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ON THE MULTIPLICATIVE PROPERTIES OF ARITHMETIC FUNCTIONS

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In this paper we define a generalization of the set $\mathscr{M}$ of all pairs of relatively prime natural numbers and then define a limit process to measure the multiplicativity of an arithmetic function with respect to this generalized set of pairs. In so doing we gain useful information about that most important special case, namely, functions which are multiplicative in the usual sense.

2. **Preliminary definitions and results.** By an *arithmetic function* we shall mean a real-valued function f whose domain is the set of natural numbers. We will deal only with arithmetic functions and, furthermore, we will assume throughout this paper that no function is eventually zero; that is, given any arithmetic function f and any number N, there is a natural number $k \geqq N$ such that $f(k) \neq 0$.

Closely connected with the multiplicative properties of an arithmetic function is the concept of a basic sequence, which is defined as follows: A *basic sequence* is a set $\mathscr{B}$ of pairs of natural numbers (a, b) with the following three properties:

(i) If $(a, b) \in \mathscr{B}$, then $(b, a) \in \mathscr{B}$;

(ii) $(a, bc) \in \mathscr{B}$ if and only if $(a, b) \in \mathscr{B}$ and $(a, c) \in \mathscr{B}$;

(iii) $(1, k) \in \mathscr{B}$, $k = 1, 2, 3, \cdots$.

We denote by $B_k (k = 1, 2, \cdots)$ the set of pairs $(a, b) \in \mathscr{B}$ such that $ab = k$.

Let an arithmetic function f and a basic sequence $\mathscr{B}$ be given. In order to measure the multiplicativity of f with respect to $\mathscr{B}$ we first define

$$\alpha_f(m, n) = \begin{cases} \dfrac{f(m)f(n) - f(mn)}{|f(m)f(n)| + |f(mn)|} & \text{if } \ |f(m)f(n)| + |f(mn)| > 0 \,, \\ 0 & \text{if } \ f(m)f(n) = f(mn) = 0 \,. \end{cases}$$

Next we set

$$\begin{aligned} \bar{\alpha}(k; f, \mathscr{B}) &= \max \{\alpha_f(m, n) \mid (m, n) \in B_k\} \,, \\ \underline{\alpha}(k; f, \mathscr{B}) &= \min \{\alpha_f(m, n) \mid (m, n) \in B_k\} \,, \\ \bar{\alpha}(f, \mathscr{B}) &= \limsup_{k \to \infty} \bar{\alpha}(k; f, \mathscr{B}) \,, \\ \underline{\alpha}(f, \mathscr{B}) &= \liminf_{k \to \infty} \underline{\alpha}(k; f, \mathscr{B}) \,. \end{aligned} \tag{2.1}$$

Finally, we define the *index of multiplicativity* of f *with respect*