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Research Article

Strong Convergence of Generalized Projection Algorithms for Nonlinear Operators

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We establish strong convergence theorems for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of two relatively nonexpansive mappings in a Banach space by using a new hybrid method. Moreover we apply our main results to obtain strong convergence for a maximal monotone operator and two nonexpansive mappings in a Hilbert space.

1. Introduction

Let E be a real Banach space with $\|\cdot\|$ and let C be a nonempty closed convex subset of E. A mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T; that is, $F(T) = \{x \in C : x = Tx\}$. A mapping T of C into itself is called *quasinonexpansive* if $F(T)$ is nonempty and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$. For two mappings S and T of C into itself, Das and Debata [1] considered the following iteration scheme: $x_0 \in C$ and

$$x_{n+1} = \alpha_n S\big(\beta_n T x_n + (1 - \beta_n) x_n\big) + (1 - \alpha_n) x_n, \quad n \ge 0, \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. In this case of $S = T$, such an iteration process was considered by Ishikawa [2]; see also Mann [3]. Das and Debata [1] proved the strong convergence of the iterates $\{x_n\}$ defined by (1.1) in the case when E is strictly convex and S, T are quasinonexpansive mappings. Fixed point iteration processes for nonexpansive mappings in a Hilbert space and a Banach space including Das and Debata's iteration and

Ishikawa's iteration have been studied by many researchers to approximating a common fixed point of two mappings; see, for instance, Takahashi and Tamura [4].

Let A be a maximal monotone operator from E to E^*, where E^* is the dual space of E. It is well known that many problems in nonlinear analysis and optimization can be formulated as follows. Find a point $u \in E$ satisfying

$$0 \in Au. \tag{1.2}$$

We denote by $A^{-1}0$ the set of all points $u \in C$ such that $0 \in Au$. Such a problem contains numerous problems in economics, optimization, and physics. A well-known method to solve this problem is called the proximal point algorithm: $x_0 \in E$ and

$$x_{n+1} = J_{r_n} x_n, \quad n = 0, 1, 2, 3, \ldots, \tag{1.3}$$

where $\{r_n\} \subset (0, \infty)$ and J_{r_n} are the resovents of A. Many researchers have studied this algorithm in a Hilbert space; see, for instance, [5–8] and in a Banach space; see, for instance, [9–11].

Next, we recall that for all $x \in E$ and $x^* \in E^*$, we denote the value of x^* at x by $\langle x, x^* \rangle$. Then, the normalized duality mapping J on E is defined by

$$Jx = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}, \quad \forall x \in E. \tag{1.4}$$

We know that if E is smooth, then the duality mapping J is single valued. Next, we assume that E is a smooth Banach space and define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2, \quad \forall y, x \in E. \tag{1.5}$$

A point $u \in C$ is said to be an *asymptotic* fixed point of T [12] if C contains a sequence $\{x_n\}$ which converges weakly to u and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\widehat{F}(T)$. A mapping $T : C \to C$ is said to be *relatively nonexpansive* [13–15] if $\widehat{F}(T) = F(T) \neq \emptyset$ and $\phi(u, Tx) \leq \phi(u, x)$ for all $u \in F(T)$ and $x \in C$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [13–15].

In 2004, Matsushita and Takahashi [15] proposed the following modification of Mann's iteration for a relatively nonexpansive mapping by using the hybrid method in a Banach space. Four years later, Qin and Su [16] have adapted Matsushita and Takahashi's idea [15] to modify Halpern's iteration and Ishikawa's iteration for a relatively nonexpansive mapping in a Banach space. In particular, in a Hilbert space Mann's iteration, Halpern's iteration, and Ishikawa's iteration were considered by many researchers.

Very recently, Inoue et al. [17] proved the following strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping by using the hybrid method.

Theorem 1.1 (Inoue et al. [17]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $S : C \to C$ be a relatively* *nonexpansive mapping such that* $F(S) \cap A^{-1}0 \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 = x \in C$ *and*

$$\begin{aligned} u_n &= J^{-1}\left(\beta_n J x_n + (1-\beta_n) J S J_{r_n} x_n\right), \\ C_n &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n &= \{z \in C : \langle x_n - z, J x_0 - J x_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0 \end{aligned} \tag{1.6}$$

for all $n \in \mathbb{N} \cup \{0\}$, *where* J *is the duality mapping on* E, $\{\beta_n\} \subset [0,1]$, *and* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$. *If* $\liminf_{n\to\infty}(1-\beta_n) > 0$, *then* $\{x_n\}$ *converges strongly to* $\Pi_{F(S)\cap A^{-1}0} x_0$, *where* $\Pi_{F(S)\cap A^{-1}0}$ *is the generalized projection of* E *onto* $F(S) \cap A^{-1}0$.

The purpose of this paper is to employ the idea of Inoue et al. [17] and Das and Debata [1] to introduce a new hybrid method for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of two relatively nonexpansive mappings. We prove a strong convergence theorem of the new hybrid method. Moreover we apply our main results to obtain strong convergence for a maximal monotone operator and two nonexpansive mappings in a Hilbert space.

2. Preliminaries

Throughout this paper, all linear spaces are real. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of all positive integers and real numbers, respectively. Let E be a Banach space and let E^* be the dual space of E. For a sequence $\{x_n\}$ of E and a point $x \in E$, the *weak* convergence of $\{x_n\}$ to x and the *strong* convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

Let $S(E)$ be the unit sphere centered at the origin of E. Then the space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S(E)$. It is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in S(E)$. A Banach space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be *uniformly convex* if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that $\|(x+y)/2\| < 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \epsilon$. We know the following [18]:

(i) if E is smooth, then J is single-valued;

(ii) if E is reflexive, then J is onto;

(iii) if E is strictly convex, then J is one to one;

(iv) if E is strictly convex, then J is strictly monotone;

(v) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

A Banach space E is said to have the *Kadec-Klee* property if for a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [18, 19] for more details. Let E be a smooth, strictly convex, and reflexive Banach space and let C be a closed convex subset of E. Throughout this paper, define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \quad \forall y, x \in E. \tag{2.2}$$

Observe that, in a Hilbert space H, (2.2) reduces to $\phi(x,y) = \|x-y\|^2$, for all $x, y \in H$. It is obvious from the definition of the function ϕ that, for all $x, y \in E$,

(1) $(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2$,

(2) $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle$,

(3) $\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \le \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$.

Following Alber [20], the generalized projection Π_C from E onto C is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y,x)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y \in C} \phi(y,x). \tag{2.3}$$

Existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y,x)$ and strict monotonicity of the mapping J. In a Hilbert space, Π_C is the metric projection of H onto C. We need the following lemmas for the proof of our main results.

Lemma 2.1 (Kamimura and Takahashi [6]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty}\phi(x_n, y_n) = 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 2.2 (Matsushita and Takahashi [15]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.3 (Alber [20] and Kamimura and Takahashi [6]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space, $x \in E$ and let $z \in C$. Then, $z = \Pi_C x$ if and only if $\langle y - z, Jx - Jz\rangle \le 0$ for all $y \in C$.*

Lemma 2.4 (Alber [20] and Kamimura and Takahashi [6]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x,y), \quad \forall x \in C,\ y \in E. \tag{2.4}$$

Let E be a smooth, strictly convex, and reflexive Banach space, and let A be a set-valued mapping from E to E^* with graph $G(A) = \{(x, x^*) : x^* \in Ax\}$, domain $D(A) = \{z \in E : Az \neq \emptyset\}$, and range $R(A) = \cup\{Az : z \in D(A)\}$. We denote a set-valued operator A from E to E^* by $A \subset E \times E^*$. A is said to be *monotone* if $\langle x - y, x^* - y^*\rangle \ge 0$, for all $(x, x^*), (y, y^*) \in A$. A monotone operator $A \subset E \times E^*$ is said to be *maximal monotone* if its graph is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $A^{-1}0 = \{z \in D(A) : 0 \in Az\}$ is closed and convex. The following theorem is well known.

Lemma 2.5 (Rockafellar [21]). *Let E be a smooth, strictly convex, and reflexive Banach space and let $A \subset E \times E^*$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Let E be a smooth, strictly convex, and reflexive Banach space, let C be a nonempty closed convex subset of E and let $A \subset E \times E^*$ be a monotone operator satisfying

$$D(A) \subset C \subset J^{-1}\left(\bigcap_{r>0} R(J + rA)\right). \tag{2.5}$$

Then we can define the resolvent $J_r : C \to D(A)$ of A by

$$J_r x = \{z \in D(A) : Jx \in Jz + rAz\}, \quad \forall x \in C. \tag{2.6}$$

We know that $J_r x$ consists of one point. For $r > 0$, the Yosida approximation $A_r : C \to E^*$ is defined by $A_r x = (Jx - JJ_r x)/r$ for all $x \in C$.

Lemma 2.6 (Kohsaka and Takahashi [22]). *Let E be a smooth, strictly convex, and reflexive Banach space, let C be a nonempty closed convex subset of E and let $A \subset E \times E^*$ be a monotone operator satisfying*

$$D(A) \subset C \subset J^{-1}\left(\bigcap_{r>0} R(J + rA)\right). \tag{2.7}$$

Let $r > 0$ and let J_r and A_r be the resolvent and the Yosida approximation of A, respectively. Then, the following hold:

(i) $\phi(u, J_r x) + \phi(J_r x, x) \le \phi(u, x)$, *for all $x \in C$, $u \in A^{-1}0$;*

(ii) $(J_r x, A_r x) \in A$, *for all $x \in C$;*

(iii) $F(J_r) = A^{-1}0$.

Lemma 2.7 (Zălinescu [23] and Xu [24]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|) \tag{2.8}$$

for all $x, y \in B_r(0)$ and $t \in [0,1]$, where $B_r(0) = \{z \in E : \|z\| \le r\}$.

3. Main Results

In this section, we prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of two relatively nonexpansive mappings in a Banach space by using the hybrid method.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let S and T be relatively nonexpansive mappings from C into itself such that $\Omega = F(S) \cap F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and*

$$\begin{aligned}
u_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T z_n),\\
z_n &= J^{-1}(\beta_n J x_n + (1-\beta_n) J S J_{r_n} x_n),\\
C_n &= \{z \in C : \phi(z, u_n) \le \phi(z, x_n)\},\\
Q_n &= \{z \in C : \langle x_n - z, J x_0 - J x_n\rangle \ge 0\},\\
x_{n+1} &= \Pi_{C_n \cap Q_n} x_0
\end{aligned} \tag{3.1}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$ and $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection of E onto Ω.

Proof. We first show that C_n and Q_n are closed and convex for each $n \ge 0$. From the definitions of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \ge 0$. Next, we prove that C_n is convex. Since $\phi(z, u_n) \le \phi(z, x_n)$ is equivalent to

$$0 \le \|x_n\|^2 - \|u_n\|^2 - 2\langle z, J x_n - J u_n\rangle, \tag{3.2}$$

which is affine in z, and hence C_n is convex. So, $C_n \cap Q_n$ is a closed and convex subset of E for all $n \ge 0$. Next, we show that $\Omega \subset C_n$ for all $n \ge 0$. Indeed, let $u \in \Omega$ and $y_n = J_{r_n} x_n$ for all $n \ge 0$. Since J_{r_n} are relatively nonexpansive mappings, we have

$$\begin{aligned}
\phi(u, z_n) &= \phi\Big(u, J^{-1}(\beta_n J x_n + (1-\beta_n) J S y_n)\Big)\\
&= \|u\|^2 - 2\langle u, \beta_n J x_n + (1-\beta_n) J S y_n\rangle + \big\|\beta_n J x_n + (1-\beta_n) J S y_n\big\|^2\\
&\le \|u\|^2 - 2\beta_n \langle u, J x_n\rangle - 2(1-\beta_n)\langle u, J S y_n\rangle + \beta_n \|x_n\|^2 + (1-\beta_n)\big\|S y_n\big\|^2\\
&= \beta_n \phi(u, x_n) + (1-\beta_n)\phi(u, S y_n)\\
&\le \beta_n \phi(u, x_n) + (1-\beta_n)\phi(u, y_n)\\
&= \beta_n \phi(u, x_n) + (1-\beta_n)\phi(u, J_{r_n} x_n)\\
&\le \beta_n \phi(u, x_n) + (1-\beta_n)\phi(u, x_n)\\
&= \phi(u, x_n).
\end{aligned} \tag{3.3}$$

It follows that

$$\begin{aligned}
\phi(u,u_n) &= \phi\Big(u, J^{-1}(\alpha_n J x_n + (1-\alpha_n) JTz_n)\Big) \\
&= \|u\|^2 - 2\langle u, \alpha_n J x_n + (1-\alpha_n) JTz_n\rangle + \|\alpha_n J x_n + (1-\alpha_n) JTz_n\|^2 \\
&\le \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2(1-\alpha_n)\langle u, JTz_n\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Tz_n\|^2 \\
&= \alpha_n\phi(u,x_n) + (1-\alpha_n)\phi(u,Tz_n) \\
&\le \alpha_n\phi(u,x_n) + (1-\alpha_n)\phi(u,z_n) \\
&\le \alpha_n\phi(u,x_n) + (1-\alpha_n)\phi(u,x_n) \\
&= \phi(u,x_n).
\end{aligned} \tag{3.4}$$

So, $u \in C_n$ for all $n \ge 0$, which implies that $\Omega \subset C_n$. Next, we show that $\Omega \subset Q_n$ for all $n \ge 0$. We prove by induction. For $n = 0$, we have $\Omega \subset C = Q_0$. Assume that $\Omega \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 2.3 we have

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1}\rangle \ge 0, \quad \forall z \in C_n \cap Q_n. \tag{3.5}$$

As $\Omega \subset C_n \cap Q_n$ by the induction assumptions, we have

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1}\rangle \ge 0, \quad \forall z \in \Omega. \tag{3.6}$$

This together with definition of Q_{n+1} implies that $\Omega \subset Q_{n+1}$ and hence $\Omega \subset Q_n$ for all $n \ge 0$. So, we have that $\Omega \subset C_n \cap Q_n$ for all $n \ge 0$. This implies that $\{x_n\}$ is well defined. From definition of Q_n that $x_n = \Pi_{Q_n} x_0$ and $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n \cap Q_n \subset Q_n$, we have

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0), \quad \forall n \ge 0. \tag{3.7}$$

Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows from Lemma 2.4 and $x_n = \Pi_{Q_n} x_0$ that

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \le \phi(u, x_0) - \phi(u, \Pi_{Q_n} x_0) \le \phi(u, x_0) \tag{3.8}$$

for all $u \in \Omega \subset Q_n$. Therefore, $\{\phi(x_n, x_0)\}$ is bounded. Moreover, by definition of ϕ, we know that $\{x_n\}$ is bounded. So, we have $\{y_n\}$ and $\{z_n\}$ are bounded. So, the limit of $\{\phi(x_n, x_0)\}$ exists. From $x_n = \Pi_{Q_n} x_0$ and Lemma 2.4, we have

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{Q_n} x_0) \le \phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n} x_0, x_0) = \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \tag{3.9}$$

for all $n \geq 0$. This implies that $\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0$. From $x_{n+1} = \Pi_{C_n\cap Q_n}x_0 \in C_n$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n). \tag{3.10}$$

Therefore, we have $\lim_{n\to\infty}\phi(x_{n+1}, u_n) = 0$.

Since $\lim_{n\to\infty}\phi(x_{n+1}, x_n) = \lim_{n\to\infty}\phi(x_{n+1}, u_n) = 0$ and E is uniformly convex and smooth, we have from Lemma 2.1 that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|x_{n+1} - u_n\| = 0. \tag{3.11}$$

So, we have $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty}\|Jx_{n+1} - Ju_n\| = \lim_{n\to\infty}\|Jx_n - Ju_n\| = 0. \tag{3.12}$$

On the other hand, we have

$$\begin{aligned}\|Jx_{n+1} - Ju_n\| &= \|Jx_{n+1} - \alpha_n Jx_n - (1-\alpha_n)JTz_n\| \\ &= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1} - JTz_n)\| \\ &= \|(1-\alpha_n)(Jx_{n+1} - JTz_n) - \alpha_n(Jx_n - Jx_{n+1})\| \\ &\geq (1-\alpha_n)\|Jx_{n+1} - JTz_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|.\end{aligned} \tag{3.13}$$

This follows that

$$\|Jx_{n+1} - JTz_n\| \leq \frac{1}{1-\alpha_n}\left(\|Jx_{n+1} - Ju_n\| + \alpha_n\|Jx_n - Jx_{n+1}\|\right). \tag{3.14}$$

From (3.12) and $\liminf_{n\to\infty}(1-\alpha_n) > 0$, we obtain that $\lim_{n\to\infty}\|Jx_{n+1} - JTz_n\| = 0$.

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|x_{n+1} - Tz_n\| = 0. \tag{3.15}$$

From

$$\|x_n - Tz_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tz_n\|, \tag{3.16}$$

we have

$$\lim_{n\to\infty}\|x_n - Tz_n\| = 0. \tag{3.17}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, we also obtain that $\{Jx_n\}$ and $\{JSy_n\}$ are bounded. So, there exists $r > 0$ such that $\{Jx_n\}, \{JSy_n\} \subset B_r(0)$. Therefore Lemma 2.7 is applicable and we observe that

$$\begin{aligned}
\phi(u, z_n) &= \phi\Big(u, J^{-1}(\beta_n Jx_n + (1-\beta_n)JSy_n)\Big) \\
&= \|u\|^2 - 2\langle u, \beta_n Jx_n + (1-\beta_n)JSy_n\rangle + \|\beta_n Jx_n + (1-\beta_n)JSy_n\|^2 \\
&\leq \|u\|^2 - 2\beta_n\langle u, Jx_n\rangle - 2(1-\beta_n)\langle u, JSy_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|Sy_n\|^2 \\
&\quad - \beta_n(1-\beta_n)g(\|Jx_n - JSy_n\|) \\
&= \beta_n\phi(u, x_n) + (1-\beta_n)\phi(u, Sy_n) - \beta_n(1-\beta_n)g(\|Jx_n - JSy_n\|) \\
&= \beta_n\phi(u, x_n) + (1-\beta_n)\phi(u, SJ_{r_n}x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JSy_n\|) \\
&\leq \beta_n\phi(u, x_n) + (1-\beta_n)\phi(u, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JSy_n\|) \\
&= \phi(u, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JSy_n\|),
\end{aligned} \tag{3.18}$$

where $g : [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing, and convex function with $g(0) = 0$. That is

$$\beta_n(1-\beta_n)g(\|Jx_n - JSy_n\|) \leq \phi(u, x_n) - \phi(u, z_n). \tag{3.19}$$

Let $\{\|x_{n_k} - Sy_{n_k}\|\}$ be any subsequence of $\{\|x_n - Sy_n\|\}$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n'_j}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty} \phi\Big(u, x_{n'_j}\Big) = \limsup_{k\to\infty} \phi(u, x_{n_k}) = a, \tag{3.20}$$

where $u \in \Omega$. By (2) and (3), we have

$$\begin{aligned}
\phi\Big(u, x_{n'_j}\Big) &= \phi\Big(u, Tz_{n'_j}\Big) + \phi\Big(Tz_{n'_j}, x_{n'_j}\Big) + 2\langle u - Tz_{n'_j}, JTz_{n'_j} - Jx_{n'_j}\rangle \\
&\leq \phi\Big(u, z_{n'_j}\Big) + \Big\|Tz_{n'_j}\Big\|\Big\|JTz_{n'_j} - Jx_{n'_j}\Big\| + \Big\|Tz_{n'_j} - x_{n'_j}\Big\|\Big\|x_{n'_j}\Big\| \\
&\quad + 2\Big\|u - Tz_{n'_j}\Big\|\Big\|JTz_{n'_j} - Jx_{n'_j}\Big\|.
\end{aligned} \tag{3.21}$$

Since $\lim_{n\to\infty}\|x_n - Tz_n\| = 0$ and hence $\lim_{n\to\infty}\|Jx_n - JTz_n\| = 0$, it follows that

$$a = \liminf_{j\to\infty} \phi\left(u, x_{n'_j}\right) \leq \liminf_{j\to\infty} \phi\left(u, z_{n'_j}\right). \tag{3.22}$$

We also have from (3.3) that

$$\limsup_{j\to\infty} \phi\left(u, z_{n'_j}\right) \leq \limsup_{j\to\infty} \phi\left(u, x_{n'_j}\right) = a, \tag{3.23}$$

and hence

$$\lim_{j\to\infty} \phi\left(u, x_{n'_j}\right) = \lim_{j\to\infty} \phi\left(u, z_{n'_j}\right) = a. \tag{3.24}$$

Since $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, it follows from (3.19) that $\lim_{j\to\infty} g(\|Jx_{n'_j} - JSy_{n'_j}\|) = 0$. By properties of the function g, we have $\lim_{j\to\infty}\|Jx_{n'_j} - JSy_{n'_j}\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain $\lim_{j\to\infty}\|x_{n'_j} - Sy_{n'_j}\| = 0$ and then

$$\lim_{n\to\infty} \left\|x_n - Sy_n\right\| = 0. \tag{3.25}$$

So, we have $\lim_{n\to\infty}\|Jx_n - JSy_n\| = 0$. Since

$$\begin{aligned} \|Jz_n - Jx_n\| &= \left\|\beta_n Jx_n + (1-\beta_n)JSy_n - Jx_n\right\| \\ &= (1-\beta_n)\left\|JSy_n - Jx_n\right\| \leq \left\|JSy_n - Jx_n\right\|, \end{aligned} \tag{3.26}$$

it follows that $\lim_{n\to\infty}\|Jz_n - Jx_n\| = 0$, and hence

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.27}$$

From (3.3), we have

$$\frac{1}{1-\beta_n}\left(\phi(u, z_n) - \beta_n\phi(u, x_n)\right) \leq \phi(u, y_n). \tag{3.28}$$

Using $y_n = J_{r_n}x_n$ and Lemma 2.6, we have

$$\phi(y_n, x_n) = \phi(J_{r_n}x_n, x_n) \leq \phi(u, x_n) - \phi(u, J_{r_n}x_n) = \phi(u, x_n) - \phi(u, y_n). \tag{3.29}$$

It follows that

$$\begin{aligned}
\phi(y_n, x_n) &\le \phi(u, x_n) - \phi(u, y_n) \\
&\le \phi(u, x_n) - \frac{1}{1-\beta_n}(\phi(u, z_n) - \beta_n \phi(u, x_n)) \\
&= \frac{1}{1-\beta_n}(\phi(u, x_n) - \phi(u, z_n)) \\
&= \frac{1}{1-\beta_n}\left(\|x_n\|^2 - \|z_n\|^2 - 2\langle u, Jx_n - Jz_n\rangle\right) \\
&\le \frac{1}{1-\beta_n}\left(\left|\|x_n\|^2 - \|z_n\|^2\right| + 2|\langle u, Jx_n - Jz_n\rangle|\right) \\
&\le \frac{1}{1-\beta_n}(|\|x_n\| - \|z_n\||(\|x_n\| + \|z_n\|) + 2\|u\|\|Jx_n - Jz_n\|) \\
&\le \frac{1}{1-\beta_n}(\|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|u\|\|Jx_n - Jz_n\|).
\end{aligned} \tag{3.30}$$

Since $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, we have that $\liminf_{n\to\infty}(1-\beta_n) > 0$. So, we have $\lim_{n\to\infty}\phi(y_n, x_n) = 0$. Since E is uniformly convex and smooth, we have from Lemma 2.1 that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.31}$$

Since

$$\begin{aligned}
\|z_n - Tz_n\| &\le \|z_n - x_n\| + \|x_n - Tz_n\|, \\
\|y_n - Sy_n\| &\le \|y_n - x_n\| + \|x_n - Sy_n\|,
\end{aligned} \tag{3.32}$$

from (3.17), (3.25), (3.27), and (3.31), we obtain that

$$\lim_{n\to\infty}\|z_n - Tz_n\| = \lim_{n\to\infty}\|y_n - Sy_n\| = 0. \tag{3.33}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v$. From $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and $\lim_{n\to\infty}\|x_n - z_n\| = 0$, we have $y_{n_k} \rightharpoonup v$ and $z_{n_k} \rightharpoonup v$. Since S and T are relatively nonexpansive, we have that $v \in \widehat{F}(S) \cap \widehat{F}(T) = F(S) \cap F(T)$. Next, we show $v \in A^{-1}0$. Since J is uniformly norm-to-norm continuous on bounded sets, from (3.31) we have

$$\lim_{n\to\infty}\|Jx_n - Jy_n\| = 0. \tag{3.34}$$

From $r_n \geq a$, we have

$$\lim_{n\to\infty} \frac{1}{r_n} \| Jx_n - Jy_n \| = 0. \tag{3.35}$$

Therefore, we have

$$\lim_{n\to\infty} \| A_{r_n} x_n \| = \lim_{n\to\infty} \frac{1}{r_n} \| Jx_n - Jy_n \| = 0. \tag{3.36}$$

For $(p, p^*) \in A$, from the monotonicity of A, we have $\langle p - y_n, p^* - A_{r_n} x_n \rangle \geq 0$ for all $n \geq 0$. Replacing n by n_k and letting $k \to \infty$, we get $\langle p - v, p^* \rangle \geq 0$. From the maximallity of A, we have $v \in A^{-1}0$, that is, $v \in \Omega$.

Finally, we show that $x_n \to \Pi_\Omega x_0$. Let $w = \Pi_\Omega x_0$. From $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$ and $w \in \Omega \subset C_n \cap Q_n$, we obtain that

$$\phi(x_{n+1}, x_0) \leq \phi(w, x_0). \tag{3.37}$$

Since the norm is weakly lower semicontinuous, we have

$$\begin{aligned}
\phi(v, x_0) &= \|v\|^2 - 2\langle v, Jx_0 \rangle + \|x_0\|^2 \\
&\leq \liminf_{k\to\infty} \left(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0 \rangle + \|x_0\|^2 \right) \\
&= \liminf_{k\to\infty} \phi(x_{n_k}, x_0) \leq \limsup_{k\to\infty} \phi(x_{n_k}, x_0) \leq \phi(w, x_0).
\end{aligned} \tag{3.38}$$

From the definition of Π_Ω, we obtain $v = w$. This implies that

$$\lim_{k\to\infty} \phi(x_{n_k}, x_0) = \phi(w, x_0). \tag{3.39}$$

Therefore we have

$$\begin{aligned}
0 &= \lim_{k\to\infty} \left(\phi(x_{n_k}, x_0) - \phi(w, x_0) \right) \\
&= \lim_{k\to\infty} \left(\|x_{n_k}\|^2 - \|w\|^2 - 2\langle x_{n_k} - w, Jx_0 \rangle \right) \\
&= \lim_{k\to\infty} \left(\|x_{n_k}\|^2 - \|w\|^2 \right).
\end{aligned} \tag{3.40}$$

Since E has the Kadec-Klee property, we obtain that $x_{n_k} \to w = \Pi_\Omega x_0$. Since $\{x_{n_k}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$. This completes the proof. □

As direct consequences of Theorem 3.1, we can obtain the following corollaries.

Corollary 3.2. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let T be a relatively nonexpansive mapping from C into itself such that $\Omega = F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and*

$$\begin{aligned}
u_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) JTz_n),\\
z_n &= J^{-1}(\beta_n J x_n + (1-\beta_n) JTJ_{r_n} x_n),\\
C_n &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\},\\
Q_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\},\\
x_{n+1} &= \Pi_{C_n \cap Q_n} x_0
\end{aligned} \tag{3.41}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$, and $\{r_n\} \subset [a,\infty)$ for some $a > 0$. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$ and $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection of E onto Ω.

Proof. Putting $S = T$ in Theorem 3.1, we obtain Corollary 3.2. □

Corollary 3.3. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $S : C \to C$ be a relatively nonexpansive mapping such that $\Omega = F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and*

$$\begin{aligned}
u_n &= J^{-1}(\beta_n J x_n + (1-\beta_n) JSJ_{r_n} x_n),\\
C_n &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\},\\
Q_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\},\\
x_{n+1} &= \Pi_{C_n \cap Q_n} x_0
\end{aligned} \tag{3.42}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\beta_n\} \subset [0,1]$, and $\{r_n\} \subset [a,\infty)$ for some $a > 0$. If $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection of E onto Ω.

Proof. Putting $T = I$ and $\alpha_n = 0$ in Theorem 3.1, we obtain Corollary 3.3. □

Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Define the subdifferential of f as follows:

$$\partial f(x) = \{x^* \in E : f(y) \geq \langle y - x, x^*\rangle + f(x),\ \forall y \in E\} \tag{3.43}$$

for each $x \in E$. Then, we know that ∂f is a maximal monotone operator; see [18] for more details.

Corollary 3.4. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let S and T be relatively nonexpansive mappings from C into itself such that $\Omega = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and*

$$\begin{aligned}
u_n &= J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTz_n),\\
z_n &= J^{-1}(\beta_n Jx_n + (1-\beta_n)JSx_n),\\
C_n &= \{z \in C : \phi(z, u_n) \le \phi(z, x_n)\},\\
Q_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\},\\
x_{n+1} &= \Pi_{C_n \cap Q_n} x_0
\end{aligned} \tag{3.44}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$ and $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection of E onto Ω.

Proof. Set $A = \partial i_C$ in Theorem 3.1, where i_C is the indicator function; that is,

$$i_C(x)\begin{cases} 0, & x \in C,\\ \infty, & \text{otherwise.}\end{cases} \tag{3.45}$$

Then, we have that A is a maximal monotone operator and $J_r = \Pi_C$ for $r > 0$. In fact, for any $x \in E$ and $r > 0$, we have from Lemma 2.3 that

$$\begin{aligned}
z = J_r x &\Longleftrightarrow Jz + r\partial i_C(z) \ni Jx\\
&\Longleftrightarrow Jx - Jz \in r\partial i_C(z)\\
&\Longleftrightarrow i_C(y) \ge \left\langle y - z, \frac{Jx - Jz}{r}\right\rangle + i_C(z), \quad \forall y \in E\\
&\Longleftrightarrow 0 \ge \langle y - z, Jx - Jz\rangle, \quad \forall y \in C\\
&\Longleftrightarrow z = \arg\min_{y\in C} \phi(y, x)\\
&\Longleftrightarrow z = \Pi_C x.
\end{aligned} \tag{3.46}$$

So, from Theorem 3.1, we obtain Corollary 3.4. □

4. Applications

In this section, we discuss the problem of strong convergence concerning a maximal monotone operator and two nonexpansive mappings in a Hilbert space. Using Theorem 3.1, we obtain the following results.

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $A \subset H \times H$ be a monotone operator satisfying $D(A) \subset C$ and let $J_r = (I + rA)^{-1}$ for all $r > 0$. Let S and T be nonexpansive mappings from C into itself such that $\Omega = F(S) \cap F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and*

$$\begin{aligned}
u_n &= \alpha_n x_n + (1 - \alpha_n) T z_n, \\
z_n &= \beta_n x_n + (1 - \beta_n) S J_{r_n} x_n, \\
C_n &= \{z \in C : \|z - u_n\| \leq \|z - x_n\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0
\end{aligned} \tag{4.1}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\liminf_{n\to\infty}(1 - \alpha_n) > 0$ and $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, then $\{x_n\}$ converges strongly to $P_\Omega x_0$, where P_Ω is the metric projection of H onto Ω.

Proof. We know that every nonexpansive mapping with a fixed point is a relatively nonexpansive one. We also know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Using Theorem 3.1, we are easily able to obtain the desired conclusion by putting $J = I$. This completes the proof. □

The following corollary follows from Theorem 4.1.

Corollary 4.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $A \subset H \times H$ be a monotone operator satisfying $D(A) \subset C$ and let $J_r = (I + rA)^{-1}$ for all $r > 0$. Let T be a nonexpansive mapping from C into itself such that $\Omega = F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and*

$$\begin{aligned}
u_n &= \alpha_n x_n + (1 - \alpha_n) T z_n, \\
z_n &= \beta_n x_n + (1 - \beta_n) T J_{r_n} x_n, \\
C_n &= \{z \in C : \|z - u_n\| \leq \|z - x_n\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0
\end{aligned} \tag{4.2}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\liminf_{n\to\infty}(1 - \alpha_n) > 0$ and $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, then $\{x_n\}$ converges strongly to $P_\Omega x_0$, where P_Ω is the metric projection of H onto Ω.

Proof. Putting $S = T$ in Theorem 4.1, we obtain Corollary 4.2. □

Corollary 4.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $A \subset H \times H$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (I + rA)^{-1}$ for all $r > 0$. Let S be* *a nonexpansive mapping from C into itself such that* $\Omega = F(S) \cap A^{-1}0 \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in C$ *and*

$$
\begin{aligned}
u_n &= \beta_n x_n + (1-\beta_n) S J_{r_n} x_n, \\
C_n &= \{z \in C : \|z - u_n\| \leq \|z - x_n\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0
\end{aligned}
\tag{4.3}
$$

for all $n \in \mathbb{N} \cup \{0\}$, *where* $\{\beta_n\} \subset [0,1]$ *and* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$. *If* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ *then* $\{x_n\}$ *converges strongly to* $P_\Omega x_0$, *where* P_Ω *is the metric projection of* H *onto* Ω.

Proof. Putting $T = I$ and $\alpha_n = 0$ in Theorem 4.1, we obtain Corollary 4.3. □

Corollary 4.4. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* S *and* T *be nonexpansive mappings from* C *into itself such that* $\Omega = F(S) \cap F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 = x \in C$ *and*

$$
\begin{aligned}
u_n &= \alpha_n x_n + (1-\alpha_n) T z_n, \\
z_n &= \beta_n x_n + (1-\beta_n) S x_n, \\
C_n &= \{z \in C : \|z - u_n\| \leq \|z - x_n\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0
\end{aligned}
\tag{4.4}
$$

for all $n \in \mathbb{N} \cup \{0\}$, *where* $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. *If* $\liminf_{n\to\infty}(1-\alpha_n) > 0$ *and* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, *then* $\{x_n\}$ *converges strongly to* $P_\Omega x_0$, *where* P_Ω *is the metric projection of* H *onto* Ω.

Proof. Set $A = \partial i_C$ in Theorem 4.1, where i_C is the indicator function; that is,

$$
i_C(x) \begin{cases} 0, & x \in C, \\ \infty, & \text{otherwise.} \end{cases}
\tag{4.5}
$$

Then, we have that A is a maximal monotone operator and $J_r = P_C$ for $r > 0$. In fact, for any $x \in E$ and $r > 0$, we have that

$$
\begin{aligned}
z = J_r x &\Longleftrightarrow z + r\partial i_C(z) \ni x \\
&\Longleftrightarrow x - z \in r\partial i_C(z) \\
&\Longleftrightarrow i_C(y) \geq \left\langle y - z, \frac{x-z}{r} \right\rangle + i_C(z), \quad \forall y \in E \\
&\Longleftrightarrow 0 \geq \langle y - z, x - z \rangle, \quad \forall y \in C \\
&\Longleftrightarrow z = P_C x.
\end{aligned}
\tag{4.6}
$$

So, from Theorem 4.1, we obtain Corollary 4.4. □

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