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DERIVATIONS OF C^*-ALGEBRAS HAVE SEMI-CONTINUOUS GENERATORS

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For each derivation δ of a C^*-algebra A with $\delta(x^*) = -\delta(x)^*$ there exists a minimal positive element h in the enveloping von Neumann algebra A'' such that $\delta(x) = hx - xh$. It is shown that the generator h belongs to the class of lower semi-continuous elements in A''. From this it follows that if the function $\pi \to \|\pi \circ \delta\|$ is continuous on the spectrum of A then h multiplies A. This immediately implies that each derivation of a simple C^*-algebra is given by a multiplier of the algebra. Another application shows that each derivation of a countably generated monotone sequentially closed C^*-algebra is inner.

A linear operator δ on a C^*-algebra A is called a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ for all a and b in A. If $\delta^* = -\delta$ (i.e., $\delta(a)^* = -\delta(a^*)$) then $\alpha_t(a) = \exp(it\delta)a$ defines a norm-continuous one-parameter group of $*$-automorphisms of A. Conversely, each such group can be written as $\exp(it\delta)$ for a suitable derivation δ of A. After a number of partial results, notably by I. Kaplansky and R. V. Kadison, it was proved by S. Sakai that every derivation of a von Neumann algebra A is inner, i.e., $\delta(a) = ha - ah$ for some h in A (see [9, III.9.3. Théorème 1]). Recently W. B. Arveson ([3])—see also [4]—gave a new proof of this result, using the theory of spectral subspaces associated with a one-parameter group of automorphisms. The powerful techniques developed in [3] enabled the first author to show that each derivation of an AW^*-algebra is inner ([12]).

In this paper we use Arveson's technique to show that if δ is a derivation of a C^*-algebra A with $\delta^* = -\delta$ then the minimal positive generator for δ, or rather for its extension to a derivation of the enveloping von Neumann algebra A'' of A, is the limit of an increasing net of self-adjoint operators from $\tilde{A}$. This shows that the function $\pi \to \|\pi \circ \delta\|$ on the spectrum $\hat{A}$ of A is lower semi-continuous and that it is continuous if and only if the minimal positive generators for δ and $-\delta$ both multiplies A. This last result was first proved in [2] and has as an immediate consequence that every derivation of a simple C^*-algebra is given by a multiplier ([17]). We finally show that every derivation of a countably generated monotone sequentially closed C^*-algebra is inner.

The possibility of using [12] to show that derivations of C^*-algebras have measurable generators was pointed out to us by E. B. Davies.