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PROBABILITIES OF WIENER PATHS CROSSING DIFFERENTIABLE CURVES

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Let $\{W(t); t \geqq 0\}$ be the standard Wiener process. The probabilities $P[\sup_{0 \leqq t \leqq T} W(t) \geqq b]$ and $P[\sup_{0 \leqq t \leqq T} W(t) - at \geqq b]$ are well known. This paper gives the probabilities of the type $P[\sup_{0 \leqq t \leqq T} W(t) - f(t) \geqq b]$ for a large class of differentiable functions $f(t)$ by the use of integral equation techniques.

1. Introduction. Let $\{W(t), t \geqq 0\}$ be the standard Wiener process such that (i) $P[W(0) = 0] = 1$, (ii) $EW(t) = 0$ for all $t \geqq 0$, and (iii) $\mathrm{Cov}\,[W(s), W(t)] = \min\,(s, t)$. It is well known that for $b \geqq 0$

$$P[\sup_{0 \leqq t \leqq T} W(t) \geqq b] = 2P[W(T) \geqq b] = 2\Psi(bT^{-1/2}) \tag{1.1}$$

where

$$\Psi(x) = (2\pi)^{-1/2} \int_x^\infty \exp\,(-u^2/2)du\ ,$$

and that

$$\begin{aligned} &P[\sup_{0 \leqq t \leqq T} W(t) - at \geqq b] \\ &= \Psi[(aT + b)T^{-1/2}] + \exp\,(-2ab)\Phi[(aT - b)T^{-1/2}]\ , \end{aligned} \tag{1.2}$$

where $\Phi(x) = 1 - \Psi(x)$.

The identity (1.1) can be found in [2:392], [5:286], and [11:256] while the identity (1.2) can be found in [6], [7:348-349], and [9:80-82]. Doob [3:397-399] gives a very interesting proof of (1.2) for $T = \infty$ case only. Shepp's proof for (1.2) is based on his transformation theorem in [7]. Cameron-Martin translation theorem in [1] also gives the same result using Shepp's argument.

The main purpose of this paper is to find the probability $P[\sup_{0 \leqq t \leqq T} W(t) - f(t) \geqq b]$ for a large class of functions $f(t)$ differentiable in $(0, T]$, which is a generalization of the results (1.1) and (1.2). Durbin [4] gave an integral equation whose solution would be the required probability. However, it turned out to be that his integral equation could not be solved analytically, and hence he presented a numerical approximation method. After that Smith [8] introduced some new techniques to obtain an approximation for the probability. The present authors' integral equation gives explicit expression for the solution, while Durbin's and Smith's do not.

2. Statement of the result and proof.

THEOREM. *For each $T > 0$ let $f(t)$ be continuous on* $[0, T]$,