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A NOTE ON THE ATIYAH-BOTT FIXED POINT FORMULA

L. M. SIBNER and R. J. SIBNER

Let f be a holomorphic self map of a compact complex analytic manifold X. The differential of f commutes with $\bar{\partial}$ and, hence, induces an endomorphism of the $\bar{\partial}$-complex of X. If f has isolated simple fixed points, the Lefschetz formula of Atiyah-Bott expresses the Lefschetz number of this endomorphism in terms of local data involving only the map f near the fixed points. For example, if X is a curve, this Lefschetz number is the sum of the residues of $(z-f(z))^{-1}$ at the fixed points.

Using a well-known technique of Atiyah-Bott for computing trace formulas, we shall, in this note, give a direct analytic derivation of the Lefschetz number as a residue formula. The formula is valid for holomorphic maps having isolated, but not necessarily simple fixed points.

1. Let E be the $\bar{\partial}$-complex of a compact complex analytic manifold X of dimension n.

$$E: 0 \longrightarrow \Gamma(\Lambda^{0,0}) \xrightarrow{\bar{\partial}} \Gamma(\Lambda^{0,1}) \longrightarrow \cdots \xrightarrow{\partial} \Gamma(\Lambda^{0,n}) \longrightarrow 0 .$$

Since E is elliptic, $H^i(X) = \ker \bar{\partial}_i / im \bar{\partial}_{i-1}$ is finite dimensional. Denote by $T = \{T_i\}$ the endomorphism induced on E by the holomorphic map f, and by $H^j T$ the resulting endomorphism on $H^i(X)$.

The Lefschetz number of f is then defined by

$$L(f) = \sum_{i=0}^{n} (-1)^i tr H^i T$$

and the finite dimensionality of the spaces $H^i(X)$ insures that this number is finite.

The Atiyah-Bott method of computing trace formulas reduces the problem of calculating $L(f)$ to that of finding a good parametrix for the $\bar{\partial}$-operator. In fact, let us suppose we can find operators $P_i: \Gamma(\Lambda^{0,i}) \rightarrow \Gamma(\Lambda^{0,i-1})$, $i = 1, \cdots, n$, having the property that

$$P_{i+1}\bar{\partial}_i + \bar{\partial}_{i-1}P_i = I - S_i \tag{1}$$

where $S_i: \Gamma(\Lambda^{0,i}) \rightarrow \Gamma(\Lambda^{0,i})$ are integral operators with sufficiently smooth kernels. Observe that if $\omega \in \Gamma(\Lambda^{0,i})$ is in the kernel of $\bar{\partial}_i$, then the left-hand side of (1) is a co-boundary. Hence, $H^i I - H^i S$ is the zero-endomorphism on homology. Similarly, since T commutes