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Research Article

Pairwise Weakly Regular-Lindelöf Spaces

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We will introduce and study the pairwise weakly regular-Lindelöf bitopological spaces and obtain some results. Furthermore, we study the pairwise weakly regular-Lindelöf subspaces and subsets, and investigate some of their characterizations. We also show that a pairwise weakly regular-Lindelöf property is not a hereditary property. Some counterexamples will be considered in order to establish some of their relations.

1. Introduction

The study of bitopological spaces was first initiated by Kelly [1] in 1963 and thereafter a large number of papers have been done to generalize the topological concepts to bitopological setting. In literature, there are several generalizations of the notion of Lindelöf spaces, and these are studied separately for different reasons and purposes. In 1959, Frolík [2] introduced the notion of weakly Lindelöf spaces and in 1996, Cammaroto and Santoro [3] studied and gave further new results about these spaces followed by Kılıçman and Fawakhreh [4]. In the same paper, Cammaroto and Santoro introduced the notion of weakly regular-Lindelöf spaces by using regular covers and leave open the study of this new concept. In 2001, Fawakhreh and Kılıçman [5] studied this new generalization of Lindelöf spaces and obtained some results. Then, Kılıçman and Fawakhreh [6] studied subspaces of this spaces and obtained some results.

Recently, the authors studied pairwise Lindelöfness in [7] and introduced and studied the notion of pairwise weakly Lindelöf spaces in bitopological spaces, see [8], where the authors extended some results that were due to Cammaroto and Santoro [3], Kılıçman and Fawakhreh [4], and Fawakhreh [9]. In [10], the authors also studied the mappings and pairwise continuity on pairwise Lindelöf bitopological spaces. The purpose of this paper is to define the notion of weakly regular-Lindelöf property in bitopological spaces, which we will call pairwise weakly regular- spaces and investigate some of their characterizations. Moreover, we study the pairwise weakly regular-Lindelöf subspaces and subsets and also investigate some of their characterizations.

In Section 3, we will introduce the concept of pairwise weakly regular-Lindelöf bitopological spaces by using pairwise regular cover. This study begin by investigating the ij-weakly regular-Lindelöf property and some results obtained. Furthermore, we study the relation between ij-nearly Lindelöf, ij-almost Lindelöf, ij-weakly Lindelöf, ij-almost regular-Lindelöf, ij-nearly regular-Lindelöf, and ij-weakly regular-Lindelöf spaces, where $i, j = 1$ or 2, $i \neq j$.

In Section 4, we will define the concept of pairwise weakly regular-Lindelöf subspaces and subsets. We will define the concept of pairwise weakly regular-Lindelöf relative to a bitopological space by investigating the ij-weakly regular-Lindelöf property and obtain some results. The main result obtained is pairwise, and weakly regular-Lindelöf property is not a hereditary property by a counterexample given.

2. Preliminaries

Throughout this paper, all spaces (X, τ) and (X, τ_1, τ_2) (or simply X) are always mean topological spaces and bitopological spaces, respectively, unless explicitly stated. We always use ij- to denote the certain properties with respect to topology τ_i and τ_j, where $i, j \in \{1, 2\}$ and $i \neq j$. By i-int(A) and i-cl(A), we will mean the interior and the closure of a subset A of X with respect to topology τ_i, respectively. We denote by int(A) and cl(A) for the interior and the closure of a subset A of X with respect to topology τ_i for each $i = 1, 2$, respectively.

If $S \subseteq A \subseteq X$, then i-int$_A(S)$ and i-cl$_A(S)$ will be used to denote the interior and closure of S with respect to topology τ_i in the subspace A, respectively. By i-open cover of X, we mean that the cover of X by i-open sets in X; similar for the ij-regular open cover of X and so forth. We will use the notation, X is i-Lindelöf space which mean that (X, τ_i) is a Lindelöf space, where $i \in \{1, 2\}$.

Definition 2.1 (see [11, 12]). A subset S of a bitopological space (X, τ_1, τ_2) is said to be ij-regular open (resp., ij-regular closed) if i-int$(j$-cl$(S)) = S$ (resp., i-cl$(j$-int$(S)) = S$), and S is said pairwise regular open (resp., pairwise regular closed) if it is both ij-regular open and ji-regular open (resp., ij-regular closed and ji-regular closed).

Definition 2.2. Let (X, τ_1, τ_2) be a bitopological space. A subset F of X is said to be

(i) i-open if F is open with respect to τ_i in X, F is said open in X if it is both 1-open and 2-open in X, or equivalently, $F \in \tau_1 \cap \tau_2$;

(ii) i-closed if F is closed with respect τ_i in X, F is said closed in X if it is both 1-closed and 2-closed in X, or equivalently, $X \setminus F \in \tau_1 \cap \tau_2$;

(iii) i-clopen if F is both i-closed and i-open set in X, F is said clopen in X if it is both 1-clopen and 2-clopen in X;

(iv) ij-clopen if F is i-closed and j-open set in X, F is said clopen if it is both ij-clopen and ji-clopen in X.

Definition 2.3 (see [13]). A bitopological space (X, τ_1, τ_2) is said to be Lindelöf if the topological space (X, τ_1) and (X, τ_2) are both Lindelöf. Equivalently, (X, τ_1, τ_2) is Lindelöf if every i-open cover of X has a countable subcover for each $i = 1, 2$.

Definition 2.4 (see [1, 11]). A bitopological space (X, τ_1, τ_2) is said to be ij-regular if for each point $x \in X$ and for each i-open set V of X containing x there exists an i-open set U such that $x \in U \subseteq j\text{-cl}(U) \subseteq V$, and X is said to be pairwise regular if it is both ij-regular and ji-regular.

Definition 2.5 (see [11, 14]). A bitopological space X is said to be ij-almost regular if for each $x \in X$ and for each ij-regular open set V of X containing x there is an ij-regular open set U such that $x \in U \subseteq j\text{-cl}(U) \subseteq V$, then X is said to be pairwise almost regular if it is both ij-almost regular and ji-almost regular.

Definition 2.6 (see [11, 12]). A bitopological space X is said to be ij-semiregular if for each $x \in X$ and for each i-open set V of X containing x there is an i-open set U such that $x \in U \subseteq i\text{-int}(j\text{-cl}(U)) \subseteq V$, and X is said pairwise semiregular if it is both ij-semiregular and ji-semiregular.

Definition 2.7. A bitopological space X is said to be ij-nearly Lindelöf [15] (resp., ij-almost Lindelöf [16], ij-weakly Lindelöf [8]) if for every i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that

$$X = \bigcup_{n\in\mathbb{N}} i\text{-int}\big(j\text{-cl}(U_{\alpha_n})\big) \quad \left(\text{resp., } X = \bigcup_{n\in\mathbb{N}} j\text{-cl}(U_{\alpha_n}),\ X = j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}\right)\right), \tag{2.1}$$

and X is said pairwise nearly Lindelöf (resp., pairwise almost Lindelöf, pairwise weakly Lindelöf) if it is both ij-nearly Lindelöf (resp., ij-almost Lindelöf, ij-weakly Lindelöf) and ji-nearly Lindelöf (resp., ji-almost Lindelöf, ji-weakly Lindelöf).

Definition 2.8 (see [8]). A subset S of a bitopological space X is said to be ij-weakly Lindelöf relative to X if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of S by i-open subsets of X such that $S \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$ there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $S \subseteq j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})$. S is said pairwise weakly Lindelöf relative to X if it is both ij-weakly Lindelöf relative to X and ji-weakly Lindelöf relative to X.

Definition 2.9 (see [8]). A bitopological space X is said to be ij-nearly paracompact if every cover of X by ij-regular open sets admits a locally finite refinement. X is said pairwise nearly paracompact if it is both ij-nearly paracompact and ji-nearly paracompact.

3. Pairwise weakly regular-Lindelöf spaces

Definition 3.1 (see [17]). An i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of a bitopological space X is said to be ij-regular cover if for every $\alpha \in \Delta$ there exists a nonempty ji-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and X$= \bigcup_{\alpha\in\Delta} i\text{-int}(C_\alpha)$. $\{U_\alpha : \alpha \in \Delta\}$ is said pairwise regular cover if it is both ij-regular cover and ji-regular cover.

Definition 3.2. A bitopological space X is said to be ij-almost regular-Lindelöf [17] (resp., ij-nearly regular-Lindelöf [18]) if for every ij-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that

$$X = \bigcup_{n\in\mathbb{N}} j\text{-cl}(U_{\alpha_n}) \quad \left(\text{resp., } X = \bigcup_{n\in\mathbb{N}} i\text{-int}\big(j\text{-cl}(U_{\alpha_n})\big)\right), \tag{3.1}$$

then X is said pairwise almost regular-Lindelöf (resp., pairwise nearly regular-Lindelöf) if it is both ij-almost regular-Lindelöf (resp., ij-nearly regular-Lindelöf) and ji-almost regular-Lindelöf (resp., ji-nearly regular-Lindelöf).

Definition 3.3. A bitopological space X is said to be ij-weakly regular-Lindelöf if for every ij-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})$. X is said pairwise weakly regular-Lindelöf if it is both ij-weakly regular-Lindelöf and ji-weakly regular-Lindelöf.

Obviously, every ij-weakly Lindelöf space is ij-weakly regular-Lindelöf, and every ij-almost regular-Lindelöf space is ij-weakly regular-Lindelöf.

Question 1. Is ij-weakly regular-Lindelöf spaces implies ij-weakly Lindelöf?

Question 2. Is ij-weakly regular-Lindelöf spaces implies ij-almost regular-Lindelöf?

The authors expected that the answer of these questions is no. We can answer Question 1. by some restrictions on the space with the following proposition. First of all, we need the following lemmas.

Lemma 3.4 (see [17]). *Let X be an ij-almost regular space. Then, for each $x \in X$ and for each ij-regular open subset W of X containing x there exist two ij-regular open subsets U and V of X such that $x \in U \subseteq j\text{-cl}(U) \subseteq V \subseteq j\text{-cl}(V) \subseteq W$.*

Lemma 3.5 (see [17]). *A space X is ij-regular if and only if it is ij-almost regular and ij-semiregular.*

Proposition 3.6. *An ij-weakly regular-Lindelöf and ij-regular space X is ij-weakly Lindelöf.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an ij-regular open cover of X. For each $x \in X$, there exists $\alpha_x \in \Delta$ such that $x \in U_{\alpha_x}$. Since X is ij-almost regular, there exist two ij-regular open subsets V_{α_x} and W_{α_x} of X such that $x \in V_{\alpha_x} \subseteq j\text{-cl}(V_{\alpha_x}) \subseteq W_{\alpha_x} \subseteq j\text{-cl}(W_{\alpha_x}) \subseteq U_{\alpha_x}$ by Lemma 3.4. Since for each $\alpha \in \Delta$, there exists a ji-regular closed set $j\text{-cl}(V_{\alpha_x})$ in X such that $j\text{-cl}(V_{\alpha_x}) \subseteq W_{\alpha_x}$ and $X = \bigcup_{\alpha\in\Delta} V_{\alpha_x} = \bigcup_{\alpha\in\Delta} i\text{-int}(j\text{-cl}(V_{\alpha_x}))$, the family $\{W_{\alpha_x} : x \in X\}$ is an ij-regular cover of X. Since X is ij-weakly regular-Lindelöf, there exists a countable set of points $\{x_n : n \in \mathbb{N}\}$ of X such that $X = j\text{-cl}(\bigcup_{n\in\mathbb{N}} W_{\alpha_{x_n}}) \subseteq j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}})$. So, $X = j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}})$ and since X is ij-semiregular, therefore X is ij-weakly Lindelöf. □

Corollary 3.7. *A pairwise weakly regular-Lindelöf and pairwise regular space X is pairwise weakly Lindelöf.*

Proposition 3.6 implies the following corollaries.

Corollary 3.8. *Let X be an ij-regular space. Then, X is ij-weakly regular-Lindelöf if and only if it is ij-weakly Lindelöf.*

Corollary 3.9. *Let X be a pairwise regular space. Then, X is pairwise weakly regular-Lindelöf if and only if it is pairwise weakly Lindelöf.*

Definition 3.10 (see [8]). A bitopological space X is called ij-weak P-space if for each countable family $\{U_n : n \in \mathbb{N}\}$ of i-open sets in X, we have $j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}) = \bigcup_{n\in\mathbb{N}} j\text{-cl}(U_{\alpha_n})$ then X is called pairwise weak P-space if it is both ij-weak P-space and ji-weak P-space.

The following proposition shows that in ij-weak P-spaces, ij-almost regular-Lindelöf property equivalent to ij-weakly regular-Lindelöf property.

Proposition 3.11. *Let X be an ij-weak P-spaces. Then, X is ij-almost regular-Lindelöf if and only if X is ij-weakly regular-Lindelöf.*

Proof. The proof follows immediately from the fact that in ij-weak P-spaces, $\bigcup_{n\in\mathbb{N}} j\text{-cl}(U_{\alpha_n}) = j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})$ for any countable family $\{U_n : n \in \mathbb{N}\}$ of i-open sets in X. □

Corollary 3.12. *Let X be a pairwise weak P-spaces. Then, X is pairwise almost regular-Lindelöf if and only if X is pairwise weakly regular-Lindelöf.*

If X is an ij-almost regular space, then X is ij-almost regular-Lindelöf if and only if it is ij-nearly Lindelöf (see [17]). Thus, we have the following corollary.

Corollary 3.13. *In ij-almost regular and ij-weak P-spaces, ij-weakly regular-Lindelöf property is equivalent to ij-nearly Lindelöf property.*

Proof. This is a direct consequence of Proposition 3.11 and the previous fact. □

Corollary 3.14. *In pairwise almost regular and pairwise weak P-spaces, pairwise weakly regular-Lindelöf property is equivalent to pairwise nearly Lindelöf property.*

Lemma 3.15 (see [17]). *An ij-regular and ij-almost regular-Lindelöf space X is i-Lindelöf.*

Corollary 3.16. *In ij-regular and ij-weak P-spaces, ij-weakly regular-Lindelöf property is equivalent to i-Lindelöf property.*

Proof. This is a direct consequence of Proposition 3.11 and Lemma 3.15. □

Corollary 3.17. *In pairwise regular and pairwise weak P-spaces, pairwise weakly regular-Lindelöf property is equivalent to Lindelöf property.*

Definition 3.18 (see [8]). A subset E of a bitopological space X is said to be i-dense in X or is an i-dense subset of X if $i\text{-cl}(E) = X$. E is said dense in X or is a dense subset of X if it is i-dense in X or is an i-dense subset of X for each $i = 1, 2$.

Definition 3.19 (see [8]). A bitopological space X is said to be i-separable if there exists a countable i-dense subset of X. X is said separable if it is i-separable for each $i = 1, 2$.

Lemma 3.20 (see [8]). *If the bitopological space X is j-separable, then it is ij-weakly Lindelöf.*

Lemma 3.21 (see [18]). *An ij-regular and ij-nearly regular-Lindelöf space X is i-Lindelöf.*

It is clear that every ij-nearly regular-Lindelöf is ij-weakly regular-Lindelöf and every ij-almost regular-Lindelöf space is ij-weakly regular-Lindelöf, but the converses are not true in general as the following example show.

Example 3.22. Let $\mathcal{B}$ be the collection of closed-open intervals in the real line $\mathbb{R}$:

$$\mathcal{B} = \{[a, b) : a, b \in \mathbb{R},\ a < b\}. \tag{3.2}$$

Hence, $\mathcal{B}$ is a base for the lower limit topology τ_1 on $\mathbb{R}$. Choose usual topology as topology τ_2 on $\mathbb{R}$. Thus, $(\mathbb{R}, \tau_1, \tau_2)$ is a Lindelöf bitopological space (see [19]). Note that, sets of the form

$(-\infty, a)$, $[a, b)$ or $[a, \infty)$ are both 1-open and 1-closed in $\mathbb{R}$, and sets of the form (a, b) and (a, ∞) are 1-open in $\mathbb{R}$ (see [19]). It is easy to check that $(\mathbb{R}, \tau_1, \tau_2)$ is 12-regular since for each $x \in \mathbb{R}$ and for each 1-open set of the form $[a, b)$ in $\mathbb{R}$ containing x, there exists a 1-open set $[a, b\text{-}\epsilon)$ with $\epsilon > 0$ such that $x \in [a, b\text{-}\epsilon) \subseteq 2\text{-cl}[a, b\text{-}\epsilon) = [a, b\text{-}\epsilon] \subseteq [a, b)$. We left to the reader to check for other forms of 1-open sets in $\mathbb{R}$. It is clear that $\mathbb{R}$ is 2-separable since the rational numbers are a countable 2-dense subset of $\mathbb{R}$. So $(\mathbb{R}\times\mathbb{R}, \tau_1\times\tau_1, \tau_2\times\tau_2)$ is 12-regular and 2-separable. Thus, $\mathbb{R}\times\mathbb{R}$ is 12-weakly Lindelöf by Lemma 3.20, and so $\mathbb{R} \times \mathbb{R}$ is 12-weakly regular-Lindelöf. It is known that $\mathbb{R} \times \mathbb{R}$ is not 1-Lindelöf since the 1-closed subspace $L = \{(x, y) : y = -x\}$ is not 1-Lindelöf for it is a discrete subspace (see [19]). Since $\mathbb{R} \times \mathbb{R}$ is 12-regular, but not 1-Lindelöf, then it is neither 12-almost regular-Lindelöf nor 12-nearly regular-Lindelöf by Lemmas 3.15 and 3.21.

It is clear that every ij-almost Lindelöf is ij-weakly Lindelöf, but the converse is not true as in the following example show.

Lemma 3.23 (see [16]). *An ij-regular space is ij-almost Lindelöf if and only if it is i-Lindelöf.*

Example 3.24. Let $(\mathbb{R}, \tau_1, \tau_2)$ be a bitopological space defined as in Example 3.22 above. Example 3.22 shows that $\mathbb{R} \times \mathbb{R}$ is 12-weakly Lindelöf, but not 1-Lindelöf. Since $\mathbb{R} \times \mathbb{R}$ is 12-regular, but not 1-Lindelöf, then it is nor 12-almost Lindelöf by Lemma 3.23.

Remark 3.25. Example 3.24 solves the open problem in [8, Question 1].

Lemma 3.26 (see [8]). *An ij-weakly Lindelöf, ij-regular, and ij-nearly paracompact bitopological space X is i-Lindelöf.*

Proposition 3.27. *Let X be an ij-regular and ij-nearly paracompact spaces. Then, X is i-Lindelöf if and only if X is ij-weakly regular-Lindelöf.*

Proof. Let X be an ij-regular, ij-nearly paracompact, and ij-weakly regular-Lindelöf space. Then, X is ij-weakly Lindelöf by Proposition 3.6. So X is i-Lindelöf by Lemma 3.26. The converse is obvious. □

Corollary 3.28. *Let X be a pairwise regular and pairwise nearly paracompact spaces. Then, X is Lindelöf if and only if X is pairwise weakly regular-Lindelöf.*

Now, we give a characterization of ij-weakly regular-Lindelöf spaces.

Theorem 3.29. *A bitopological spaces X is ij-weakly regular-Lindelöf if and only if for every family $\{C_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X such that for each $\alpha \in \Delta$, there exists a j-open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha\in\Delta} i\text{-cl}(A_\alpha) = \varnothing$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $j\text{-int}(\bigcap_{n\in\mathbb{N}} C_{\alpha_n}) = \varnothing$.*

Proof. Let $\{C_\alpha : \alpha \in \Delta\}$ be a family of i-closed subsets of X such that for each $\alpha \in \Delta$ there exists a j-open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha\in\Delta} i\text{-cl}(A_\alpha) = \varnothing$. It follows that $X = X \setminus (\bigcap_{\alpha\in\Delta} i\text{-cl}(A_\alpha)) = \bigcup_{\alpha\in\Delta}(X \setminus i\text{-cl}(A_\alpha)) = \bigcup_{\alpha\in\Delta} i\text{-int}(X \setminus A_\alpha)$. Since $C_\alpha \subseteq A_\alpha \subseteq j\text{-int}(i\text{-cl}(A_\alpha)) \subseteq i\text{-cl}(A_\alpha)$, then $X \setminus i\text{-cl}(A_\alpha) \subseteq X \setminus j\text{-int}(i\text{-cl}(A_\alpha)) \subseteq X \setminus C_\alpha$, that is, $i\text{-int}(X \setminus A_\alpha) \subseteq j\text{-cl}(i\text{-int}(X \setminus A_\alpha)) \subseteq X \setminus C_\alpha$. Therefore,

$$X = \bigcup_{\alpha\in\Delta} i\text{-int}(X \setminus A_\alpha) \subseteq \bigcup_{\alpha\in\Delta}(X \setminus C_\alpha). \tag{3.3}$$

So $X = \bigcup_{\alpha\in\Delta}(X \setminus C_\alpha)$ and the family $\{X \setminus C_\alpha : \alpha \in \Delta\}$ is an ij-regular cover of X. Since X is ij-weakly regular-Lindelöf, there exists a countable subfamily $\{X \setminus C_{\alpha_n} : n \in \mathbb{N}\}$ such that

$$X = j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} X \setminus C_{\alpha_n}\right) = j\text{-cl}\left(X \setminus \left(\bigcap_{n\in\mathbb{N}} C_{\alpha_n}\right)\right) = X \setminus \left(j\text{-int}\left(\bigcap_{n\in\mathbb{N}} C_{\alpha_n}\right)\right). \tag{3.4}$$

Therefore, $j\text{-int}(\bigcap_{n\in\mathbb{N}} C_{\alpha_n}) = \varnothing$.

Conversely, let $\{U_\alpha : \alpha \in \Delta\}$ be an ij-regular cover of X. By Definition 3.1, for each $\alpha \in \Delta$, U_α is i-open set in X and there exists a ji-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $X = \bigcup_{\alpha\in\Delta} i\text{-int}(C_\alpha)$. The family $\{X \setminus U_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X is satisfying the condition, for each $\alpha \in \Delta$, there exists a j-open subset $X \setminus C_\alpha$ of X such that $X \setminus C_\alpha \supseteq X \setminus U_\alpha$ and

$$\bigcap_{\alpha\in\Delta} i\text{-cl}(X \setminus C_\alpha) = \bigcap_{\alpha\in\Delta}(X \setminus i\text{-int}(C_\alpha)) = X \setminus \left(\bigcup_{\alpha\in\Delta} i\text{-int}(C_\alpha)\right) = X \setminus X = \varnothing. \tag{3.5}$$

By hypothesis, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $j\text{-int}(\bigcap_{n\in\mathbb{N}}(X \setminus U_{\alpha_n})) = \varnothing$, that is, $j\text{-int}(X \setminus \bigcup_{n\in\mathbb{N}} U_{\alpha_n}) = \varnothing$. So $X \setminus j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}) = \varnothing$ and, therefore, $X = j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})$. This completes the proof. □

Corollary 3.30. *A bitopological spaces X is pairwise weakly regular-Lindelöf if and only if for every family $\{C_\alpha : \alpha \in \Delta\}$ of closed subsets of X such that for each $\alpha \in \Delta$, there exists an open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha\in\Delta}\text{cl}(A_\alpha) = \varnothing$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\text{int}(\bigcap_{n\in\mathbb{N}} C_{\alpha_n}) = \varnothing$.*

The following diagram illustrates the relationship among the generalizations of pairwise Lindelöf spaces and the generalizations of pairwise regular-Lindelöf spaces in terms of ij-:

$$\begin{array}{ccccc} ij\text{-nearly Lindelöf} & \Longrightarrow & ij\text{-almost Lindelöf} & \Longrightarrow & ij\text{-weakly Lindelöf} \\ \Downarrow & & \Downarrow & & \Downarrow \\ ij\text{-nearly} & \Longrightarrow & ij\text{-almost} & \Longrightarrow & ij\text{-weakly} \\ \text{regular-Lindelöf} & & \text{regular-Lindelöf} & & \text{regular-Lindelöf} \end{array} \tag{3.6}$$

4. Pairwise weakly regular-Lindelöf subspaces and subsets

A subset S of a bitopological space X is said to be ij-weakly regular-Lindelöf (resp., pairwise weakly regular-Lindelöf) if S is ij-weakly regular-Lindelöf (resp., pairwise weakly regular-Lindelöf) as a subspace of X, that is, S is ij-weakly regular-Lindelöf (resp., pairwise weakly regular-Lindelöf) with respect to the inducted bitopology from the bitopology of X.

Definition 4.1 (see [17]). Let S be a subset of a bitopological space X. A cover $\{U_\alpha : \alpha \in \Delta\}$ of S by i-open subsets of X such that $S \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$ is said to be ij-regular cover of S by i-open subsets of X if for each $\alpha \in \Delta$, there exists a nonempty ji-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $S \subseteq \bigcup_{\alpha\in\Delta} i\text{-int}(C_\alpha)$. $\{U_\alpha : \alpha \in \Delta\}$ is said pairwise regular cover by open subsets of X if it is both ij-regular cover of S by i-open subsets of X and ji-regular cover of S by j-open subsets of X.

Definition 4.2 (see [17]). A subset S of a bitopological space X is said to be ij-almost regular-Lindelöf relative to X if for every ij-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of S by i-open subsets of X there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $S \subseteq \bigcup_{n\in\mathbb{N}} j\text{-cl}(U_{\alpha_n})$. S is said pairwise almost regular-Lindelöf relative to X if it is both ij-almost regular-Lindelöf relative to X and ji-almost regular-Lindelöf relative to X.

Definition 4.3. A subset S of a bitopological space X is said to be ij-weakly regular-Lindelöf relative to X if for every ij-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of S by i-open subsets of X there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $S \subseteq j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})$. S is said pairwise weakly regular-Lindelöf relative to X if it is both ij-weakly regular-Lindelöf relative to X and ji-weakly regular-Lindelöf relative to X.

Obviously, every ij-weakly Lindelöf relative to the space is ij-weakly regular-Lindelöf relative to the space and every ij-almost regular-Lindelöf relative to the space is ij-weakly regular-Lindelöf relative to the space.

Question 3. Is ij-weakly regular-Lindelöf relative to the space implies ij-weakly Lindelöf relative to the space?

Question 4. Is ij-weakly regular-Lindelöf relative to the space implies ij-almost regular-Lindelöf relative to the space?

The authors expected that the answer of both questions is no.

Theorem 4.4. *A subset S of a bitopological spaces X is ij-weakly regular-Lindelöf relative to X if and only if for every family $\{C_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X such that for each $\alpha \in \Delta$ there exists a j-open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $(\bigcap_{\alpha\in\Delta} i\text{-cl}(A_\alpha)) \cap S = \varnothing$ there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $(j\text{-int}(\bigcap_{n\in\mathbb{N}} C_{\alpha_n})) \cap S = \varnothing$.*

Proof. Let $\{C_\alpha : \alpha \in \Delta\}$ be a family of i-closed subsets of X such that for each $\alpha \in \Delta$ there exists a j-open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $(\bigcap_{\alpha\in\Delta} i\text{-cl}(A_\alpha)) \cap S = \varnothing$. It follows that $S \subseteq X \setminus (\bigcap_{\alpha\in\Delta} i\text{-cl}(A_\alpha)) = \bigcup_{\alpha\in\Delta}(X \setminus i\text{-cl}(A_\alpha)) = \bigcup_{\alpha\in\Delta} i\text{-int}(X \setminus A_\alpha)$. Since $C_\alpha \subseteq A_\alpha \subseteq j\text{-int}(i\text{-cl}(A_\alpha)) \subseteq i\text{-cl}(A_\alpha)$, then $X \setminus i\text{-cl}(A_\alpha) \subseteq X \setminus j\text{-int}(i\text{-cl}(A_\alpha)) \subseteq X \setminus C_\alpha$, that is, $i\text{-int}(X \setminus A_\alpha) \subseteq j\text{-cl}(i\text{-int}(X \setminus A_\alpha)) \subseteq X \setminus C_\alpha$. Therefore, $S \subseteq \bigcup_{\alpha\in\Delta} i\text{-int}(X \setminus A_\alpha) \subseteq \bigcup_{\alpha\in\Delta}(X \setminus C_\alpha)$. So $j\text{-cl}(i\text{-int}(X \setminus A_\alpha))$ is a ji-regular closed subset of X satisfying the condition of Definition 4.1. Thus, the family $\{X \setminus C_\alpha : \alpha \in \Delta\}$ is an ij-regular cover of S by i-open subsets of X. Since X is ij-weakly regular-Lindelöf relative to X, there exists a countable subfamily $\{X \setminus C_{\alpha_n} : n \in \mathbb{N}\}$ such that

$$S \subseteq j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} (X \setminus C_{\alpha_n})\right) = j\text{-cl}\left(X \setminus \bigcap_{n\in\mathbb{N}} C_{\alpha_n}\right) = X \setminus j\text{-int}\left(\bigcap_{n\in\mathbb{N}} C_{\alpha_n}\right). \tag{4.1}$$

Therefore, $(j\text{-int}(\bigcap_{n\in\mathbb{N}} C_{\alpha_n})) \cap S = \varnothing$.

Conversely, let $\{U_\alpha : \alpha \in \Delta\}$ be an ij-regular cover of S by i-open subsets of X. By Definition 4.1, for each $\alpha \in \Delta$, there exists a ji-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $S \subseteq \bigcup_{\alpha\in\Delta} i\text{-int}(C_\alpha)$. The family $\{X \setminus U_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X is satisfying the condition, for each $\alpha \in \Delta$, there exists a j-open set $X \setminus C_\alpha \supseteq X \setminus U_\alpha$ with

$$S \subseteq \bigcup_{\alpha\in\Delta} i\text{-int}(C_\alpha) = X \setminus \left(\bigcap_{\alpha\in\Delta} X \setminus i\text{-int}(C_\alpha)\right) = X \setminus \left(\bigcap_{\alpha\in\Delta} i\text{-cl}(X \setminus C_\alpha)\right), \tag{4.2}$$

then it follows that, $(\bigcap_{\alpha\in\Delta} i\text{-cl}(X \setminus C_\alpha)) \cap S = \varnothing$. By hypothesis, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that

$$\left(j\text{-int}\left(\bigcap_{n\in\mathbb{N}}(X \setminus U_{\alpha_n})\right)\right) \cap S = \varnothing, \quad \text{that is,} \left(j\text{-int}\left(X \setminus \bigcup_{n\in\mathbb{N}} U_{\alpha_n}\right)\right) \cap S = \varnothing. \tag{4.3}$$

Thus we have, $(X \setminus j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})) \cap S = \varnothing$ and, therefore, $S \subseteq j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})$. This completes the proof. □

Corollary 4.5. *A subset S of a bitopological spaces X is pairwise weakly regular-Lindelöf relative to X if and only if for every family $\{C_\alpha : \alpha \in \Delta\}$ of closed subsets of X such that for each $\alpha \in \Delta$ there exists an open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $(\bigcap_{\alpha\in\Delta} \text{cl}(A_\alpha)) \cap S = \varnothing$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $(\text{int}(\bigcap_{n\in\mathbb{N}} C_{\alpha_n})) \cap S = \varnothing$.*

Proposition 4.6. *A subset S of a space X is ij-weakly regular-Lindelöf relative to X if and only if for every family $\{U_\alpha : \alpha \in \Delta\}$ of ij-regular open subsets of X satisfying the conditions $S \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$ and for each $\alpha \in \Delta$ there exists a nonempty ji-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $S \subseteq \bigcup_{\alpha\in\Delta} i\text{-int}(C_\alpha)$, then there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $S \subseteq j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})$.*

Proof. The necessity is obvious by the Definitions 4.1 and 4.2 since every ij-regular open set in X is i-open. For the sufficiency, let $\{U_\alpha : \alpha \in \Delta\}$ be a family of i-open sets in X satisfying the conditions of Definition 4.1 above. Then $\{i\text{-int}(j\text{-cl}(U_\alpha)) : \alpha \in \Delta\}$ is a family of ij-regular open sets in X satisfying the conditions of the theorem, since for each $\alpha \in \Delta$, we have $C_\alpha \subseteq U_\alpha \subseteq i\text{-int}(j\text{-cl}(U_\alpha))$. By hypothesis, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that

$$\begin{aligned} S &\subseteq j\text{-cl}\left(\bigcup_{n\in\mathbb{N}}(i\text{-int}(j\text{-cl}(U_{\alpha_n})))\right) \subseteq j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} j\text{-cl}(U_{\alpha_n})\right) \\ &\subseteq j\text{-cl}\left(j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}\right)\right) = j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}\right). \end{aligned} \tag{4.4}$$

This implies that S is ij-weakly regular-Lindelöf relative to X and completes the proof. □

Corollary 4.7. *A subset S of a space X is pairwise weakly regular-Lindelöf relative to X if and only if for every family $\{U_\alpha : \alpha \in \Delta\}$ of pairwise regular open subsets of X satisfying the conditions $S \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$ and for each $\alpha \in \Delta$ there exists a nonempty pairwise regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $S \subseteq \bigcup_{\alpha\in\Delta} \text{int}(C_\alpha)$, then there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $S \subseteq \text{cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})$.*

Proposition 4.8. *If $\{A_k : k \in \mathbb{N}\}$ is a countable family of subsets of a space X such that each A_k is ij-weakly regular-Lindelöf relative to X, then $\bigcup\{A_k : k \in \mathbb{N}\}$ is ij-weakly regular-Lindelöf relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an ij-regular cover of $\bigcup\{A_k : k \in \mathbb{N}\}$ by i-open subsets of X. Then for each $\alpha \in \Delta$, there exists a nonempty ji-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $\bigcup_{k\in\mathbb{N}} A_k \subseteq \bigcup_{\alpha\in\Delta} i\text{-int}(C_\alpha)$. Let $\Delta_k = \{\alpha \in \Delta : U_\alpha \cap A_k \neq \varnothing\}$, then for each $\alpha_k \in \Delta_k \subseteq \Delta$ there exists a nonempty ji-regular closed subset C_{α_k} of X such that $C_{\alpha_k} \subseteq U_{\alpha_k}$ and $A_k \subseteq \bigcup_{\alpha_k\in\Delta_k} i\text{-int}(C_{\alpha_k})$. So $\{U_{\alpha_k} : \alpha_k \in \Delta_k\}$ is an ij-regular cover of A_k by i-open subsets of X. Since A_k is ij-weakly regular-Lindelöf relative to X, there exists a countable subfamily $\{U_{\alpha_{k_n}} : n \in \mathbb{N}\}$ such that $A_k \subseteq j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_{k_n}})$. But a countable union of countable sets is countable, so

$$\bigcup_{k\in\mathbb{N}} A_k \subseteq \bigcup_{k\in\mathbb{N}} \left(j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} U_{\alpha_{k_n}} \right) \right) \subseteq j\text{-cl}\left(\bigcup_{k\in\mathbb{N}} \left(\bigcup_{n\in\mathbb{N}} U_{\alpha_{k_n}} \right) \right) = j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} U_{\alpha_{k_n}} \right). \tag{4.5}$$

This implies that $\bigcup\{A_k : k \in \mathbb{N}\}$ is ij-weakly regular-Lindelöf relative to X and completes the proof. □

Corollary 4.9. *If $\{A_k : k \in \mathbb{N}\}$ is a countable family of subsets of a space X such that each A_k is pairwise weakly regular-Lindelöf relative to X, then $\bigcup\{A_k : k \in \mathbb{N}\}$ is pairwise weakly regular-Lindelöf relative to X.*

Proposition 4.10. *If S is an ij-weakly regular-Lindelöf subspace of a bitopological space X, then S is ij-weakly regular-Lindelöf relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an ij-regular cover of S by i-open subsets of X. Then, for each $\alpha \in \Delta$ there exists a nonempty ji-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $S \subseteq \bigcup_{\alpha\in\Delta} i\text{-int}_X(C_\alpha)$. For each $\alpha \in \Delta$, we have $i\text{-int}_X(C_\alpha) \cap S$ and $U_\alpha \cap S$ are i-open sets in S, and $C_\alpha \cap S$ is j-closed set in S. Since for each $\alpha \in \Delta$, there exists a ji-regular closed set $j\text{-cl}_S(i\text{-int}_X(C_\alpha) \cap S)$ in S such that $j\text{-cl}_S(i\text{-int}_X(C_\alpha) \cap S) \subseteq C_\alpha \cap S \subseteq U_\alpha \cap S$ and

$$S = \left(\bigcup_{\alpha\in\Delta} i\text{-}\underset{X}{\text{int}}(C_\alpha) \right) \cap S = \bigcup_{\alpha\in\Delta} \left(i\text{-}\underset{X}{\text{int}}(C_\alpha) \cap S \right) \subseteq \bigcup_{\alpha\in\Delta} i\text{-}\underset{S}{\text{int}}\left(j\text{-}\underset{S}{\text{cl}}\left(i\text{-}\underset{X}{\text{int}}(C_\alpha) \cap S \right) \right), \tag{4.6}$$

that is, $S = \bigcup_{\alpha\in\Delta} i\text{-int}_S(j\text{-cl}_S(i\text{-int}_X(C_\alpha) \cap S))$, then the family $\{U_\alpha \cap S : \alpha \in \Delta\}$ is an ij-regular cover of S. Since S is an ij-weakly regular-Lindelöf subspace of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that

$$S = j\text{-}\underset{S}{\text{cl}}\left(\bigcup_{n\in\mathbb{N}} \left(U_{\alpha_n} \cap S \right) \right) = \left(j\text{-}\underset{X}{\text{cl}}\left(\bigcup_{n\in\mathbb{N}} \left(U_{\alpha_n} \cap S \right) \right) \right) \cap S \subseteq j\text{-}\underset{X}{\text{cl}}\left(\bigcup_{n\in\mathbb{N}} U_{\alpha_n} \right). \tag{4.7}$$

This shows that S is ij-weakly regular-Lindelöf relative to X. □

Corollary 4.11. *If S is a pairwise weakly regular-Lindelöf subspace of a bitopological space X, then S is pairwise weakly regular-Lindelöf relative to X.*

Question 5. Is the converse of Proposition 4.10 above true?

The authors expected that the answer is no.

Theorem 4.12. *If every ij-regular closed proper subset of a bitopological space X is ij-weakly regular-Lindelöf relative to X, then X is ij-weakly regular-Lindelöf.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an ij-regular cover of X. For each $\alpha \in \Delta$, there exists a nonempty ji-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $X = \bigcup_{\alpha\in\Delta} i\text{-int}(C_\alpha)$. Fix an arbitrary $\alpha_0 \in \Delta$ and let $\Delta^* = \Delta \setminus \{\alpha_0\}$. Put $K = X \setminus (i\text{-int}(C_{\alpha_0}))$, then K is an ij-regular closed subset of X and $K \subseteq \bigcup_{\alpha\in\Delta^*} i\text{-int}(C_\alpha)$. Therefore, $\{U_\alpha : \alpha \in \Delta^*\}$ is an ij-regular cover of K by i-open subsets of X by Definition 4.1. By hypothesis, K is ij-weakly regular-Lindelöf relative to X, hence there exists a countable subset $\{\alpha_n : n \in \mathbb{N}^*\}$ of Δ^* such that $K \subseteq j\text{-cl}(\bigcup_{n\in\mathbb{N}^*} U_{\alpha_n})$. So, we have

$$\begin{aligned} X &= K \cup (i\text{-int}(C_{\alpha_0})) \subseteq K \cup (j\text{-cl}(U_{\alpha_0})) \subseteq \left(j\text{-cl}\left(\bigcup_{n\in\mathbb{N}^*} U_{\alpha_n} \right) \right) \cup (j\text{-cl}(U_{\alpha_0})) \\ &= \bigcup_{n\in\mathbb{N}} j\text{-cl}(U_{\alpha_n}). \end{aligned} \tag{4.8}$$

So $X = j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})$ and this shows that X is ij-weakly regular-Lindelöf. □

Corollary 4.13. *If every pairwise regular closed proper subset of a bitopological space X is pairwise weakly regular-Lindelöf relative to X, then X is pairwise weakly regular-Lindelöf.*

It is very clear that Theorem 4.12 implies the following corollaries.

Corollary 4.14. *If every ij-regular closed subset of a bitopological space X is ij-weakly regular-Lindelöf relative to X, then X is ij-weakly regular-Lindelöf.*

Corollary 4.15. *If every pairwise regular closed subset of a bitopological space X is pairwise weakly regular-Lindelöf relative to X, then X is pairwise weakly regular-Lindelöf.*

Note that, the space X in above propositions is any bitopological space. If we consider X itself is an ij-weakly regular-Lindelöf, we have the following results.

Theorem 4.16. *Let X be an ij-weakly regular-Lindelöf space. If A is a proper ij-clopen subset of X, then A is ij-weakly regular-Lindelöf relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an ij-regular cover of A by i-open subsets of X. Hence the family $\{U_\alpha : \alpha \in \Delta\} \cup \{X \setminus A\}$ is an ij-regular cover of X since $X \setminus A$ is a proper ji-clopen subset of X is also a ji-regular closed subset of X. Since X is ij-weakly regular-Lindelöf, there exists a countable subfamily $\{X \setminus A, U_{\alpha_1}, U_{\alpha_2}, \dots\}$ such that

$$X = j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} U_{\alpha_n} \right) \cup j\text{-cl}(X \setminus A) = \left(j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} U_{\alpha_n} \right) \right) \cup (X \setminus A). \tag{4.9}$$

But A and $X \setminus A$ are disjoint; therefore, we have $A \subseteq j\text{-cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})$. This completes the proof. □

Corollary 4.17. *Let X be a pairwise weakly regular-Lindelöf space. If A is a proper clopen subset of X, then A is pairwise weakly regular-Lindelöf relative to X.*

It is very clear that Theorem 4.16 implies the following corollary.

Corollary 4.18. *Let X be an ij-weakly regular-Lindelöf space. If A is an ij-clopen subset of X, then A is ij-weakly regular-Lindelöf relative to X.*

Corollary 4.19. *Let X be a pairwise weakly regular-Lindelöf space. If A is a clopen subset of X, then A is pairwise weakly regular-Lindelöf relative to X.*

Question 6. Is i-closed subspace of an ij-weakly regular-Lindelöf space X ij-weakly regular-Lindelöf?

Question 7. Is ij-regular closed subspace of an ij-weakly regular-Lindelöf space X ij-weakly regular-Lindelöf?

The authors expected that the answer of both questions is no. Observe that the condition in Theorem 4.16 that a subset should be ij-clopen is necessary and it is not sufficient to be only i-open or ij-regular open as example below shows. Arbitrary subspaces of ij-weakly regular-Lindelöf spaces need not be ij-weakly regular-Lindelöf nor ij-weakly regular-Lindelöf relative to the spaces. An i-open or ij-regular open subset of an ij-weakly regular-Lindelöf space is neither ij-weakly regular-Lindelöf nor ij-weakly regular-Lindelöf relative to the spaces as in the following example also show. We need the following lemma (see [20, page 11]).

Lemma 4.20. *If A is a countable subset of ordinals Ω not containing ω_1, where ω_1 being the first uncountable ordinal, then* $\sup A < \omega_1$.

Example 4.21. Let Ω denote the set of ordinals which are less than or equal to the first uncountable ordinal number ω_1, that is, $\Omega = [1, \omega_1]$. This Ω is an uncountable well-ordered set with a largest element ω_1, having the property that if $\alpha \in \Omega$ with $\alpha < \omega_1$, then $\{\beta \in \Omega : \beta \leq \alpha\}$ is countable. Since Ω is a totally ordered space, it can be provided with its order topology. Let us denote this order topology by τ_1. Choose discrete topology as another topology for Ω denoted by τ_2. So (Ω, τ_1, τ_2) form a bitopological space. Now it is known that Ω is a 1-Lindelöf space [20], so it is 12-weakly Lindelöf and thus 12-weakly regular-Lindelöf. The subspace $\Omega_0 = \Omega \setminus \{\omega_1\} = [1, \omega_1)$, however, is not 1-Lindelöf (see [20]). We notice that Ω_0 is 1-open subspace of Ω and also 12-regular open subset of Ω. Observe that Ω_0 is not 12-weakly regular-Lindelöf by Corollary 3.16 since it is 12-regular and 12-weak P-space. Moreover, Ω_0 is not 12-weakly regular-Lindelöf relative to Ω. In fact, the family $\{[1, \alpha) : \alpha \in \Omega_0\}$ of 1-open sets in Ω is 12-regular cover of Ω_0 by 1-open subsets of Ω because $\Omega_0 \subseteq \bigcup_{\alpha \in \Omega_0} [1, \alpha)$ and for each $\alpha \in \Omega_0$, there exists a nonempty 21-regular closed subset $[1, \alpha)$ of Ω such that $[1, \alpha) \subseteq [1, \alpha)$ and $\Omega_0 \subseteq \bigcup_{\alpha \in \Omega_0} [1, \alpha) = \bigcup_{\alpha \in \Omega_0} 1\text{-int}([1, \alpha))$. But the family $\{[1, \alpha) : \alpha \in \Omega_0\}$ has no countable subfamily $\{[1, \alpha_n) : n \in \mathbb{N}\}$ such that $\Omega_0 \subseteq 2\text{-cl}(\bigcup_{n \in \mathbb{N}} [1, \alpha_n)) = \bigcup_{n \in \mathbb{N}} [1, \alpha_n)$. For if $\{[1, \alpha_1), [1, \alpha_2), \ldots\}$ satisfy the condition: 2-closures of unions of it elements cover Ω_0, then $\sup\{\alpha_1, \alpha_2, \ldots\} = \omega_1$ which is impossible by Lemma 4.20.

So we can conclude that an ij-weakly regular-Lindelöf property is not hereditary property and, therefore, pairwise weakly regular-Lindelöf property is not so.

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