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Research Article

Some New Wilker-Type Inequalities for Circular and Hyperbolic Functions

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In this paper, we give some new Wilker-type inequalities for circular and hyperbolic functions in exponential form by using generalizations of Cusa-Huygens inequality and Cusa-Huygens-type inequality.

1. Introduction

Wilker [1] proposed two open questions, the first of which was the following statement.

Problem 1. Let $0 < x < \pi/2$. Then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \tag{1.1}$$

holds.

Sumner et al. [2] proved inequality (1.1). Guo et al. [3] gave a new proof of inequality (1.1). Zhu [4, 5] showed two new simple proofs of Wilker's inequality above, respectively.

Recently, Wu and Srivastava [6] obtained Wilker-type inequality as follows:

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2, \quad 0 < x < \frac{\pi}{2}. \tag{1.2}$$

Baricz and Sandor [7] found that inequality (1.2) can be proved by using inequality (1.1).

On the other hand, in the form of inequality (1.1), Zhu [5] obtained the following Wilker type inequality:

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}>2,\quad x>0. \tag{1.3}$$

In fact, we can obtain further results:

$$\begin{gathered}\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}>2,\quad 0<x<\frac{\pi}{2},\\ \left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}>\left(\frac{x}{\sinh x}\right)^2+\frac{x}{\tanh x}>2,\quad x>0.\end{gathered} \tag{1.4}$$

In this note, we establish the following four new Wilker type inequalities in exponential form for circular and hyperbolic functions.

Theorem 1.1. *Let* $0<x<\pi/2$, $\alpha\in\mathbb{R}$ *and* $\alpha\neq 0$. *Then*

(i) *when* $\alpha>0$, *the inequality*

$$\left(\frac{\sin x}{x}\right)^{2\alpha}+\left(\frac{\tan x}{x}\right)^{\alpha}>\left(\frac{x}{\sin x}\right)^{2\alpha}+\left(\frac{x}{\tan x}\right)^{\alpha} \tag{1.5}$$

holds;

(ii) *when* $\alpha<0$, *inequality* (1.5) *is revered.*

Theorem 1.2. *Let* $0<x<\pi/2$ *and* $\alpha\geq 1$. *Then the inequality*

$$\left(\frac{\sin x}{x}\right)^{2\alpha}+\left(\frac{\tan x}{x}\right)^{\alpha}>\left(\frac{x}{\sin x}\right)^{2\alpha}+\left(\frac{x}{\tan x}\right)^{\alpha}>2 \tag{1.6}$$

holds.

Theorem 1.3. *Let* $x>0$, $\alpha\in\mathbb{R}$ *and* $\alpha\neq 0$. *Then*

(i) *when* $\alpha>0$, *the inequality*

$$\left(\frac{\sinh x}{x}\right)^{2\alpha}+\left(\frac{\tanh x}{x}\right)^{\alpha}>\left(\frac{x}{\sinh x}\right)^{2\alpha}+\left(\frac{x}{\tanh x}\right)^{\alpha} \tag{1.7}$$

holds;

(ii) *when* $\alpha<0$, *inequality* (1.7) *is revered.*

Theorem 1.4. *Let $x > 0$ and $\alpha \geq 1$. Then the inequality*

$$\left(\frac{\sinh x}{x}\right)^{2\alpha} + \left(\frac{\tanh x}{x}\right)^{\alpha} > \left(\frac{x}{\sinh x}\right)^{2\alpha} + \left(\frac{x}{\tanh x}\right)^{\alpha} > 2 \tag{1.8}$$

holds.

2. Lemmas

Lemma 2.1 (see [8–24]). *Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions which are differentiable on (a, b). Further, let $g' \neq 0$ on (a, b). If f'/g' is increasing (or decreasing) on (a, b), then the functions $(f(x) - f(b))/(g(x) - g(b))$ and $(f(x) - f(a))/(g(x) - g(a))$ are also increasing (or decreasing) on (a, b).*

Lemma 2.2 (see [25–27]). *Let a_n and b_n $(n = 0, 1, 2, \ldots)$ be real numbers, and let the power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$ be convergent for $|t| < R$. If $b_n > 0$ for $n = 0, 1, 2, \ldots$, and if a_n/b_n is strictly increasing (or decreasing) for $n = 0, 1, 2, \ldots$, then the function $A(t)/B(t)$ is strictly increasing (or decreasing) on $(0, R)$.*

Lemma 2.3 (see [28, 29]). *Let $|x| < \pi$, then the inequality*

$$\frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n} \tag{2.1}$$

holds.

Lemma 2.4. *Let $|x| < \pi$, then the inequality*

$$\frac{1}{\sin^2 x} = csc^2 x = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| (2n - 1) x^{2n-2} \tag{2.2}$$

holds.

Proof. The following power series expansion can be found in [30, 1.3.1.4 (3)]

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1}, \quad |x| < \pi. \tag{2.3}$$

Then

$$\frac{1}{\sin^2 x} = \csc^2 x = -(\cot x)' = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| (2n - 1) x^{2n-2}, \quad |x| < \pi. \tag{2.4}$$

□

Lemma 2.5 (see [5, 31]). *Let $0 < x < \pi/2$. Then the inequality*

$$\left(\frac{\sin x}{x}\right)^3 > \cos x \tag{2.5}$$

holds.

Lemma 2.6 (see [5, 31, 32]). *Let $x > 0$. Then the inequality*

$$\left(\frac{\sinh x}{x}\right)^3 > \cosh x \tag{2.6}$$

holds.

Lemma 2.7. *Let $0 < x < \pi/2$. Then the function $G(\alpha) = ((\sin x/x)^{2\alpha} + (\tan x/x)^{\alpha})/((x/\sin x)^{2\alpha} + (x/\tan x)^{\alpha})$ increases as α increases on $(-\infty, +\infty)$.*

Lemma 2.8. *Let $x > 0$. Then the function $H(\alpha) = ((\sinh x/x)^{2\alpha} + (tanh\, x/x)^{\alpha})/((x/\sinh x)^{2\alpha} + (x/tanh\, x)^{\alpha})$ increases as α increases on $(-\infty, +\infty)$.*

Lemma 2.9 (a generalization of Cusa-Huygens inequality). *Let $0 < x < \pi/2$ and $\alpha \geq 1$. Then the inequality*

$$2\left(\frac{x}{\sin x}\right)^{\alpha} + \left(\frac{x}{\tan x}\right)^{\alpha} > 3 \tag{2.7}$$

or

$$\left(\frac{\sin x}{x}\right)^{\alpha} < \frac{2}{3} + \frac{1}{3}(\cos x)^{\alpha} \tag{2.8}$$

holds.

Lemma 2.10 (a generalization of Cusa-Huygens type inequality). *Let $x > 0$ and $\alpha \geq 1$. Then the inequality*

$$2\left(\frac{x}{\sinh x}\right)^{\alpha} + \left(\frac{x}{tanh\, x}\right)^{\alpha} > 3 \tag{2.9}$$

or

$$\left(\frac{\sinh x}{x}\right)^{\alpha} < \frac{2}{3} + \frac{1}{3}(\cosh x)^{\alpha} \tag{2.10}$$

holds.

3. Proofs of Lemma 2.7 and Theorem 1.1

Proof of Lemma 2.7. Direct calculation yields $G'(\alpha) = J(\alpha)/[(x/\sin x)^{2\alpha} + (x/\tan x)^{\alpha}]^2$, where

$$\begin{aligned}
J(\alpha) &= \left[\left(\frac{\tan x}{x}\right)^{\alpha}\left(\frac{x}{\sin x}\right)^{2\alpha} - \left(\frac{x}{\tan x}\right)^{\alpha}\left(\frac{\sin x}{x}\right)^{2\alpha} + 2\right]\log\frac{\tan x}{x} \\
&\quad + 2\left[\left(\frac{\tan x}{x}\right)^{\alpha}\left(\frac{x}{\sin x}\right)^{2\alpha} - \left(\frac{x}{\tan x}\right)^{\alpha}\left(\frac{\sin x}{x}\right)^{2\alpha} - 2\right]\log\frac{x}{\sin x} \\
&= \left[\left(\frac{2x}{\sin 2x}\right)^{\alpha} - \left(\frac{\sin 2x}{2x}\right)^{\alpha} + 2\right]\log\frac{\tan x}{x} + 2\left[\left(\frac{2x}{\sin 2x}\right)^{\alpha} - \left(\frac{\sin 2x}{2x}\right)^{\alpha} - 2\right]\log\frac{x}{\sin x} \\
&= \log\left[\left(\frac{\tan x}{x}\right)^{(2x/\sin 2x)^{\alpha}-(\sin 2x/2x)^{\alpha}+2}\left(\frac{x^2}{\sin^2 x}\right)^{(2x/\sin 2x)^{\alpha}-(\sin 2x/2x)^{\alpha}-2}\right] \\
&= \log\left[\left(\frac{2x}{\sin 2x}\right)^{(2x/\sin 2x)^{\alpha}-(\sin 2x/2x)^{\alpha}}\left(\left(\frac{\sin x}{x}\right)^{3}\frac{1}{\cos x}\right)^{2}\right].
\end{aligned} \tag{3.1}$$

First, we have $[(\sin x/x)^3(1/\cos x)]^2 > 1$ by Lemma 2.5. Second, when letting $2x/\sin 2x = t$ for $0 < x < \pi/2$, we have $t > 1$, and $t^{\alpha} - t^{-\alpha} > 0$ for $\alpha > 0$, so $t^{t^{\alpha}-t^{-\alpha}} > 1$ and $(2x/\sin 2x)^{(2x/\sin 2x)^{\alpha}-(\sin 2x/2x)^{\alpha}}[(\sin x/x)^3(1/\cos x)]^2 > 1$. Thus $J(\alpha) > 0$ and $G'(\alpha) > 0$. The proof of Lemma 2.7 is complete. □

Proof of Theorem 1.1. From Lemma 2.7 we have $G(\alpha) > G(0) = 1$ for $\alpha > 0$. That is, (1.5) holds. At the same time, we have $G(\alpha) < G(0) = 1$ for $\alpha < 0$. That is, (1.5) is revered. □

4. Proofs of Lemma 2.9 and Theorem 1.2

Proof of Lemma 2.9. Let $F(x) = ((\sin x/x)^{\alpha} - 1)/((\cos x)^{\alpha} - 1) =: f(x)/g(x)$, where $f(x) = (\sin x/x)^{\alpha} - 1$, and $g(x) = (\cos x)^{\alpha} - 1$. Then

$$k(x) = \frac{f'(x)}{g'(x)} = \left(\frac{\sin x}{x\cos x}\right)^{\alpha-1}\frac{\sin x - x\cos x}{x^2\sin x}, \qquad k'(x) = \left(\frac{\sin x}{x\cos x}\right)^{\alpha-2}\frac{u(x)}{x^4\sin x\cos^2 x}, \tag{4.1}$$

where

$$\begin{aligned}
u(x) &= (\alpha-1)(x-\sin x\cos x)(\sin x - x\cos x) + \cos x\left(x^2 - 2\sin^2 x + x\sin x\cos x\right) \\
&= \left(x\sin x - \sin^2 x\cos x - x^2\cos x + x\cos^2 x\sin x\right)\alpha \\
&\quad - \left(x\sin x + \sin^2 x\cos x - 2x^2\cos x\right) \\
&= \left(x\sin x - \sin^2 x\cos x - x^2\cos x + x\cos^2 x\sin x\right)(\alpha - G(x)),
\end{aligned} \tag{4.2}$$

where $G(x) = (x\sin x + \sin^2 x\cos x - 2x^2\cos x)/(x\sin x - \sin^2 x\cos x - x^2\cos x + x\cos^2 x\sin x)$. Then

$$G(x) = \frac{2x/\sin 2x + 1 - 2x^2/\sin^2 x}{2x/\sin 2x - 1 - (x/\sin x)^2 + x\cot x} := \frac{A(x)}{B(x)}, \tag{4.3}$$

where $A(x) = 2x/\sin 2x + 1 - (2x^2/\sin^2 x)$, and $B(x) = 2x/\sin 2x - 1 - (x/\sin x)^2 + x\cot x$. By (2.1), (2.2), and (2.3), we have

$$\begin{aligned} A(x) &= 1 + \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|(2x)^{2n} + 1 - 2\left(1 + \sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|(2n-1)x^{2n}\right) \\ &= \sum_{n=1}^{\infty}\frac{(2^{2n}-4n)2^{2n}}{(2n)!}|B_{2n}|x^{2n} = \sum_{n=2}^{\infty}\frac{(2^{2n}-4n)2^{2n}}{(2n)!}|B_{2n}|x^{2n} =: \sum_{n=2}^{\infty} a_n x^{2n}, \\ B(x) &= 1 + \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|(2x)^{2n} - 1 - \left(1 + \sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|(2n-1)x^{2n}\right) + 1 - \sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n} \\ &= \sum_{n=1}^{\infty}\frac{(2^{2n}-2n-2)2^{2n}}{(2n)!}|B_{2n}|x^{2n} = \sum_{n=2}^{\infty}\frac{(2^{2n}-2n-2)2^{2n}}{(2n)!}|B_{2n}|x^{2n} =: \sum_{n=2}^{\infty} b_n x^{2n}, \end{aligned} \tag{4.4}$$

where $a_n = ((2^{2n}-4n)2^{2n}/(2n)!)|B_{2n}|$ and $b_n = ((2^{2n}-2n-2)2^{2n}/(2n)!)|B_{2n}| > 0$.

When setting $c_n = a_n/b_n$, we have that $c_n = (2^{2n}-4n)/(2^{2n}-2n-2)$ is increasing for $n = 2, 3, \ldots$, $A(x)/B(x)$ is increasing from $(0, \pi/2)$ onto $(4/5, 1)$ by Lemma 2.2. When $\alpha \geq 1$, we have $u(x) \geq 0$. So $k(x)$ is increasing on $(0, \pi/2)$. This leads to that $f'(x)/g'(x)$ is increasing on $(0, \pi/2)$. Thus $F(x) = f(x)/g(x) = (f(x) - f(0^+))/(g(x) - g(0^+))$ is increasing on $(0, \pi/2)$ by Lemma 2.1. At the same time, $\lim_{x\to 0^+} F(x) = 1/3$. So the proof of Lemma 2.9 is complete. □

Proof of Theorem 1.2. From Theorem 1.1, when $\alpha \geq 1$ we have

$$\left(\frac{\sin x}{x}\right)^{2\alpha} + \left(\frac{\tan x}{x}\right)^{\alpha} > \left(\frac{x}{\sin x}\right)^{2\alpha} + \left(\frac{x}{\tan x}\right)^{\alpha}. \tag{4.5}$$

On the other hand, when $\alpha \geq 1$ we can obtain

$$1 + \left(\frac{x}{\sin x}\right)^{2\alpha} + \left(\frac{x}{\tan x}\right)^{\alpha} \geq 2\left(\frac{x}{\sin x}\right)^{\alpha} + \left(\frac{x}{\tan x}\right)^{\alpha} > 3 \tag{4.6}$$

by the arithmetic mean-geometric mean inequality and Lemma 2.9. So

$$\left(\frac{x}{\sin x}\right)^{2\alpha} + \left(\frac{x}{\tan x}\right)^{\alpha} > 2 \tag{4.7}$$

holds.

Combining (4.5) and (4.7) gives (1.6). □

5. Proofs of Lemma 2.8 and Theorem 1.3

Proof of Lemma 2.8. Direct calculation yields $H'(\alpha) = I(\alpha)/[(x/\sinh x)^{2\alpha} + (x/\tanh x)^{\alpha}]^2$, where

$$\begin{aligned} I(\alpha) &= \left[\left(\frac{\tanh x}{x}\right)^{\alpha}\left(\frac{x}{\sinh x}\right)^{2\alpha} + \left(\frac{x}{\tanh x}\right)^{\alpha}\left(\frac{\sinh x}{x}\right)^{2\alpha} + 2\right]\log\frac{\tanh x}{x} \\ &\quad + 2\left[\left(\frac{\tanh x}{x}\right)^{\alpha}\left(\frac{x}{\sinh x}\right)^{2\alpha} + \left(\frac{x}{\tanh x}\right)^{\alpha}\left(\frac{\sinh x}{x}\right)^{2\alpha} + 2\right]\log\frac{\sinh x}{x} \\ &= \left[\left(\frac{\tanh x}{x}\right)^{\alpha}\left(\frac{x}{\sinh x}\right)^{2\alpha} + \left(\frac{x}{\tanh x}\right)^{\alpha}\left(\frac{\sinh x}{x}\right)^{2\alpha} + 2\right]\log\left[\left(\frac{\sinh x}{x}\right)^{3}\frac{1}{\cosh x}\right]. \end{aligned} \tag{5.1}$$

First, $(\tanh x/x)^{\alpha}(x/\sinh x)^{2\alpha} + (x/\tanh x)^{\alpha}(\sinh x/x)^{2\alpha} + 2 > 0$ for $x > 0$. Second, we have $\log[(\sinh x/x)^3(1/\cosh x)] > 0$ by Lemma 2.6. Thus $I(\alpha) > 0$ and $H'(\alpha) > 0$. The proof of Lemma 2.8 is complete. □

Proof of Theorem 1.3. From Lemma 2.8 we have $H(\alpha) > H(0) = 1$ for $\alpha > 0$. That is, (1.7) holds. At the same time, we have $H(\alpha) < H(0) = 1$ for $\alpha < 0$. That is, (1.7) is revered. □

6. Proofs of Lemma 2.10 and Theorem 1.4

Proof of Lemma 2.10. Let $Q(x) = ((\sinh x/x)^{\alpha} - 1)/((\cosh x)^{\alpha} - 1) =: f(x)/g(x)$, where $f(x) = (\sinh x/x)^{\alpha} - 1$, and $g(x) = (\cosh x)^{\alpha} - 1$. Then

$$k(x) =: \frac{f'(x)}{g'(x)} = \left(\frac{\sinh x}{t\cosh x}\right)^{\alpha-1}\frac{x\cosh x - \sinh x}{x^2\sinh x} =: \left(\frac{\sinh x}{t\cosh x}\right)^{\alpha-1}\frac{A(x)}{B(x)}, \tag{6.1}$$

where $A(x) = x\cosh x - \sinh x$ and $B(x) = x^2\sinh x$. Since

$$\begin{aligned} A(x) &= x\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} - \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{(2n)x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty}\frac{(2n+2)x^{2n+3}}{(2n+3)!} =: \sum_{n=0}^{\infty}a_n x^{2n+3}, \\ B(x) &= \sum_{n=0}^{\infty}\frac{x^{2n+3}}{(2n+1)!} =: \sum_{n=0}^{\infty}b_n x^{2n+3}, \end{aligned} \tag{6.2}$$

where $a_n = (2n+2)/(2n+3)!$ and $b_n = 1/(2n+1)!$.

When setting $c_n = a_n/b_n$, we have $c_n = 1/(2n+3)$ is decreasing for $n = 0, 1, 2, \ldots,$ $A(x)/B(x)$ is decreasing on $(0, +\infty)$ by Lemma 2.2. At the same time, the function $(\tanh x/x)^{\alpha-1}$ is decreasing on $(0, +\infty)$ when $\alpha \geq 1$. By (6.1), we obtain that $k(x) = f'(x)/g'(x)$ is decreasing on $(0, +\infty)$. Thus $Q(x) = f(x)/g(x) = (f(x)-f(0^+))/(g(x)-g(0^+))$ is decreasing on $(0, +\infty)$ by Lemma 2.1. At the same time, $\lim_{x\to 0^+}Q(x) = 1/3$. So the proof of Lemma 2.10 is complete. □

Proof of Theorem 1.4. By the same way as Theorem 1.2, we can prove Theorem 1.4. □

7. Open Problem

In this section, we pose the following open problem: find the respective largest range of α such that the inequalities (1.6) and (1.8) hold.

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