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*Research Article*

# Noncoherence of a Causal Wiener Algebra Used in Control Theory

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Let $\mathbb{C}_{\geq 0} := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$, and let $\mathcal{W}^+$ denote the ring of all functions $f : \mathbb{C}_{\geq 0} \to \mathbb{C}$ such that $f(s) = \hat{f_a}(s) + \sum_{k=0}^{\infty} f_k e^{-st_k}$ $(s \in \mathbb{C}_{\geq 0})$, where $f_a \in L^1(0,\infty)$, $(f_k)_{k\geq 0} \in \ell^1$, and $0 = t_0 < t_1 < t_2 < \cdots$ equipped with pointwise operations. (Here $\hat{\cdot}$ denotes the Laplace transform.) It is shown that the ring $\mathcal{W}^+$ is not coherent, answering a question of Alban Quadrat. In fact, we present two principal ideals in the domain $\mathcal{W}^+$ whose intersection is not finitely generated.



## 1. Introduction

The aim of this paper is to show that the ring $\mathcal{W}^+$ (defined below) is notcoherent.

We first recall the notion of a coherent ring.

*Definition 1.1.* Let $R$ be a commutative ring with identity element 1, and let $R^m = R \times \cdots \times R$ ($m$ times). Suppose that $f = (f_1, \ldots, f_m) \in R^m$.

(1) An element $(g_1, \ldots, g_m) \in R^m$ is called a *relation on* $f$ if

$$g_1 f_1 + \cdots + g_m f_m = 0. \tag{1.1}$$

(2) Let $f^\perp$ denote the set of all relations on $f \in R^m$. (Then $f^\perp$ is an $R$-submodule of the $R$-module $R^m$.)

(3) The ring $R$ is called *coherent* if for all $m \in \mathbb{N}$ and all $f \in R^m$, $f^\perp$ is finitely generated, that is, there exists a $d \in \mathbb{N}$ and there exist $g_j \in f^\perp$, $j \in \{1, \ldots, d\}$, such that for all $g \in f^\perp$, there exist $r_j \in R$, $j \in \{1, \ldots, d\}$ such that $g = r_1 g_1 + \cdots + r_d g_d$.

An integral domain is coherent if and only if the intersection of any two finitely generated ideals of the ring is again finitely generated; see [1, Theorem 2.3.2, page 45].

The coherence of some rings of analytic functions has been investigated in earlier works. For example, McVoy and Rubel [2] showed that the Hardy algebra $H^\infty(\mathbb{D})$ is coherent, while the disc algebra $A(\mathbb{D})$ is not. Mortini and von Renteln proved that the Wiener algebra $W^+(\mathbb{D})$ (of all absolutely convergent Taylor series in the open unit disc) is not coherent [3]. In this article, we will show that the ring $\mathcal{W}^+$ (defined below, and which is useful in control theory) is not coherent.

*Notation 1.* Throughout the article, we will use the following notation:

$$\mathbb{C}_{\geq 0} := \{s \in \mathbb{C} \mid \operatorname{Re}(s) \geq 0\}. \tag{1.2}$$

*Definition 1.2.* Let $\mathcal{W}^+$ denote the Banach algebra

$$\mathcal{W}^+ = \left\{ f : \mathbb{C}_{\geq 0} \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} f(s) = \widehat{f_a}(s) + \displaystyle\sum_{k=0}^{\infty} f_k e^{-st_k} \ \left(s \in \mathbb{C}_{\geq 0}\right), \\ f_a : (0,\infty) \longrightarrow \mathbb{C},\ f_a \in L^1(0,\infty), \\ \forall k \geq 0,\ \ f_k \in \mathbb{C},\ \left(f_k\right)_{k\geq 0} \in \ell^1, \\ \forall k \geq 0,\ \ t_k \in \mathbb{R},\ 0 = t_0 < t_1 < t_2 < \cdots \end{array} \right\} \tag{1.3}$$

equipped with pointwise operations and the norm

$$\|f\|_{\mathcal{W}^+} := \left\|f_a\right\|_{L^1} + \left\|\left(f_k\right)_{k\geq 0}\right\|_{\ell^1}. \tag{1.4}$$

Here $\widehat{f_a}$ denotes the Laplace transform of $f_a$, given by

$$\widehat{f_a}(s) = \int_0^\infty e^{-st} f_a(t)\,dt, \quad s \in \mathbb{C}_{\geq 0}. \tag{1.5}$$

The above algebra arises as a natural class of transfer functions of stable distributed parameter systems in control theory; see [4, 5].

Our main result is the following.

**Theorem 1.3.** *The ring $\mathcal{W}^+$ is not coherent.*

The relevance of the coherence property in control theory can be found in [6, 7]. We will prove Theorem 1.3 following the same method as in the proof of the noncoherence of $W^+(\mathbb{D})$ given by Mortini and von Renteln in [3].

In Section 3, we will give the proof of Theorem 1.3. But before doing that, in Section 2, we first prove a few technical results needed in the sequel.

## 2. Preliminaries

We first recall the definition of the Hardy algebra $H^\infty$ of the open right half plane.

*Definition 2.1.* Let $H^\infty$ denote the Hardy space of all bounded analytic functions in the open right half plane equipped with the norm

$$\|\varphi\|_\infty := \sup_{\mathrm{Re}(s)>0} |\varphi(s)|, \quad \varphi \in H^\infty. \tag{2.1}$$

In order to prove our main result (Theorem 1.3), we will use the relation between the convergence in $H^\infty$ versus that in $\mathcal{W}^+$.

**Lemma 2.2.** *If $f \in \mathcal{W}^+$, then $f \in H^\infty$ and $\|f\|_\infty \le \|f\|_{\mathcal{W}^+}$.*

*Proof.* Let

$$f(s) = \widehat{f_a}(s) + \sum_{k=0}^{\infty} f_k e^{-st_k} \quad (s \in \mathbb{C}_{\ge 0}). \tag{2.2}$$

For $s \in \mathbb{C}_{\ge 0}$, we have

$$\left|\widehat{f_a}(s)\right| = \left|\int_0^\infty e^{-st} f_a(t)dt\right| \le \int_0^\infty e^{-\mathrm{Re}(s)t} |f_a(t)| dt \le \int_0^\infty 1 \cdot |f_a(t)| dt = \|f_a\|_{L^1}, \tag{2.3}$$

and moreover,

$$\left|\sum_{k=0}^{\infty} f_k e^{-st_k}\right| \le \sum_{k=0}^{\infty} |f_k| e^{-\mathrm{Re}(s)t_k} \le \sum_{k=0}^{\infty} |f_k| \cdot 1 = \left\|(f_k)_k\right\|_{\ell^1}. \tag{2.4}$$

So the result follows. □

The maximal ideal $\mathfrak{m}_0$ (defined below) of $\mathcal{W}^+$ will play an important role in the remainder of this article.

*Notation 2.* Let $\mathfrak{m}_0$ denote the kernel of the complex algebra homomorphism $f \mapsto f(0) : \mathcal{W}^+ \to \mathbb{C}$, that is,

$$\mathfrak{m}_0 := \{ f \in \mathcal{W}^+ \mid f(0) = 0 \}.$$

Then $\mathfrak{m}_0$ is a maximal ideal of $\mathcal{W}^+$, and this maximal ideal plays an important role in the proof of our main result in the next section. We will prove a few technical results about $\mathfrak{m}_0$ in this section, which will be used in the sequel. The following result is analogous to [3, Lemma 1].

**Lemma 2.3.** *Let $L \ne (0)$ be an ideal in $\mathcal{W}^+$ contained in the maximal ideal $\mathfrak{m}_0$. If $L = L\mathfrak{m}_0$, that is, if every function $f \in L$ can be factorized in a product $f = hg$ of two functions $h \in L$ and $g \in \mathfrak{m}_0$, then $L$ cannot be finitely generated.*

*Proof.* Suppose that

$$L = (f_1, \ldots, f_N) \ne (0) \tag{2.5}$$

is a finitely generated ideal in $\mathcal{W}^+$ contained in the maximal ideal $\mathfrak{m}_0$. By our assumption, there are functions $h_n \in L$, $g_n \in \mathfrak{m}_0$ with

$$f_n = h_n g_n \quad (n = 1, \dots, N). \tag{2.6}$$

Since $h_n \in L$, there exist functions $q_k^{(n)} \in \mathcal{W}^+$ with

$$h_n = \sum_{k=1}^{N} q_k^{(n)} f_k \quad (n = 1, \dots, N;\ k = 1, \dots, N). \tag{2.7}$$

From this it follows that

$$\sum_{n=1}^{N} |h_n| \le NC \sum_{n=1}^{N} |f_n| = NC \sum_{n=1}^{N} |h_n g_n| \quad \text{in } \mathbb{C}_{\ge 0}, \tag{2.8}$$

where $C$ is a constant chosen so that

$$\|q_k^{(n)}\|_\infty \le C, \quad \forall k \text{ and } n. \tag{2.9}$$

(Here $\|\cdot\|_\infty$ denotes the supnorm over $\mathbb{C}_{\ge 0}$.) This implies together with the Cauchy-Schwarz inequality that

$$\sum_{n=1}^{N} |h_n|^2 \le \left(\sum_{n=1}^{N} |h_n|\right)^2 \le N^2C^2 \left(\sum_{n=1}^{N} |h_n g_n|\right)^2 \le N^2C^2 \left(\sum_{n=1}^{N} |h_n|^2\right)\left(\sum_{n=1}^{N} |g_n|^2\right). \tag{2.10}$$

This inequality holds for all $s \in \mathbb{C}_{\ge 0}$. With $\delta := 1/(N^2C^2)$, we obtain the inequality

$$\delta \le \sum_{n=1}^{N} |g_n(s)|^2 \tag{2.11}$$

for all points $s \in E$, where

$$E := \left\{ s \in \mathbb{C}_{\ge 0} \;\middle|\; \sum_{n=1}^{N} |h_n(s)|^2 > 0 \right\}. \tag{2.12}$$

Since $L \ne (0)$, $E$ is a dense subset of $\mathbb{C}_{\ge 0}$ (for otherwise, if $s_0 \in \mathbb{C}_{\ge 0}$ is such that it has a neighbourhood $V$ in $\mathbb{C}_{\ge 0}$ where there is no point of $E$, then each $h_n$ is identically zero in $V$, and by the identity theorem for holomorphic functions, each $h_n$ is zero; consequently each $f_n$ is zero, and so $L = (0)$, a contradiction). So by continuity, inequality (2.11) holds in $\mathbb{C}_{\ge 0}$. But this contradicts the fact that each $g_n$ vanishes at 0. □

*Remark 2.4.* Lemma 2.3 can be proved purely algebraically using Nakayama's lemma. Indeed, it holds in the following more general algebraic situation: if $I$ is a nonzero ideal of a commutative domain $D$ contained in a maximal ideal $M$ and $I = IM$, then $I$ cannot be finitely generated. However, we have given an analytic proof in our special case above.

Since every maximal ideal is closed, $\mathfrak{m}_0$ is a commutative Banach subalgebra of $\mathcal{W}^+$, but obviously without identity element. But there is a substitute, namely the notion of the approximate identity, which turns out to be useful.

*Definition 2.5.* Let $R$ be a commutative Banach algebra (without identity element). One says that $R$ has an *approximate identity* if there exists a bounded sequence $(e_n)_n$ of elements $e_n$ in $R$ such that for any $f \in R$,

$$\lim_{n\to\infty} \left\| e_n f - f \right\| = 0. \tag{2.13}$$

We will now prove the following result, which shows that the maximal ideal $\mathfrak{m}_0$ in $\mathcal{W}^+$ has an approximate identity.

**Theorem 2.6.** *Let*

$$e_n := \frac{s}{s+1/n}, \quad n \in \mathbb{N}. \tag{2.14}$$

*Then $(e_n)_{n\in\mathbb{N}}$ is an approximate identity for $\mathfrak{m}_0$.*

The existence of an approximate identity for the maximal ideal $\mathfrak{m}_0$ in $\mathcal{W}^+$ is not obvious. In order to prove Theorem 2.6, we will need the following lemma.

**Lemma 2.7.** *Suppose $\widehat{f} \in \mathfrak{m}_0$. Then, for all $\epsilon > 0$, there exists a $\widehat{p} \in \mathfrak{m}_0$ such that $p$ has compact support in $[0,\infty)$, and $\|\widehat{f} - \widehat{p}\|_{\mathcal{W}^+} < \epsilon$.*

*Proof.* Let $\epsilon > 0$ be given. Suppose that

$$f = f_a + \sum_{k=0}^{\infty} f_k \delta(\cdot - t_k), \tag{2.15}$$

where $f_a \in L^1(0,\infty)$, $(f_k)_{k\geq 0} \in \ell^1$, and $0 = t_0 < t_1 < t_2 < \cdots$. Since $\int_0^\infty |f_a(t)|dt < \infty$, we can choose an $M > 0$ large enough such that

$$\int_M^\infty \left| f_a(t) \right| dt < \frac{\epsilon}{4}. \tag{2.16}$$

With $p_a(t) := f_a(t)$ if $t \in [0, M]$, and 0 otherwise, we have that $p_a \in L^1(0,\infty)$ is compactly supported and

$$\left\| p_a - f_a \right\|_{L^1} < \frac{\epsilon}{4}. \tag{2.17}$$

Furthermore, select $N \in \mathbb{N}$ such that

$$\sum_{k>N} \left| f_k \right| < \frac{\epsilon}{4}. \tag{2.18}$$

Now let $T \in (0,\infty)$ be any number satisfying $t_N < T < t_{N+1}$, and define

$$f_T := -\left( \int_0^\infty p_a(t)dt + \sum_{0\leq k\leq N} f_k \right). \tag{2.19}$$

Set

$$p := p_a + \sum_{0\le k\le N} f_k \delta(\cdot - t_k) + f_T \delta(\cdot - T). \tag{2.20}$$

Then $\widehat{p} \in \mathcal{W}^+$ and

$$\widehat{p}(0) = \int_0^\infty p(t)dt = \int_0^\infty p_a(t)dt + \sum_{0\le k\le N} f_k + f_T = 0. \tag{2.21}$$

So $\widehat{p} \in \mathfrak{m}_0$. Clearly $p$ has compact support contained in $[0,\infty)$. We have

$$\begin{aligned}
|f_T| &= \left| \int_0^\infty p_a(t)dt + \sum_{0\le k\le N} f_k \right| \\
&= \left| \int_0^\infty f_a(t)dt + \sum_{k=0}^{\infty} f_k + \int_0^\infty (p_a(t) - f_a(t))dt - \sum_{k>N} f_k \right| \\
&\le \left| \int_0^\infty f(t)dt \right| + \|p_a - f_a\|_{L^1} + \sum_{k>N} |f_k| \\
&= |\widehat{f}(0)| + \|p_a - f_a\|_{L^1} + \sum_{k>N} |f_k| \\
&< 0 + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\end{aligned} \tag{2.22}$$

Thus

$$\|\widehat{f} - \widehat{p}\|_{\mathcal{W}^+} = \|f_a - p_a\|_{L^1} + \sum_{k>N} |f_k| + |f_T| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon. \tag{2.23}$$

This completes the proof. □

We are now ready to prove the existence of an approximate identity for the maximal ideal $\mathfrak{m}_0$ in $\mathcal{W}^+$.

*Proof of Theorem 2.6.* We have

$$e_n = \frac{s}{s+1/n} = \frac{s+1/n-1/n}{s+1/n} = 1 - \frac{1}{n}\frac{1}{s+1/n} = 1 + \widehat{-\frac{1}{n}e^{-t/n}}. \tag{2.24}$$

Thus for an $n \in \mathbb{N}$,

$$\|e_n\|_{\mathcal{W}^+} = \left\| -\frac{1}{n}e^{-t/n} \right\|_{L^1} + |1| = 1 + 1 = 2. \tag{2.25}$$

Given $\widehat{f} \in \mathcal{W}^+$, and $\epsilon > 0$ arbitrarily small, in view of Lemma 2.7, we can find a $\widehat{p} \in \mathfrak{m}_0$ such that $p$ has compact support and $\|\widehat{f} - \widehat{p}\|_{\mathcal{W}^+} < \epsilon$. Then

$$\|e_n\widehat{f} - \widehat{f}\|_{\mathcal{W}^+} \le \|e_n\widehat{p} - \widehat{p}\|_{\mathcal{W}^+} + \|e_n\|_{\mathcal{W}^+}\|\widehat{f} - \widehat{p}\|_{\mathcal{W}^+} + \|\widehat{f} - \widehat{p}\|_{\mathcal{W}^+}. \tag{2.26}$$

So it is enough to prove that

$$\lim_{n\to\infty} \|e_n\widehat{p} - \widehat{p}\|_{\mathcal{W}^+} = 0 \tag{2.27}$$

for all $\widehat{p} \in \mathfrak{m}_0$ such that $p$ has compact support in $[0,\infty)$. We do this below.

We have

$$e_n\widehat{p} - \widehat{p} = \frac{s+1/n-1/n}{s+1/n}\widehat{p} - \widehat{p} = -\frac{1}{n}\frac{1}{s+1/n}\widehat{p} = -\frac{1}{n}(\widehat{e^{-t/n} * p}). \tag{2.28}$$

Let $C$ denote the convolution $e^{-t/n} * p$:

$$C(t) := \int_0^t e^{-(t-\tau)/n} p(\tau) d\tau. \tag{2.29}$$

We note that $C \in L^1(0,\infty)$, since $L^1(0,\infty)$ is an ideal in $\mathcal{W}^+$. Let $T > 0$ be such that

$$\operatorname{supp}(p) \subset [0,T]. \tag{2.30}$$

We have

$$\|e_n\widehat{p} - \widehat{p}\|_{\mathcal{W}^+} = \frac{1}{n}\|C\|_{L^1} = \frac{1}{n}\int_0^\infty |C(t)|dt = \underbrace{\frac{1}{n}\int_0^T |C(t)|dt}_{(I)} + \underbrace{\frac{1}{n}\int_T^\infty |C(t)|dt}_{(II)}. \tag{2.31}$$

We estimate $(I)$ as follows:

$$\begin{aligned}
(I) = \frac{1}{n}\int_0^T |C(t)|dt &= \frac{1}{n}\int_0^T \left|\int_0^t e^{-(t-\tau)/n} p(\tau)d\tau\right| dt \\
&\leq \frac{1}{n}\int_0^T \int_0^t e^{-(t-\tau)/n} |p(\tau)| d\tau\, dt \\
&\leq \frac{1}{n}\underbrace{\left(\int_0^T \int_0^t 1 \cdot |p(\tau)| d\tau\, dt\right)}_{(III)}.
\end{aligned} \tag{2.32}$$

Since the integral $(III)$ does not depend on $n$, we obtain that

$$\lim_{n\to\infty} \frac{1}{n}\int_0^T |C(t)|dt = 0. \tag{2.33}$$

Furthermore,

$$\begin{aligned}
(II) &= \frac{1}{n}\int_T^\infty |C(t)|dt \\
&= \frac{1}{n}\int_T^\infty e^{-t/n}\left|\int_0^t e^{\tau/n} p(\tau)d\tau\right| dt \\
&= \frac{1}{n}\int_T^\infty e^{-t/n}\left|\int_0^\infty e^{\tau/n} p(\tau)d\tau\right| dt \quad (\text{since } \operatorname{supp}(p) \subset [0,T]) \\
&= \frac{1}{n}\int_T^\infty e^{-t/n}\left|\widehat{p}\left(-\frac{1}{n}\right)\right| dt.
\end{aligned} \tag{2.34}$$

Since $p$ has compact support in $[0,T]$, $\widehat{p}$ is an entire function by the Payley-Wiener theorem (see, e.g., [8, Theorem 7.2.3, page 122]). Consequently,

$$(II) = \frac{1}{n}\int_T^{\infty} e^{-t/n}\left|\widehat{p}\left(-\frac{1}{n}\right)\right|dt = e^{-T/n}\left|\widehat{p}\left(-\frac{1}{n}\right)\right| \longrightarrow n \longrightarrow \infty 1\cdot\left|\widehat{p}(0)\right| = 1\cdot 0 = 0. \tag{2.35}$$

This completes the proof. □

We will also need the following lemma, which is basically a repetition of key steps from Browder's proof of Cohen's factorization theorem; see [9, Theorem 1.6.5, page 74]. We will need this version since in our application in the proof of Theorem 1.3, we are not able to use Cohen's factorization theorem directly.

**Lemma 2.8.** *Let $f_1, f_2 \in \mathfrak{m}_0$, and $\delta > 0$. Let $U(\mathcal{W}^+)$ denote the set of all invertible elements in $\mathcal{W}^+$. Then there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{W}^+$ such that*

(1) *for all $n \in \mathbb{N}$, $g_n \in U(\mathcal{W}^+)$;*

(2) *$(g_n)_{n\in\mathbb{N}}$ is convergent in $\mathcal{W}^+$ to a limit $g \in \mathfrak{m}_0$;*

(3) *for all $n \in \mathbb{N}$, $\|g_n^{-1}f_i - g_{n+1}^{-1}f_i\|_{\mathcal{W}^+} \leq \delta/2^n$, $i = 1, 2$.*

*Proof.* We will first prove two general results in steps (A) and (B), which we will use in the rest of the proof.

(A) Let $e \in \mathfrak{m}_0$ and $\|e\|_{\mathcal{W}^+} \leq K$, where $K > 1$. Then $1 - c + ce \in U(\mathcal{W}^+)$, where $c$ is a number chosen such that

$$0 < c < \frac{1}{4K} < \frac{1}{4}. \tag{2.36}$$

Indeed,

$$\left\|\frac{c}{c-1}e\right\|_{\mathcal{W}^+} < \frac{1/(4K)}{3/4}\cdot K = \frac{1}{3} < 1, \tag{2.37}$$

and so

$$(1 - c + ce)^{-1} = \frac{1}{1-c}\sum_{k=0}^{\infty}\left(\frac{c}{c-1}\right)^k e^k. \tag{2.38}$$

(B) Furthermore, under the same assumptions and notation as in (A) above, we now show that if $\|eF - F\|_{\mathcal{W}^+}$ is small for some $F$, then so is $\|EF - F\|_{\mathcal{W}^+}$, where $E := (1 - c + ce)^{-1}$. Since

$$1 = \frac{1}{1-c}\sum_{k=0}^{\infty}\left(\frac{c}{c-1}\right)^k, \tag{2.39}$$

we have

$$\|EF - F\|_{\mathcal{W}^+} = \left\|\frac{1}{1-c}\sum_{k=0}^{\infty}\left(\frac{c}{c-1}\right)^k\left(e^kF - F\right)\right\|_{\mathcal{W}^+} \leq \frac{1}{1-c}\sum_{k=0}^{\infty}\left(\frac{c}{1-c}\right)^k\left\|e^kF - F\right\|_{\mathcal{W}^+}. \tag{2.40}$$

But

$$\left\|e^kF - F\right\|_{\mathcal{W}^+} = \left\|\sum_{j=0}^{k-1}\left(e^{j+1}F - e^jF\right)\right\|_{\mathcal{W}^+} \leq \sum_{j=0}^{k-1}\left\|e^j\right\|_{\mathcal{W}^+}\|eF - F\|_{\mathcal{W}^+} \leq \|eF - F\|_{\mathcal{W}^+}\sum_{j=0}^{k-1}\|e\|_{\mathcal{W}^+}^j < \|eF - F\|_{\mathcal{W}^+}\frac{K^k}{K-1}. \tag{2.41}$$

Hence

$$\|EF - F\|_{\mathcal{W}^+} < \|eF - F\|_{\mathcal{W}^+} \frac{1}{1-c} \sum_{k=0}^{\infty} \frac{1}{K-1} \left( \frac{1}{4(1-c)} \right)^k < \frac{2}{K-1} \|eF - F\|_{\mathcal{W}^+}. \tag{2.42}$$

This estimate will be used in constructing the sequence of $g_n$'s.

Let $(e_n)_{n\in\mathbb{N}}$ denote the approximate identity for $\mathfrak{m}_0$ from Theorem 2.6. Let $K > 1$ be such that $\|e_n\|_{\mathcal{W}^+} \le K$ for all $n \in \mathbb{N}$. Choose $c$ such that

$$0 < c < \frac{1}{4K} < \frac{1}{4}. \tag{2.43}$$

We will inductively define a sequence $(e_{m_k})_{k\in\mathbb{N}}$ with terms from the approximate identity for $\mathfrak{m}_0$ such that if

$$g_n := c \sum_{k=1}^{n} (1-c)^{k-1} e_{m_k} + (1-c)^n, \tag{2.44}$$

then we have $\|f_i - g_1^{-1} f_i\|_{\mathcal{W}^+} < \delta/2$, $i = 1, 2$, and

(P1) for all $n \in \mathbb{N}$, $g_n \in U(\mathcal{W}^+)$,

(P2) for all $n \in \mathbb{N}$, $\|g_n^{-1} f_i - g_{n+1}^{-1} f_i\|_{\mathcal{W}^+} < \delta/2^n$, $i = 1, 2$.

Since $(e_n)_{n\in\mathbb{N}}$ is an approximate identity for $\mathfrak{m}_0$, we can choose $m_1$ such that

$$\left\| e_{m_1} f_i - f_i \right\|_{\mathcal{W}^+} < \frac{\delta}{4}(K-1), \quad i = 1, 2. \tag{2.45}$$

Define $g_1 = c e_{m_1} + 1 - c$. So by (A), $g_1 \in U(\mathcal{W}^+)$ and using the calculation in (B), we see that

$$\left\| f_i - g_1^{-1} f_i \right\|_{\mathcal{W}^+} < \frac{\delta}{2}, \quad i = 1, 2. \tag{2.46}$$

Suppose that $e_{m_1}, \ldots, e_{m_n}$ have been constructed, so that $g_n$ defined by (2.44) satisfies (P1) and (P2). We assert that if we choose $e_{m_{n+1}}$ such that

$$\left\| e_{m_{n+1}} f_i - f_i \right\|_{\mathcal{W}^+} \quad (i = 1, 2), \qquad \left\| e_{m_{n+1}} e_{m_k} - e_{m_k} \right\|_{\mathcal{W}^+} \quad (1 \le k \le n) \tag{2.47}$$

are sufficiently small, then $g_{n+1}$ defined by (2.44) satisfies (P1) and (P2), completing the induction step.

Indeed, if $E := (1 - c + c e_{m_{n+1}})^{-1}$, we have

$$\begin{aligned} g_n &= E^{-1} c \sum_{k=1}^{n} (1-c)^{k-1} E e_{m_k} + (1-c)^n, \\ g_{n+1} &= E^{-1} \left( c \sum_{k=1}^{n} (1-c)^{k-1} E e_{m_k} + (1-c)^n \right). \end{aligned} \tag{2.48}$$

Let $G_n$ be defined by

$$G_n = c\sum_{k=1}^{n}(1-c)^{k-1}Ee_{m_k} + (1-c)^n. \tag{2.49}$$

Then we have

$$\|G_n - g_n\|_{\mathcal{W}^+} < c\sum_{k=1}^{n}(1-c)^{k-1}\|Ee_{m_k} - e_{m_k}\|_{\mathcal{W}^+} < \max_{1\le k\le n}\|Ee_{m_k} - e_{m_k}\|_{\mathcal{W}^+} < \frac{2}{K-1}\max_{1\le k\le n}\|e_{m_{n+1}}e_{m_k} - e_{m_k}\|_{\mathcal{W}^+}. \tag{2.50}$$

Hence $G_n \in U(\mathcal{W}^+)$ and moreover $\|G_n^{-1} - g_n^{-1}\|_{\mathcal{W}^+}$ is small, provided only that $\|e_{m_{n+1}}e_{m_k} - e_{m_k}\|_{\mathcal{W}^+}$ is small for $k = 1, \dots, n$. (Indeed, this is because $U(\mathcal{W}^+)$ is an open set in $\mathcal{W}^+$.)

Since $g_{n+1} = E^{-1}G_n$, we have then $g_{n+1} \in U(\mathcal{W}^+)$, $g_{n+1}^{-1} = G_n^{-1}E$, and so for $i = 1, 2$,

$$\begin{aligned} \|g_{n+1}^{-1}f_i - g_n^{-1}f_i\|_{\mathcal{W}^+} &= \|G_n^{-1}Ef_i - g_n^{-1}f_i\|_{\mathcal{W}^+} \\ &\le \|G_n^{-1}Ef_i - g_n^{-1}Ef_i\|_{\mathcal{W}^+} + \|g_n^{-1}Ef_i - g_n^{-1}f_i\|_{\mathcal{W}^+} \\ &\le \|G_n^{-1} - g_n^{-1}\|_{\mathcal{W}^+}\|Ef_i\|_{\mathcal{W}^+} + \|g_n^{-1}\|_{\mathcal{W}^+}\|Ef_i - f_i\|_{\mathcal{W}^+}. \end{aligned} \tag{2.51}$$

Moreover, recall that by (B), we know that

$$\|Ef_i - f_i\|_{\mathcal{W}^+} \le \frac{2}{K-1}\|e_{m_{n+1}}f_i - f_i\|_{\mathcal{W}^+}, \quad i = 1, 2. \tag{2.52}$$

Thus if $\|e_{m_{n+1}}f_i - f_i\|_{\mathcal{W}^+}$ $(i = 1, 2)$ and $\|e_{m_{n+1}}e_{m_k} - e_{m_k}\|_{\mathcal{W}^+}$ $(1 \le k \le n)$ are sufficiently small, we will have $\|g_{n+1}^{-1}f_i - g_n^{-1}f_i\|_{\mathcal{W}^+}$ as small as we please. This completes the induction step.

Since $\|e_{m_k}\|_{\mathcal{W}^+} \le K$, $0 < 1 - c < 1$, and $\mathcal{W}^+$ is a Banach algebra, it follows that

$$g_n \longrightarrow c\sum_{k=1}^{\infty}(1-c)^{k-1}e_{m_k} =: g \in \mathfrak{m}_0, \tag{2.53}$$

and the proof is completed. □

## 3. Noncoherence of $\mathcal{W}^+$

*Proof of Theorem 1.3.* We will use the characterization that an integral domain is coherent if and only if the intersection of any two finitely generated ideals of the ring is again finitely generated; see [1, Theorem 2.3.2, page 45]. In fact, we present two finitely generated ideals $I$ and $J$ such that $I \cap J$ is not finitely generated.

Let $p$, $S$ be given by

$$p = \left(1 - e^{-s}\right)^3, \qquad S = e^{-(1+e^{-s})/(1-e^{-s})}. \tag{3.1}$$

Clearly we have $p \in \mathfrak{m}_0$.

It is known (see, e.g., [3, Remark after Theorem 1, page 224]) that

$$(1-z)^3 e^{-(1+z)/(1-z)} \in W^+(\mathbb{D}) := \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \ (z \in \overline{\mathbb{D}}) \,\middle|\, \sum_{n=0}^{\infty} |a_n| < \infty \right\}. \tag{3.2}$$

Here $\overline{\mathbb{D}} := \{z \in \mathbb{C} \mid |z| \leq 1\}$. So if $a_n$'s are defined via

$$(1-z)^3 e^{-(1+z)/(1-z)} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots, \quad z \in \mathbb{D}, \tag{3.3}$$

then we have

$$\sum_{k=0}^{\infty} |a_k| < \infty. \tag{3.4}$$

If $\operatorname{Re}(s) > 0$, then $e^{-s} \in \mathbb{D}$, and so from (3.3), we have

$$pS = a_0 + a_1 e^{-s} + a_2 e^{-2s} + a_3 e^{-3s} + \cdots, \quad \operatorname{Re}(s) > 0. \tag{3.5}$$

Since $\sum_{k=0}^{\infty} |a_k| < \infty$, the right-hand side in (3.5) belongs to $\mathcal{W}^+$. So $pS \in \mathcal{W}^+$.

We define the ideals $I = (p)$ and $J = (pS)$ of $\mathcal{W}^+$.

Let

$$K := \{pSf \mid f \in \mathcal{W}^+ \text{ and } Sf \in \mathcal{W}^+\}. \tag{3.6}$$

We claim that $K = I \cap J$. Trivially $K \subset I \cap J$. To prove the reverse inclusion, let $g \in I \cap J$. Then there exist two functions $f$ and $h$ in $\mathcal{W}^+$ such that $g = ph = pSf$. Since $p \neq 0$ and $\mathcal{W}^+$ is an integral domain, we obtain that $Sf = h \in \mathcal{W}^+$. So $g \in K$.

Let $L$ denote the ideal

$$L := \{f \in \mathcal{W}^+ \mid Sf \in \mathcal{W}^+\}. \tag{3.7}$$

Then $K := pSL$. Since $S$ has a singularity at $s = 0$, it follows that $L \subset \mathfrak{m}_0$. We will show that $L = L\mathfrak{m}_0$. Let $f \in L$. We would like to factor $f = hg$ with $h \in L$ and $g \in \mathfrak{m}_0$. Applying Lemma 2.8 with $f_1 := f \in \mathfrak{m}_0$ and $f_2 := Sf \in \mathfrak{m}_0$, for any $\delta > 0$, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{W}^+$ such that

(1) for all $n \in \mathbb{N}$, $g_n \in U(\mathcal{W}^+)$;

(2) $(g_n)_{n\in\mathbb{N}}$ is convergent in $\mathcal{W}^+$ to a limit $g \in \mathfrak{m}_0$;

(3) for all $n \in \mathbb{N}$,

$$\left\| g_n^{-1} f - g_{n+1}^{-1} f \right\|_{\mathcal{W}^+} \leq \frac{\delta}{2^n}, \qquad \left\| g_n^{-1} Sf - g_{n+1}^{-1} Sf \right\|_{\mathcal{W}^+} \leq \frac{\delta}{2^n}. \tag{3.8}$$

Put

$$h_n := g_n^{-1} f, \qquad H_n := g_n^{-1} Sf. \tag{3.9}$$

Then $h_n \in \mathfrak{m}_0$. Also $H_n \in \mathfrak{m}_0$, since $|S|$ is bounded by 1 on $\mathrm{Re}(s) > 0$ and $f(0) = 0$. The estimates above imply that $(h_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathcal{W}^+$. Since $\mathfrak{m}_0$ is closed, they converge to elements $h$ and $H$, respectively, in $\mathfrak{m}_0$, that is, $h_n = g_n^{-1} f \to h$ and $H_n = g_n^{-1} S f = S h_n \to H$. Since convergence in $\mathcal{W}^+$ implies convergence in $H^\infty$ (Lemma 2.2), it follows that

$$\begin{aligned} h_n &\xrightarrow{H^\infty} h \quad (\text{since } h_n \xrightarrow{\mathcal{W}^+} h), \\ S h_n &\xrightarrow{H^\infty} S h \quad (\text{since } h_n \xrightarrow{H^\infty} h,\ S \in H^\infty), \\ S h_n &\xrightarrow{H^\infty} H \quad (\text{since } H_n \xrightarrow{\mathcal{W}^+} H) \end{aligned} \tag{3.10}$$

and so by the uniqueness of the limit of the sequence $(Sh_n)_{n\in\mathbb{N}}$ in $H^\infty$, we have $Sh = H$. Also, in $\mathcal{W}^+$-norm we have

$$f = \lim_{n\to\infty} h_n g_n = hg, \tag{3.11}$$

since multiplication is continuous in the Banach algebra $\mathcal{W}^+$. Since $h$ and $Sh = H$ belong to $\mathfrak{m}_0 \subset \mathcal{W}^+$, we see that $h \in L$. Moreover, as $g \in \mathfrak{m}_0$, we have got the desired factorization and $L = L\mathfrak{m}_0$.

But $L \neq (0)$, since $p \in L$. By Lemma 2.3, it follows that $L$ cannot be finitely generated. Therefore, $pSL = I \cap J$ is not finitely generated. □

*Remark 3.1.* The ideal $L$ in the above proof can be interpreted as an *ideal of denominators*; see [10, page 396]. Using the fact that $pS \in \mathcal{W}^+$, we have $S \in Q(\mathcal{W}^+)$, where $Q(\mathcal{W}^+)$ denotes the field of fractions of $\mathcal{W}^+$. We can then consider the *fractional ideal* $M := \mathcal{W}^+ + \mathcal{W}^+ S$ of $\mathcal{W}^+$ (see [11, page 19]) and the *ideal of denominators* $L$ of $S$, namely $L = \mathcal{W}^+ : M = \{d \in \mathcal{W}^+ \mid dS \in \mathcal{W}^+\}$.

Based on the results in [12, Theorem 3, Example 3], it follows that $S \in Q(\mathcal{W}^+)$ does not admit a weak coprime factorization, since $L$ is not a principal ideal of $\mathcal{W}^+$. In particular, $S$ does not admit a coprime factorization, that is, there do not exist $d, x, y, n \in \mathcal{W}^+$ such that $d \neq 0$, $S = n/d$, and $dx - ny = 1$. Moreover, $S$ is not internally stabilizable, since otherwise $L$ would be generated by two elements. Finally, the fact that $L$ is not finitely generated implies that $\mathcal{W}^+$ is not a *greatest common divisor domain*: indeed, were it the case that $\mathcal{W}^+$ is a greatest common divisor domain, then by [12, Corollary 3], every element in $Q(\mathcal{W}^+)$ would admit a weak coprime factorization.

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