left(\int_0^{(1+\eta)/2} q(r)\, \mathrm{d}r \right) \mathrm{d}s\, \mathrm{d}\tau \\
&\le \frac{a}{A} \int_0^\eta A_2 \mathrm{d}\tau + \frac{a}{A} \int_t^1 \int_{(1+\eta)/2}^\tau A_2 \mathrm{d}s\, \mathrm{d}\tau \\
&\le \frac{a}{A} A_2 \eta + \frac{a}{A} A_2 (1-t) = w_0(t).
\end{aligned} \tag{54}
$$

So,

$$
w_1(t) \le w_0(t) \le \frac{\sqrt{2}}{2} a, \quad t \in [0,1]. \tag{55}
$$

Thus,

$$
\alpha(w_1) := \max_{0 \le t \le 1} |w_1(t)| \le \frac{\sqrt{2}}{2} a. \tag{56}
$$

From assumptions, for $t \in [0, (1+\eta)/2]$,

$$
\begin{aligned}
|w_1'(t)| &= \int_t^{(1+\eta)/2} \phi_p^{-1} \\
&\qquad \times \left(\int_s^{(1+\eta)/2} q(r)\, f\left(r, \frac{\sqrt{2}}{2} ar, \frac{\sqrt{2}}{2} a\right) \mathrm{d}r \right) \mathrm{d}s \\
&\le \int_t^{(1+\eta)/2} \phi_p^{-1} \left(\int_s^{(1+\eta)/2} q(r)\, \phi_p\left(\frac{a}{A}\right) \mathrm{d}r \right) \mathrm{d}s \\
&\le \frac{a}{A} \int_t^{(1+\eta)/2} \phi_p^{-1} \left(\int_0^{(1+\eta)/2} q(r)\, \mathrm{d}r \right) \mathrm{d}s \\
&\le \frac{a}{A} A_2 \le \frac{\sqrt{2}}{2} a = |w_0'(t)|,
\end{aligned} \tag{57}
$$

and for $t \in [(1+\eta)/2, 1]$,

$$\begin{aligned}
&\left|w_1'(t)\right| \\
&= \left|\int_{(1+\eta)/2}^{t} \phi_p^{-1}\left(\int_{(1+\eta)/2}^{s} q(r) \times f\left(r, \frac{\sqrt{2}}{2}a(1+\eta-r), -\frac{\sqrt{2}}{2}a\right) \mathrm{d}r\right) \mathrm{d}s\right| \\
&\le \int_{(1+\eta)/2}^{t} \phi_p^{-1}\left(\int_{(1+\eta)/2}^{1} q(r)\,\phi_p\left(\frac{a}{A}\right) \mathrm{d}r\right) \mathrm{d}s \\
&\le \frac{a}{A}\int_{(1+\eta)/2}^{t} \phi_p^{-1}\left(\int_{0}^{(1+\eta)/2} q(r)\,\mathrm{d}r\right) \mathrm{d}s \\
&\le \frac{a}{A}A_2 \le \frac{\sqrt{2}}{2}a = \left|w_0'(t)\right|.
\end{aligned} \tag{58}$$

Hence from (57) and (58), we have

$$\beta(w_1) := \max_{0\le t\le 1}\left|w_1'(t)\right| \le \frac{\sqrt{2}}{2}a. \tag{59}$$

Consequently, from (56) and (59), it follows that

$$\|w_1\| \le \sqrt{2}\max\{\alpha(w_1), \beta(w_1)\} \le a. \tag{60}$$

From the proof of Lemma 3, we see that w_1 is nonnegative, concave, and pseudosymmetric about η on $[0,1]$, and hence

$$w_1 \in \overline{P}_a. \tag{61}$$

Define $\{w_n\}$ as follows:

$$w_{n+1} = Tw_n = T^n w_1 = T^{n+1}w_0, \quad n = 0, 1, \ldots. \tag{62}$$

Then $\{w_n\}$ is well defined and for $n = 1, 2, \ldots$,

$$w_{n+1}(t) \le w_n(t), \quad \left|w_{n+1}'(t)\right| \le \left|w_n'(t)\right|, \quad t \in [0,1]. \tag{63}$$

In fact, for $t \in [0, (1+\eta)/2]$,

$$\begin{aligned}
w_2(t) &= Tw_1(t) \\
&= \int_0^t \int_\tau^{(1+\eta)/2} \phi_p^{-1}\left(\int_s^{(1+\eta)/2} q(r) \times f\left(r, w_1(r), w_1'(r)\right) \mathrm{d}r\right) \mathrm{d}s\,\mathrm{d}\tau \\
&\le \int_0^t \int_\tau^{(1+\eta)/2} \phi_p^{-1}\left(\int_s^{(1+\eta)/2} q(r) \times f\left(r, \frac{\sqrt{2}}{2}ar, \frac{\sqrt{2}}{2}a\right) \mathrm{d}r\right) \mathrm{d}s\,\mathrm{d}\tau \\
&= w_1(t),
\end{aligned} \tag{64}$$

$$\begin{aligned}
\left|w_2'(t)\right| &= \left|Tw_1'(t)\right| \\
&= \int_t^{(1+\eta)/2} \phi_p^{-1}\left(\int_s^{(1+\eta)/2} q(r) \times f\left(r, w_1(r), w_1'(r)\right) \mathrm{d}r\right) \mathrm{d}s \\
&\le \int_t^{(1+\eta)/2} \phi_p^{-1}\left(\int_s^{(1+\eta)/2} q(r) \times f\left(r, \frac{\sqrt{2}}{2}ar, \frac{\sqrt{2}}{2}a\right) \mathrm{d}r\right) \mathrm{d}s \\
&= w_1'(t).
\end{aligned} \tag{65}$$

For $t \in [(1+\eta)/2, 1]$, since $w_1, w_2 \in \overline{P}_a$, it follows from (64) and (65) that

$$w_2(t) = w_2(1+\eta-t) \le w_1(1+\eta-t) = w_1(t), \tag{66}$$

$$\left|w_2'(t)\right| = \left|w_2'(1+\eta-t)\right| \le \left|w_1'(1+\eta-t)\right| = \left|w_1'(t)\right|. \tag{67}$$

So from (64)–(67), we have

$$w_2(t) \le w_1(t), \quad \left|w_2'(t)\right| \le \left|w_1'(t)\right|, \quad t \in [0,1], \tag{68}$$

that is, (63) holds when $n = 1$. Assume that (63) holds when $n = k$, that is,

$$w_{k+1}(t) \le w_k(t), \quad \left|w_{k+1}'(t)\right| \le \left|w_k'(t)\right|, \quad t \in [0,1]. \tag{69}$$

Then from Lemma 5, we obtain

$$\begin{aligned} w_{k+2}(t) &= (Tw_{k+1})(t) \le (Tw_k)(t) \\ &= w_{k+1}(t), \quad t \in [0,1], \\ |w'_{k+2}(t)| &= |(Tw_{k+1})'(t)| \le |(Tw_k)'(t)| \\ &= |w'_{k+1}(t)|, \quad t \in [0,1]. \end{aligned} \tag{70}$$

So by induction (63) holds.

Since $T : \overline{P}_a \to \overline{P}_a$ is completely continuous, it follows that $\{w_n\}_{n=1}^{\infty}$ is relative compact. Then $\{w_n\}$ has a convergent subsequence $\{w_{n_k}\}$ and $w^* \in \overline{P}_a$ such that

$$w_{n_k} \longrightarrow w^* \quad (k \longrightarrow \infty), \tag{71}$$

that is,

$$\begin{aligned} &w_{n_k}(t) \rightrightarrows w^*(t) \quad (k \longrightarrow \infty), \\ &w'_{n_k}(t) \rightrightarrows w^{*\prime}(t) \quad (k \longrightarrow \infty) \text{ on } [0,1]. \end{aligned} \tag{72}$$

While from (63) and the fact for each $n = 1, 2, \ldots$, $w'_n((1+\eta)/2) = 0$ and $w''_n(t) \le 0$ on $[0,1]$, it follows that

$$\begin{aligned} &w_1(t) \ge w_2(t) \ge \cdots \ge w_n(t) \ge w_{n+1}(t) \ge \cdots, \\ &\qquad n = 1, 2, \ldots, \text{ on } [0,1], \\ &w'_1(t) \ge w'_2(t) \ge \cdots \ge w'_n(t) \ge w'_{n+1}(t) \ge \cdots, \\ &\qquad n = 1, 2, \ldots, \text{ on } \left[0, \frac{1+\eta}{2}\right], \\ &w'_1(t) \le w'_2(t) \le \cdots \le w'_n(t) \le w'_{n+1}(t) \le \cdots, \\ &\qquad n = 1, 2, \ldots, \text{ on } \left[\frac{1+\eta}{2}, 1\right]. \end{aligned} \tag{73}$$

Hence,

$$\begin{aligned} &w_n(t) \rightrightarrows w^*(t) \quad (n \longrightarrow \infty), \\ &w'_n(t) \rightrightarrows w^{*\prime}(t) \quad (n \longrightarrow \infty) \text{ on } [0,1], \end{aligned} \tag{74}$$

that is,

$$w_n \longrightarrow w^* \ (n \longrightarrow \infty). \tag{75}$$

This together with the continuity of T and $w_{n+1} = Tw_n$, implies that

$$Tw^* = w^*. \tag{76}$$

Also, since

$$\begin{aligned} 0 &\le w_n(t) \le w_0(t) \\ &= \begin{cases} \dfrac{\sqrt{2}}{2} at, & 0 \le t \le \dfrac{1+\eta}{2}, \\ \dfrac{\sqrt{2}}{2} a(1+\eta-t), & \dfrac{1+\eta}{2} \le t \le 1, \end{cases} \\ 0 &\le |w'_n(t)| \le |w'_1(t)| \le \frac{\sqrt{2}}{2} a, \quad t \in [0,1], \end{aligned} \tag{77}$$

we have

$$0 \le w^*(t) \le \frac{\sqrt{2}}{2} a, \quad 0 \le |w^{*\prime}(t)| \le \frac{\sqrt{2}}{2} a, \quad t \in [0,1]. \tag{78}$$

Furthermore, we have

$$w^*(t) > 0, \quad t \in (0,1]. \tag{79}$$

In fact, from (H_3) and $w^*(t) \not\equiv 0$ on $[0,1]$, we have $w^*((1+\eta)/2) > 0$. Since $w^*(t)$ is concave on $[0,1]$, then

$$\begin{aligned} w^*(t) &\ge \frac{w^*((1+\eta)/2) - 0}{((1+\eta)/2) - 0} t \\ &= \frac{2}{1+\eta} w^*\left(\frac{1+\eta}{2}\right) t > 0, \quad t \in \left(0, \frac{1+\eta}{2}\right]. \end{aligned} \tag{80}$$

Consequently from the fact w^* is pseudosymmetric on $[0,1]$, we have

$$w^*(t) > 0, \quad t \in (0,1]. \tag{81}$$

Let $v_0(t) \equiv 0$ on $[0,1]$, then $v_0 \in \overline{P}_a$. Set $v_{n+1} = Tv_n$, $n = 0, 1, 2, \ldots$. Then from Lemma 6, it follows that

$$v_n \in \overline{P}_a, \quad n = 1, 2, \ldots. \tag{82}$$

From Lemma 4, we see that $\{v_n\}_{n=1}^{\infty}$ is relative compact, and hence there exists a convergent subsequence $\{v_{n_k}\} \subset \{v_n\}$ and $v^* \in \overline{P}_a$ such that

$$v_{n_k} \longrightarrow v^* \quad (k \longrightarrow \infty), \tag{83}$$

that is,

$$v_{n_k}(t) \rightrightarrows v^*(t) \quad (k \longrightarrow \infty) \text{ on } [0,1], \tag{84}$$

$$v'_{n_k}(t) \rightrightarrows v^{*\prime}(t) \quad (k \longrightarrow \infty) \text{ on } [0,1]. \tag{85}$$

Since $v_1 = Tv_0 = T0 \in \overline{P}_a$, then

$$\begin{aligned} &v_1(t) = Tv_0(t) = (T0)(t) \ge 0, \quad t \in [0,1], \\ &|v'_1(t)| = |(Tv_0)'(t)| = |(T0)'(t)| \ge 0, \quad t \in [0,1]. \end{aligned} \tag{86}$$

Thus from Lemma 5,

$$\begin{aligned} &v_2(t) = Tv_1(t) \ge Tv_0(t) = v_1(t), \quad t \in [0,1], \\ &|v'_2(t)| = |(Tv_1)'(t)| \ge |(Tv_0)'(t)| = |v'_1(t)|, \quad t \in [0,1]. \end{aligned} \tag{87}$$

By induction, it is easy to see that for $n = 1, 2, \ldots$,

$$v_{n+1}(t) \ge v_n(t), \quad t \in [0,1], \tag{88}$$

$$|v'_{n+1}(t)| \ge |v'_n(t)|, \quad t \in [0,1]. \tag{89}$$

From (84)–(89), we see that

$$v_n(t) \rightrightarrows v^*(t) \quad (n \longrightarrow \infty),$$
$$v_n'(t) \rightrightarrows v^{*\prime}(t) \quad (n \longrightarrow \infty) \text{ on } [0,1]. \tag{90}$$

Therefore, $v_n \to v^*(n \to \infty)$, $v^* \in \overline{P}_a$. By the continuity of T and $v_{n+1} = Tv_n$, we have

$$Tv^* = v^*. \tag{91}$$

Again from (H_3), we have $v^*(t) > 0$ on $(0,1]$.

Since every fixed point of T in P is the solution of BVP (9), (10), then w^* and v^* are two positive, concave and pseudosymmetric solutions of BVP (9), (10). This completes the proof of the theorem. □

4. An Example

Consider the following third-order four-point boundary value problem:

$$u'''(t) + f(t, u(t), u'(t)) = 0, \quad t \in (0,1),$$
$$u(0) = 0, \qquad u(1) = u\left(\frac{1}{2}\right), \qquad u''\left(\frac{3}{4}\right) = 0, \tag{92}$$

where

$$f(t,u,v) = \frac{\sqrt{2}}{84}\left(\frac{3}{4} - t\right)\left(uv^2 + 16\sqrt{2}\right),$$
$$(t,u,v) \in [0,1] \times [0,+\infty) \times \mathbb{R}. \tag{93}$$

It is easy to see that BVP (92) corresponds to BVP (9), (10) when $p = 2$, $q(t) \equiv 1$, and $\eta = 1/2$. Take $a = 6\sqrt{2}$, and then $A = 3\sqrt{2}/4$.

Next we verify that all conditions of Theorem 7 are satisfied. In fact, obviously the conditions (H_0), (H_1), (H_2), and (H_3) hold. In addition, for $0 \le t \le 3/4$, $0 \le u_1 \le u_2 \le 6\sqrt{2}$, $0 \le |v_1| \le |v_2| \le 6\sqrt{2}$,

$$f(t, u_1, v_1) \le f(t, u_2, v_2),$$
$$\max_{0 \le t \le 3/4} f(t, a, a) = f\left(0, 6\sqrt{2}, 6\sqrt{2}\right) = 8 = \phi_2\left(\frac{a}{A}\right). \tag{94}$$

Hence, from Theorem 7, BVP (92) has two positive, concave, and pseudosymmetric solutions w^* and v^* such that

$$0 < w^*(t) \le 6, \quad 0 < \left|w^{*\prime}(t)\right| \le 6, \quad t \in [0,1],$$
$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} T^n w_0 = w^*, \tag{95}$$
$$\lim_{n\to\infty} w_n' = \lim_{n\to\infty} (T^n w_0)' = w^{*\prime},$$

where

$$w_0(t) = \begin{cases} 6t, & 0 \le t \le \dfrac{3}{4}, \\ 9 - 6t, & \dfrac{3}{4} \le t \le 1, \end{cases} \tag{96}$$

for $n = 0, 1, 2, \ldots,$

$$w_{n+1}(t) = \begin{cases} \displaystyle\int_0^t \int_\tau^{3/4} \int_s^{3/4} \frac{\sqrt{2}}{84}\left(\frac{3}{4} - r\right) \times \left(w_n(r) w_n'^2(r) + 16\sqrt{2}\right) dr\, ds\, d\tau, & 0 \le t \le \dfrac{3}{4}, \\ \displaystyle\int_0^{1/2} \int_\tau^{3/4} \int_s^{3/4} \frac{\sqrt{2}}{84}\left(\frac{3}{4} - r\right) \times \left(w_n(r) w_n'^2(r) + 16\sqrt{2}\right) dr\, ds\, d\tau \\ \displaystyle - \int_t^1 \int_{3/4}^\tau \int_{3/4}^s \frac{\sqrt{2}}{84}\left(\frac{3}{4} - r\right) \times \left(w_n(r) w_n'^2(r) + 16\sqrt{2}\right) dr\, ds\, d\tau, & \dfrac{3}{4} \le t \le 1, \end{cases}$$

$$0 < v^*(t) \le 6\sqrt{2}, \qquad 0 < \left|v^{*\prime}(t)\right| \le 6\sqrt{2}, \quad t \in [0,1],$$
$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} T^n v_0 = v^*, \qquad \lim_{n\to\infty} v_n' = \lim_{n\to\infty} (T^n v_0)' = v^{*\prime}, \tag{97}$$

where $v_0(t) \equiv 0$ on $[0,1]$ and for $n = 0, 1, 2, \ldots,$

$$v_{n+1}(t) = \begin{cases} \displaystyle\int_0^t \int_\tau^{3/4} \int_s^{3/4} \frac{\sqrt{2}}{84}\left(\frac{3}{4} - r\right) \times \left(v_n(r) v_n'^2(r) + 16\sqrt{2}\right) dr\, ds\, d\tau, & 0 \le t \le \dfrac{3}{4}, \\ \displaystyle\int_0^{1/2} \int_\tau^{3/4} \int_s^{3/4} \frac{\sqrt{2}}{84}\left(\frac{3}{4} - r\right) \times \left(v_n(r) v_n'^2(r) + 16\sqrt{2}\right) dr\, ds\, d\tau \\ \displaystyle - \int_t^1 \int_{3/4}^\tau \int_{3/4}^s \frac{\sqrt{2}}{84}\left(\frac{3}{4} - r\right) \times \left(v_n(r) v_n'^2(r) + 16\sqrt{2}\right) dr\, ds\, d\tau, & \dfrac{3}{4} \le t \le 1. \end{cases} \tag{98}$$

The first two terms of $\{w_n(t)\}$ and three terms of $\{v_n(t)\}$, respectively, are as follows:

$$w_0(t) = \begin{cases} 6t, & 0 \le t \le \dfrac{3}{4}, \\ 9 - 6t, & \dfrac{3}{4} \le t \le 1; \end{cases} \tag{99}$$

$$w_1(t)=\begin{cases}-\dfrac{3\sqrt{2}}{70}t^5+\left(\dfrac{9\sqrt{2}}{112}-\dfrac{1}{63}\right)t^4+\dfrac{1}{21}t^3\\ -\left(\dfrac{81\sqrt{2}}{896}+\dfrac{3}{56}\right)t^2\\ \quad+\left(\dfrac{243\sqrt{2}}{3584}+\dfrac{3}{112}\right)t, \quad 0\le t\le\dfrac{3}{4},\\ \dfrac{3\sqrt{2}}{70}t^5-\left(\dfrac{27\sqrt{2}}{112}+\dfrac{1}{63}\right)t^4\\ \quad+\left(\dfrac{27\sqrt{2}}{56}+\dfrac{1}{21}\right)t^3\\ -\left(\dfrac{405\sqrt{2}}{896}+\dfrac{3}{56}\right)t^2+\left(\dfrac{729\sqrt{2}}{3584}+\dfrac{3}{112}\right)t\\ -\dfrac{729\sqrt{2}}{35840}, \quad \dfrac{3}{4}\le t\le 1;\end{cases}$$

$$v_0(t)\equiv 0, \quad t\in[0,1],$$

$$v_1(t)=-\frac{1}{63}t^4+\frac{1}{21}t^3-\frac{3}{56}t^2+\frac{3}{112}t, \quad t\in[0,1],$$

$$\begin{aligned}v_2(t)={}&\frac{\sqrt{2}}{2867038902}t^{14}-\frac{\sqrt{2}}{273051324}t^{13}\\&+\frac{\sqrt{2}}{56010528}t^{12}-\frac{\sqrt{2}}{18670176}t^{11}\\&+\frac{47\sqrt{2}}{426746880}t^{10}-\frac{65\sqrt{2}}{398297088}t^{9}\\&+\frac{\sqrt{2}}{5619712}t^{8}-\frac{59\sqrt{2}}{413048832}t^{7}\\&+\frac{13\sqrt{2}}{157351936}t^{6}-\frac{51\sqrt{2}}{1573519360}t^{5}\\&+\left(\frac{9\sqrt{2}}{1258815488}-\frac{1}{63}\right)t^4+\frac{1}{21}t^3\\&-\left(\frac{81\sqrt{2}}{161128382464}+\frac{3}{56}\right)t^2\\&+\left(\frac{81\sqrt{2}}{598476849152}+\frac{3}{112}\right)t, \quad t\in[0,1].\end{aligned} \tag{100}$$

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