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Research Article

On the Generalized Hyers-Ulam-Rassias Stability of Quadratic Functional Equations

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Recommended by John Rassias

We achieve the general solution and the generalized Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias stabilities for quadratic functional equations $f(ax+by)+f(ax-by)=(b(a+b)/2)f(x+y)+(b(a+b)/2)f(x-y)+(2a^2-ab-b^2)f(x)+(b^2-ab)f(y)$ where a,b are nonzero fixed integers with $b\neq \pm a, -3a$, and $f(ax+by)+f(ax-by)=2a^2f(x)+2b^2f(y)$ for fixed integers a,b with $a,b\neq 0$ and $a\pm b\neq 0$.

1. Introduction

In 1940, Ulam [1] proposed the stability problem for functional equations in the following question regarding to the stability of group homomorphism.

Let $(G_1,\cdot)$ be a group and let $(G_2,*)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x\cdot y), h(x) * h(y)) < \delta$, for all $x,y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x),H(x)) < \epsilon$, for all $x \in G_1$? In other words, under what conditions does a homomorphism exist near an approximately homomorphism? Generally, the concept of stability for a functional equation comes up when we the functional equation is replaced by an inequality which acts as a perturbation of that equation. Hyers [2] answered to the question affirmatively in 1941 so if $f : E \to E'$ such that

$$\|f(x+y)-f(x)-f(y)\| \leq \delta, \tag{1.1}$$

for all $x,y \in E$, and for some $\delta > 0$ where E, E' are Banach spaces; then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x)-T(x)\| \leq \delta, \tag{1.2}$$

for all $x \in E$. However, if $f(tx)$ is a continuous mapping at $t \in \mathbb{R}$ for each fixed $x \in E$ then T is linear. In 1950, Hyers's theorem was generalized by Aoki [3] for additive mappings and independently, in 1978, by Rassias [4] for linear mappings considering the Cauchy difference controlled by sum of powers of norms. This stability phenomenon is called the Hyers-Ulam-Rassias stability.

On the other hand, Rassias [5–10] considered the Cauchy difference controlled by a product of different powers of norm. However, there was a singular case; for this singularity a counterexample was given by Găvruţa [11]. This stability phenomenon is called the Ulam-Găvruţa-Rassias stability (see also [12, 13]). In addition, J. M. Rassias considered the mixed product-sum of powers of norms control function [14]. This stability is called JMRassias mixed product-sum stability (see also [15–22]).

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1.3}$$

is related to symmetric biadditive function and is called a quadratic functional equation naturally, and every solution of the quadratic equation (1.3) is said to be a quadratic function. It is well known that a function f between two real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x)=B(x,x)$ for all x where

$$B(x,y)=\frac{1}{4}(f(x+y)-f(x-y)) \tag{1.4}$$

(see [23, 24]). Skof proved Hyers-Ulam-Rassias stability problem for quadratic functional equation (1.3) for a class of functions $f: A \to B$, where A is normed space and B is a Banach space, (see [25]). Cholewa [26] noticed that Skof's theorem is still true if relevant domain A alters to an abelian group. In 1992, Czerwik proved the Hyers-Ulam-Rassias stability of (1.3) (see [27]) and four years later, Grabiec [28] generalized the result mentioned above.

Throughout this paper, assume that a, b are fixed integers with $a, b \neq 0$, we introduce the following functional equations, which are different from (1.3):

$$\begin{aligned} f(ax+by)+f(ax-by) = {} & \frac{b(a+b)}{2}f(x+y)+\frac{b(a+b)}{2}f(x-y) \\ & + \left(2a^2-ab-b^2\right)f(x)+\left(b^2-ab\right)f(y), \end{aligned} \tag{1.5}$$

where $b \neq \pm a, -3a$, and

$$f(ax+by)+f(ax-by)=2a^2f(x)+2b^2f(y), \tag{1.6}$$

where $b \neq \pm a$.

In this paper, we establish the general solution and the generalized Hyers-Ulam-Rassias and Ulam-Găvruţa-Rassias stabilities problem for (1.5), (1.6) which are equivalent to (1.3).

2. Solution of (1.5), (1.6)

Let X and Y be real vector spaces. We here present the general solution of (1.5), (1.6).

Theorem 2.1. *A function $f : X \to Y$ satisfies the functional equation* (1.3) *if and only if $f : X \to Y$ satisfies the functional equation* (1.5). *Therefore, every solution of functional equation* (1.5) *is also a quadratic function.*

Proof. Let f satisfy the functional equation (1.3). Putting $x = y = 0$ in (1.3), we get $f(0) = 0$. Set $x = 0$ in (1.3) to get $f(-y) = f(y)$. Letting $y = x$ and $y = 2x$ in (1.3), respectively, we obtain that $f(2x) = 4f(x)$ and $f(3x) = 9f(x)$ for all $x \in X$. By induction, we lead to $f(kx) = k^2 f(x)$ for all positive integers k. Replacing x and y by $2x + y$ and $2x - y$ in (1.3), respectively, gives

$$f(2x+y) + f(2x-y) = 8f(x) + 2f(y) \tag{2.1}$$

for all $x, y \in X$. Using (1.3) and (2.1), we lead to

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 4f(x) - 2f(y) \tag{2.2}$$

for all $x, y \in X$. Suppose that $k \neq 0$ is a fixed integer by using (1.3), we get

$$kf(x+y) + kf(x-y) - 2kf(x) - 2kf(y) = 0 \tag{2.3}$$

for all $x, y \in X$. Using (2.2) and (2.3), we obtain

$$f(2x+y) + f(2x-y) = (2+k)f(x+y) + (2+k)f(x-y) + 2(2-k)f(x) - 2(1+k)f(y) \tag{2.4}$$

for all $x, y \in X$. Replacing x and y by $3x + y$ and $3x - y$ in (1.3), respectively, then using (1.3) and (2.3), we have

$$f(3x+y) + f(3x-y) = (3+k)f(x+y) + (3+k)f(x-y) + 2(6-k)f(x) - 2(2+k)f(y) \tag{2.5}$$

for all $x, y \in X$. By using the above method, by induction, we infer that

$$\begin{aligned} f(ax+y) + f(ax-y) &= (a+k)f(x+y) + (a+k)f(x-y) \\ &\quad + 2\left(a^2 - a - k\right)f(x) - 2(a+k-1)f(y) \end{aligned} \tag{2.6}$$

for all $x, y \in X$ and each positive integer $a \geq 1$. For a negative integer $a \leq -1$, replacing a by $-a$ one can easily prove the validity of (2.6). Therefore (1.3) implies (2.6) for any integer $a \neq 0$. First, it is noted that (2.6) also implies the following equation

$$\begin{aligned} f(bx+y) + f(bx-y) = (b+k)f(x+y) + (b+k)f(x-y) \\ + 2\left(b^2 - b - k\right)f(x) - 2(b+k-1)f(y) \end{aligned} \tag{2.7}$$

for all integers $b \neq 0$. Setting $y = 0$ in (2.7) gives $f(bx) = b^2 f(x)$. Substituting y with by into (2.7), one gets

$$\begin{aligned} (b+k)f(x+by) + (b+k)f(x-by) = b^2 f(x+y) + b^2 f(x-y) \\ - 2\left(b^2 - b - k\right)f(x) + 2b^2(b+k-1)f(y) \end{aligned} \tag{2.8}$$

for all $x, y \in X$. Replacing y by by in (2.6), we observe that

$$\begin{aligned} f(ax+by) + f(ax-by) = (a+k)f(x+by) + (a+k)f(x-by) \\ + 2\left(a^2 - a - k\right)f(x) - 2(a+k-1)f(by) \end{aligned} \tag{2.9}$$

for all $x, y \in X$. Hence, according to (2.8) and (2.9), we get

$$\begin{aligned} (b+k)f(ax+by) + (b+k)f(ax-by) = b^2(a+k)f(x+y) + b^2(a+k)f(x-y) \\ + 2\left(a^2(b+k) - b^2(a+k)\right)f(x) - 2b^2(a-b)f(y) \end{aligned} \tag{2.10}$$

for all $x, y \in X$. In particular, if we substitute $k := b$ in (2.10) and dividing it by $2b$, we conclude that f satisfies (1.5).

Let f satisfy the functional equation (1.5), for nonzero fixed integers a,b with $b \neq \pm a, -3a$. Putting $x = y = 0$ in (1.5), we get

$$\left(2a^2 - ba + b^2 - 2\right)f(0) = 0, \tag{2.11}$$

so

$$\left(2a - \frac{b + \sqrt{16 - 7b^2}}{2}\right)\left(a - \frac{b - \sqrt{16 - 7b^2}}{4}\right)f(0) = 0, \tag{2.12}$$

but since $a, b \neq 0$ and $b \neq \pm a, -3a$, therefore $f(0) = 0$. Setting $y = 0$ in (1.5) gives $f(ax) = a^2 f(x)$ for all $x \in X$. Letting $y = -y$ in (1.5), we get

$$\begin{aligned} f(ax - by) + f(ax + by) = {} & \frac{b(a+b)}{2} f(x-y) + \frac{b(a+b)}{2} f(x+y) \\ & + \left(2a^2 - ab - b^2\right) f(x) + \left(b^2 - ab\right) f(-y) \end{aligned} \tag{2.13}$$

for all $x, y \in X$. If we compare (1.5) with (2.13), then since $a, b \neq 0$ and $b \neq \pm a, -3a$, we conclude that $f(-y) = f(y)$ for all $y \in X$. Letting $x = 0$ in (1.5) and using the evenness of f give $f(by) = b^2 f(y)$ for all $y \in X$. Therefore for all $x \in X$, we get $f(abx) = a^2 b^2 f(x)$. Replacing x and y by bx and ay in (1.5), respectively, we have

$$\begin{aligned} a^2 b^2 f(x+y) + a^2 b^2 f(x-y) = {} & \frac{b(a+b)}{2} f(bx + ay) + \frac{b(a+b)}{2} f(bx - ay) \\ & + b^2\left(2a^2 - ab - b^2\right) f(x) + a^2\left(b^2 - ab\right) f(y) \end{aligned} \tag{2.14}$$

for all $x, y \in X$. On the other hand, if we interchange x with y in (1.5), we obtain

$$\begin{aligned} f(ay + bx) + f(ay - bx) = {} & \frac{b(a+b)}{2} f(y+x) + \frac{b(a+b)}{2} f(y-x) \\ & + \left(2a^2 - ab - b^2\right) f(y) + \left(b^2 - ab\right) f(x) \end{aligned} \tag{2.15}$$

for all $x, y \in X$. But since f is even, it follows from (2.15) that

$$\begin{aligned} f(bx + ay) + f(bx - ay) = {} & \frac{b(a+b)}{2} f(x+y) + \frac{b(a+b)}{2} f(x-y) \\ & + \left(b^2 - ab\right) f(x) + \left(2a^2 - ab - b^2\right) f(y) \end{aligned} \tag{2.16}$$

for all $x, y \in X$. Hence, according to (2.14) and (2.16), we obtain that

$$\begin{aligned} a^2 b^2 f(x+y) + a^2 b^2 f(x-y) = {} & \frac{b(a+b)}{2} \left[\frac{b(a+b)}{2} \left(f(x+y) + f(x-y) \right) \right. \\ & \left. + \left(b^2 - ab\right) f(x) + \left(2a^2 - ab - b^2\right) f(y) \right] \\ & + b^2\left(2a^2 - ab - b^2\right) f(x) + a^2\left(b^2 - ab\right) f(y) \end{aligned} \tag{2.17}$$

for all $x, y \in X$. So from (2.17), we have

$$\frac{b^2}{4}\left(4a^2-(a+b)^2\right)(f(x+y)+f(x-y)) = \frac{b^2}{2}\left(3a^2-2ab-b^2\right)f(x) + \frac{b^2}{2}\left(3a^2-2ab-b^2\right)f(y) \tag{2.18}$$

for all $x, y \in X$. But since $a, b \neq 0$ and $b \neq \pm a, -3a$, we conclude that

$$f(x+y)+f(x-y) = 2f(x)+2f(y) \tag{2.19}$$

for all $x, y \in X$. Therefore, f satisfies (1.3). □

Theorem 2.2. *A function $f : X \to Y$ satisfies the functional equation* (1.3) *if and only if $f : X \to Y$ satisfies the functional equation* (1.6)*. Therefore, every solution of functional equation* (1.6) *is also a quadratic function.*

Proof. If f satisfies the functional equation (1.3), then f satisfies the functional equation (1.5). Now combining (1.3) with (1.5), we have

$$f(ax+by)+f(ax-by) = \frac{b(a+b)}{2}(2f(x)+2f(y)) + \left(2a^2-ab-b^2\right)f(x) + \left(b^2-ab\right)f(y) \tag{2.20}$$

for all $x, y \in X$. So from (2.20), we conclude that f satisfies (1.6).

Let f satisfy the functional equation (1.6) for fixed integers a, b with $a \neq 0$, $b \neq 0$ and $a \pm b \neq 0$. Putting $x = y = 0$ in (1.6), we get $(2(a^2+b^2)-2)f(0) = 0$, and since $a \neq 0$, $b \neq 0$, therefore $f(0) = 0$. Setting $y = 0$ in (1.6) gives $f(ax) = a^2 f(x)$ for all $x \in X$. Letting $y := -y$ in (1.6), we have

$$f(ax-by)+f(ax+by) = 2a^2 f(x)+2b^2 f(-y) \tag{2.21}$$

for all $x, y \in X$. If we compare (1.6) with (2.21), then since $a, b \neq 0$ and $a \pm b \neq 0$, we obtain that $f(-y) = f(y)$ for all $y \in X$. Letting $x = 0$ in (1.6) and using the evenness of f gives $f(by) = b^2 f(y)$ for all $y \in X$. Therefore for all $x \in X$, we get $f(abx) = a^2b^2 f(x)$. Replacing x and y by bx and ay in (1.6), respectively, we have

$$f(abx-aby)+f(abx+aby) = 2a^2 f(bx)+2b^2 f(ay) \tag{2.22}$$

for all $x, y \in X$. Now, by using $f(ax) = a^2 f(x), f(bx) = b^2 f(x)$ and $f(abx) = a^2b^2 f(x)$, it follows from (2.22) that

$$f(x+y)+f(x-y) = 2f(x)+2f(y) \tag{2.23}$$

for all $x, y \in X$. Which completes the proof of the theorem. □

Corollary 2.3 ([29, Proposition 2.1]). *A function $f : X \to Y$ satisfies the following functional equation:*

$$f(ax+y) + f(ax-y) = 2a^2 f(x) + 2f(y) \tag{2.24}$$

for all $x, y \in X$ if and only if $f : X \to Y$ satisfies the functional equation (1.3) *for all $x, y \in X$.*

Proof. Assume that $b = 1$ in functional equation (1.6) and apply Theorem 2.2. □

3. Stability

We now investigate the generalized Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias stabilities problem for functional equations (1.5), (1.6). From this point on, let X be a real vector space and let Y be a Banach space. Before taking up the main subject, we define the difference operator $\Delta_f : X \times X \to Y$ by

$$\begin{aligned}\Delta_f(x,y) = f(ax+by) + f(ax-by) - \frac{b(a+b)}{2} f(x+y) - \frac{b(a+b)}{2} f(x-y) \\ - \left(2a^2 - ab - b^2\right) f(x) - \left(b^2 - ab\right) f(y)\end{aligned} \tag{3.1}$$

for all $x, y \in X$ and a, b fixed integers such that $a, b \neq 0$ and $a \pm b \neq 0$ where $f : X \to Y$ is a given function.

Theorem 3.1. *Let $j \in \{-1, 1\}$ be fixed, and let $\varphi : X \times X \to [0, \infty)$ be a function such that*

$$\tilde{\varphi}(x) := \sum_{i=(1-j)/2}^{\infty} \frac{1}{a^{2ij}} \varphi\left(a^{ij}x, 0\right) < \infty \tag{3.2}$$

$$\lim_{n \to \infty} \frac{1}{a^{2nj}} \varphi\left(a^{nj}x, a^{nj}y\right) = 0 \tag{3.3}$$

for all $x, y \in X$. Suppose that $f : X \to Y$ be a function satisfies

$$\left\| \Delta_f(x,y) \right\| \leq \varphi(x,y) \tag{3.4}$$

for all $x, y \in X$. Furthermore, assume that $f(0) = 0$ in (3.4) *for the case $j = 1$. Then there exists a unique quadratic function $Q : X \to Y$ such that*

$$\| f(x) - Q(x) \| \leq \frac{1}{2a^{1+j}} \tilde{\varphi}\left(\frac{x}{a^{(1-j)/2}}\right), \tag{3.5}$$

for all $x \in X$.

Proof. For $j = 1$, putting $y = 0$ in (3.4), we have

$$\left\| 2f(ax) - 2a^2 f(x) \right\| \leq \varphi(x, 0) \tag{3.6}$$

for all $x \in X$. So

$$\left\| f(x) - \frac{1}{a^2} f(ax) \right\| \leq \frac{1}{2a^2} \varphi(x, 0) \tag{3.7}$$

for all $x \in X$. Replacing x by ax in (3.7) and dividing by a^2 and summing the resulting inequality with (3.7), we get

$$\left\| f(x) - \frac{1}{a^4} f\left(a^2 x\right) \right\| \leq \frac{1}{2a^2} \left(\varphi(x, 0) + \frac{\varphi(ax, o)}{a^2} \right) \tag{3.8}$$

for all $x \in X$. Hence

$$\left\| \frac{1}{a^{2k}} f\left(a^k x\right) - \frac{1}{a^{2m}} f(a^m x) \right\| \leq \frac{1}{2a^2} \sum_{i=k}^{m-1} \frac{1}{a^{2i}} \varphi\left(a^i x, 0\right) \tag{3.9}$$

for all nonnegative integers m and k with $m > k$ and for all $x \in X$. It follows from (3.2) and (3.9) that the sequence $\{(1/a^{2n}) f(a^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(1/a^{2n}) f(a^n x)\}$ converges. So one can define the function $Q : X \to Y$ by

$$Q(x) := \lim_{n \to \infty} \frac{1}{a^{2n}} f(a^n x) \tag{3.10}$$

for all $x \in X$. By (3.3) for $j = 1$ and (3.4),

$$\| \Delta_Q(x, y) \| = \lim_{n \to \infty} \frac{1}{a^{2n}} \| \Delta_f(a^n x, a^n y) \| \leq \lim_{n \to \infty} \frac{1}{a^{2n}} \varphi(a^n x, a^n y) = 0 \tag{3.11}$$

for all $x, y \in X$. So $\Delta_Q(x, y) = 0$. By Theorem 2.1, the function $Q : X \to Y$ is quadratic. Moreover, letting $k = 0$ and passing the limit $m \to \infty$ in (3.9), we get the inequality (3.5) for $j = 1$.

Now, let $Q' : X \to Y$ be another quadratic function satisfying (1.5) and (3.5). Then we have

$$\begin{aligned} \left\| Q(x) - Q'(x) \right\| &= \frac{1}{a^{2n}} \left\| Q(a^n x) - Q'(a^n x) \right\| \\ &\leq \frac{1}{a^{2n}} \left(\| Q(a^n x) - f(a^n x) \| + \left\| Q'(a^n x) - f(a^n x) \right\| \right) \\ &\leq \frac{1}{a^2 a^{2n}} \widetilde{\varphi}(a^n x, 0), \end{aligned} \tag{3.12}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of Q.

Also, for $j = -1$, it follows from (3.6) that

$$\left\| f(x) - a^2 f\left(\frac{x}{a}\right) \right\| \le \frac{1}{2}\varphi\left(\frac{x}{a}, 0\right) \tag{3.13}$$

for all $x \in X$. Hence

$$\left\| a^{2k} f\left(\frac{x}{a^k}\right) - a^{2m} f\left(\frac{x}{a^m}\right) \right\| \le \frac{1}{2}\sum_{i=k}^{m-1} a^{2i}\varphi\left(\frac{x}{a^{i+1}}, 0\right) \tag{3.14}$$

for all nonnegative integers m and k with $m > k$ and for all $x \in X$. It follows from (3.14) that the sequence $\{a^{2n} f(x/a^n)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{a^{2n} f(x/a^n)\}$ converges. So one can define the function $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} a^{2n} f\left(\frac{x}{a^n}\right) \tag{3.15}$$

for all $x \in X$. By (3.3) for $j = -1$ and (3.4),

$$\left\| \Delta_Q(x, y) \right\| = \lim_{n\to\infty} a^{2n} \left\| \Delta_f\left(\frac{x}{a^n}, \frac{y}{a^n}\right) \right\| \le \lim_{n\to\infty} a^{2n}\varphi\left(\frac{x}{a^n}, \frac{y}{a^n}\right) = 0, \tag{3.16}$$

for all $x, y \in X$. So $\Delta_Q(x, y) = 0$. By Theorem 2.1, the function $Q : X \to Y$ is quadratic. Moreover, letting $k = 0$ and passing the limit $m \to \infty$ in (3.14), we get the inequality (3.5) for $j = -1$. The rest of the proof is similar to the proof of previous section. □

From Theorem 3.1, we obtain the following corollaries concerning the JMRassias mixed product-sum stability of the functional equation (1.5).

Corollary 3.2. *Let $\varepsilon, p, q \ge 0$ and $r, s > 0$ be real numbers such that $p, q < 2$ and $r + s \ne 2$. Suppose that a function $f : X \to Y$ satisfies*

$$\left\| \Delta_f(x, y) \right\| \le \varepsilon\left(\|x\|^p + \|y\|^q + \|x\|^r \|y\|^s\right) \tag{3.17}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\left\| f(x) - Q(x) \right\| \le \frac{\varepsilon}{2(a^2 - a^p)} \|x\|^p \tag{3.18}$$

for all $x \in X$.

Proof. In Theorem 3.1, put $j := 1$ and $\varphi(x, y) := \varepsilon(\|x\|^p + \|y\|^q + \|x\|^r \|y\|^s)$. □

Corollary 3.3. *Let $\varepsilon, p, q \geq 0$ and $r, s > 0$ be real numbers such that $p, q > 2$ and $r + s \neq 2$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies (3.17) for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{\varepsilon}{2(a^p - a^2)}\|x\|^p \tag{3.19}$$

for all $x \in X$.

Proof. In Theorem 3.1, put $j := -1$ and $\varphi(x, y) := \varepsilon(\|x\|^p + \|y\|^q + \|x\|^r\|y\|^s)$. □

Theorem 3.4. *Let $j \in \{-1, 1\}$ be fixed, and let $\varphi : X \times X \to [0, \infty)$ be a function such that*

$$\begin{gathered}\widetilde{\varphi}(x) := \sum_{i=(1-j)/2}^{\infty} \frac{1}{a^{2ij}}\varphi\left(a^{ij}x, 0\right) < \infty, \\ \lim_{n\to\infty} \frac{1}{a^{2nj}}\varphi\left(a^{nj}x, a^{nj}y\right) = 0\end{gathered} \tag{3.20}$$

for all $x, y \in X$. Suppose that $f : X \to Y$ be a function satisfies

$$\left\|f(ax + by) + f(ax - by) - 2a^2 f(x) - 2b^2 f(y)\right\| \leq \varphi(x, y) \tag{3.21}$$

for all $x, y \in X$. Furthermore, assume that $f(0) = 0$ in (3.21) for the case $j = 1$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{2a^{1+j}}\widetilde{\varphi}\left(\frac{x}{a^{1-j/2}}\right), \tag{3.22}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.1. □

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References

[1] S. M. Ulam, *Problems in Modern Mathematics*, chapter 6, John Wiley & Sons, New York, NY, USA, 1940.
[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, no. 4, pp. 222–224, 1941.
[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] Th. M. Rassias, “On the stability of the linear mapping in Banach spaces,” *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
[5] J. M. Rassias, “On approximation of approximately quadratic mappings by quadratic mappings,” *Annales Mathematicae Silesianae*, no. 15, pp. 67–78, 2001.
[6] J. M. Rassias, “On approximation of approximately linear mappings by linear mappings,” *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.
[7] J. M. Rassias, “On approximation of approximately linear mappings by linear mappings,” *Bulletin des Sciences Mathématiques*, vol. 108, no. 4, pp. 445–446, 1984.
[8] J. M. Rassias, “On a new approximation of approximately linear mappings by linear mappings,” *Discussiones Mathematicae*, vol. 7, pp. 193–196, 1985.
[9] J. M. Rassias, “Solution of a problem of Ulam,” *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.
[10] J. M. Rassias, “Solution of a stability problem of Ulam,” *Discussiones Mathematicae*, vol. 12, pp. 95–103, 1992.
[11] P. Găvruţa, “An answer to a question of John M. Rassias concerning the stability of Cauchy equation,” in *Advances in Equations and Inequalities*, Hadronic Mathematics, pp. 67–71, Hadronic Press, Palm Harbor, Fla, USA, 1999.
[12] Z. Gajda, “On stability of additive mappings,” *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.
[13] P. Găvruţa, “A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
[14] K. Ravi, M. Arunkumar, and J. M. Rassias, “Ulam stability for the orthogonally general Euler-Lagrange type functional equation,” *International Journal of Mathematics and Statistics*, vol. 3, no. A08, pp. 36–46, 2008.
[15] K.-W. Jun, H.-M. Kim, and J. M. Rassias, “Extended Hyers-Ulam stability for Cauchy-Jensen mappings,” *Journal of Difference Equations and Applications*, vol. 13, no. 12, pp. 1139–1153, 2007.
[16] H.-M. Kim, K.-W. Jun, and J. M. Rassias, “Extended stability problem for alternative Cauchy-Jensen mappings,” *Journal of Inequalities in Pure and Applied Mathematics*, vol. 8, no. 4, article 120, pp. 1–17, 2007.
[17] C.-G. Park and J. M. Rassias, “Hyers-Ulam stability of an Euler-Lagrange type additive mapping,” *International Journal of Applied Mathematics & Statistics*, vol. 7, no. Fe07, pp. 112–125, 2007.
[18] J. M. Rassias, “Solution of a quadratic stability Hyers-Ulam type problem,” *Ricerche di Matematica*, vol. 50, no. 1, pp. 9–17, 2001.
[19] J. M. Rassias, “On the stability of a multi-dimensional Cauchy type functional equation,” in *Geometry, Analysis and Mechanics*, pp. 365–376, World Scientific, River Edge, NJ, USA, 1994.
[20] J. M. Rassias, “On the stability of the Euler-Lagrange functional equation,” *Chinese Journal of Mathematics*, vol. 20, no. 2, pp. 185–190, 1992.
[21] J. M. Rassias, “Refined Hyers-Ulam approximation of approximately Jensen type mappings,” *Bulletin des Sciences Mathématiques*, vol. 131, no. 1, pp. 89–98, 2007.
[22] J. M. Rassias, “Solution of a Cauchy-Jensen stability Ulam type problem,” *Archivum Mathematicum*, vol. 37, no. 3, pp. 161–177, 2001.
[23] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1989.
[24] Pl. Kannappan, “Quadratic functional equation and inner product spaces,” *Results in Mathematics*, vol. 27, no. 3-4, pp. 368–372, 1995.
[25] F. Skof, “Proprieta’ locali e approssimazione di operatori,” *Rendiconti del Seminario Matemàtico e Fisico di Milano*, vol. 53, no. 1, pp. 113–129, 1983.
[26] P. W. Cholewa, “Remarks on the stability of functional equations,” *Aequationes Mathematicae*, vol. 27, no. 1-2, pp. 76–86, 1984.
[27] St. Czerwik, “On the stability of the quadratic mapping in normed spaces,” *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, no. 1, pp. 59–64, 1992.
[28] A. Grabiec, “The generalized Hyers-Ulam stability of a class of functional equations,” *Publicationes Mathematicae Debrecen*, vol. 48, no. 3-4, pp. 217–235, 1996.
[29] J. R. Lee, J. S. An, and C. Park, “On the stability of quadratic functional equations,” *Abstract and Applied Analysis*, vol. 2008, Article ID 628178, 8 pages, 2008.