PACIFIC JOURNAL OF MATHEMATICS
Vol. 21, No. 3, 1967

INVARIANT MEANS AND THE STONE-ČECH COMPACTIFICATION

CARROLL WILDE AND KLAUS WITZ

In the first part of this paper the Arens multiplication on a space of bounded functions is used to simplify and extend results by Day and Frey on amenability of subsemigroups and ideals of a semigroup. For example it is shown that if S is a left amenable cancellation semigroup then a subsemigroup A of S is left amenable if and only if each two right ideals of A intersect. The remainder and major portion of this paper is devoted to relations between left invariant means on $m(S)$ and left ideals of βS (=the Stone-Čech compactification of S). We find: If μ is a left invariant mean on $m(S)$ and if S has left cancellation then $\mathscr{S}(\mu)$, the support of μ considered as a Borel measure on $\beta(S)$, is a left ideal of $\beta(S)$. An application is that if S is a left amenable semigroup and I is a left ideal of βS, then $K(I)$, the w^*-closed convex hull of I, contains an extreme left invariant mean; if in addition S has cancellation then $K(I)$ contains a left invariant mean which is the w^*-limit of a net of unweighted finite averages.

2. Preliminaries. For standard notation and terminology we follow Day [2] in functional analysis and Kelley [6] in topology. Specific terms and notation in amenable semigroups follow Day [1].

Let S be any set, and let $m(S)$ be the Banach space of all bounded, real-valued functions on S, equipped with the supremum norm. A *mean* on $m(S)$ is a positive linear functional on $m(S)$ which has norm one; every mean μ satisfies $\mu(e) = 1$, where e is the function which is identically one on S. We denote by $M(=M(S))$ the set of all means on $m(S)$. Then M is a nonempty, convex subset of $m(S)^*$; it is also compact in the w^*-topology, the only topology we consider in $m(S)^*$.

For each $s \in S$, $q(s)$ denotes the evaluation functional at s:

$$qs(x) = x(s) \, (x \in m(S)) .$$

We have $qs \in M (s \in S)$ and βS, the Stone-Čech compactification of the discrete space S, coincides with the (w^*-) closure of qS in $m(S)^*$, so that $\beta S \subseteq M$. The symbols $k(T), K(T)$ will always indicate the convex hull, resp. the (w^*-) closed convex hull, of any subset T of $m(S)^*$; in particular, we write kA for $k(qA)$ and KA for $K(qA)$ when $A \subseteq S$. Then we have $M = KS = K(\beta S)$.

Now suppose S is a semigroup. Then each $s \in S$ determines two mappings, l_s and r_s, on $m(S)$ defined by $l_s x(t) = x(st)$ and $r_s x(t) = x(ts) \, (t \in S, x \in m(S))$. A mean μ on $m(S)$ is *left* [*right*] *invariant* if