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A NEW PROOF OF THE MAXIMUM PRINCIPLE FOR DOUBLY-HARMONIC FUNCTIONS

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Let f be a real-valued Lebesque integrable function on a domain Ω in Euclidean space E_{2m}, and let f be doubly-harmonic on Ω so that it satisfies

$$\frac{\partial^2 f}{\partial x_{2k-1}^2} + \frac{\partial^2 f}{\partial x_{2k}^2} = 0 \qquad \text{for } k=1, 2, \cdots, m .$$

In this paper, a new proof of the maximum principle is given for nonconstant functions f satisfying the preceding conditions.

The proof depends on the fact that the associated forms

$$\varphi_p(H; f) = \sum_{r_1+\cdots+r_n=p} \frac{h_1^{r_1}\cdots h_n^{r_n}}{r_1!\cdots r_n!}\left(\frac{\partial^p f}{\partial x_1^{r_1}\cdots \partial x_n^{r_n}}\right)_{x=A},$$

where $A\in\Omega$, are either indefinite or identically 0 for each $p\geqq 1$. The authors previously proved this under weaker hypotheses on f, but the proof used the strong form of the maximum principle for solutions of linear elliptic partial differential equations of the second order with constant coefficients. By means of the theory of distributions, the authors now prove that the $\varphi_p(H; f)$ have the stated property without using the maximum principle. Consequently, they obtain a new proof of this principle.

1. Introduction. We say that f is *doubly-harmonic* on a domain $\Omega \subset E_{2m}$ if it is a real-valued function defined on Ω such that the equations

$$\frac{\partial^2 f}{\partial x_{2k-1}^2} + \frac{\partial^2 f}{\partial x_{2k}^2} = 0 , \qquad k = 1, 2, \cdots, m \tag{1}$$

hold for all $(x_1, \cdots, x_{2m}) \in \Omega$. Such a function f is necessarily harmonic on Ω since on adding the m equations (1) we see that f satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_{2m-1}^2} + \frac{\partial^2 f}{\partial x_{2m}^2} = 0 . \tag{2}$$

Moreover, the class of doubly-harmonic functions contains each function that is the real part of a function of m complex variables which is holomorphic on Ω; this can be seen from the Cauchy-Riemann equations applied to each complex variable $x_{2k-1} + ix_{2k}$ separately. Obviously, if $m = 1$, the class of doubly-harmonic functions coincides with the class of harmonic functions.

2. Two lemmas. Throughout, we use the notation of our earlier