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NORMAL EXPECTATIONS IN VON NEUMANN ALGEBRAS

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Let h and k be two Hilbert spaces, $h \otimes k$ will denote the tensor product of h and k. Let $\mathscr{A}$ be a von Neumann algebra acting on h. Let ψ be an ampliation of $\mathscr{A}$ in $h \otimes k$, i.e., ψ is a map of $\mathscr{A}$ into bounded linear operators of $h \otimes k$ and $\psi(\mathscr{A}) = \mathscr{A} \otimes I_k$ (I_k is the identity map on k). Let $\tilde{\mathscr{A}}$ be the image of $\mathscr{A}$ by ψ.

The purpose of this paper is to prove the following result: If $\mathscr{B}$ is a subalgebra of $\mathscr{A}$ and if $\mathscr{B}$ is the range of a normal expectation φ defined on $\mathscr{A}$, then there exists an ampliation of $\mathscr{A}$ in $h \otimes k$, independent of $\mathscr{B}$ and of φ, such that $\varphi \otimes I_k$ is a spatial isomorphism of $\tilde{\mathscr{A}}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be two C^* algebras with identity. Suppose $\mathscr{B} \subset \mathscr{A}$. Let φ be a positive linear map of $\mathscr{A}$ on $\mathscr{B}$ such that φ preserves the identity and such that $\varphi(BX) = B\varphi(X)$ for all B in $\mathscr{B}$ and all X in $\mathscr{A}$. φ is then defined to be an expectation of $\mathscr{A}$ on $\mathscr{B}$. The extension of the notion of an expectation in the probability theory sense, to expectations on finite von Neumann algebra is largely due to J. Dixmier and H. Umegaki [1]. In [4] Tomiyama considers an expectation on von Neumann algebras to be a projection of norm one. If φ is an expectation in the sense $\varphi(BX) = B\varphi(X)$, φ positive and φ preserves identities, then $\varphi(XB) = \varphi(X)B$ for all X in $\mathscr{A}$, B in $\mathscr{B}$. $\mathscr{B}$ is the set of fixed points of φ. By writing $\varphi[(X - \varphi(X))^* (X - \varphi(x))] \geqq 0$ we have $\varphi(X^*X) \geqq \varphi(X)^*\varphi(X)$. In particular φ is a bounded map. The result stated in the previous paragraph extends a result by Nakamura, Takesaki, and Umegaki [2], who consider the case when $\mathscr{A}$ is a finite von Neumann algebra.

2. Preliminaries.

Basic definitions and some essentially known results will now be given for ready reference. Let M and N be C^* algebras and φ a positive linear map of M on N. Let M_n be the set of all $n \times n$ matrices whose entries are elements of M, call those entries $A_{i,j}$. Define for each n, $\varphi^{(n)}(A_{i,j}) = (\varphi(A_{i,j}))$; φ^n is then a map of M_n on N_n. φ is called *completely positive* if each φ^n is.

Let $\mathscr{A}$ and $\mathscr{B}$ be two von Neumann algebras, with $\mathscr{B} \subset \mathscr{A}$. Let φ be an expectation of $\mathscr{A}$ on $\mathscr{B}$. φ is called *faithful* if for any T in $\mathscr{A}$, $\varphi(TT^*) = 0$ implies $T = 0$. Let A_α be a net of uniformly bounded self adjoint operators in $\mathscr{A}$. φ is called *normal* if

$$\sup_\alpha \varphi(A_\alpha) = \varphi(\sup_\alpha A_\alpha) .$$