PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 2, 1974

ON THE IMPOSSIBILITY OF OBTAINING $S^2 \times S^1$ BY ELEMENTARY SURGERY ALONG A KNOT

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Elementary surgery along a knot has been used in an attempt to construct a counterexample to the Poincaré Conjecture. Certain classes of knots have been examined, but no counterexample has yet been found. Another, and perhaps as interesting a question, is whether $S^2 \times S^1$ can be obtained by elementary surgery along a knot. In this paper the question is answered in the negative for knots with nontrivial Alexander polynomial, for composite knots, and for a large class of knots with trivial Alexander polynomial—the simply doubled knots.

By a knot we will mean a polygonal simple closed curve in the 3-sphere S^3. A solid torus T is a 3-manifold homeomorphic to $S^1 \times D^2$. The boundary of T is a torus, a 2-manifold homeomorphic to $S^1 \times S^1$. A meridian of T is a simple closed curve on Bd T which bounds a disk in T but is not homologous to zero on Bd T. A meridianal disk of T is a disk D in T such that $D \cap \mathrm{Bd}\, T = \mathrm{Bd}\, D$, and $\mathrm{Bd}\, D$ is a meridian of T. A longitude of T is a simple closed curve on Bd T which is transverse to a meridian of T and is null-homologous in $\overline{S^3 - T}$.

The basic construction, elementary surgery along a knot, is now described: Let N be a regular neighborhood of a knot K, m an oriented meridianal curve on Bd N, and l an oriented curve on Bd N which is transverse to m and bounds an orientable surface in $\overline{S^3 - N}$. Let T be a solid torus and let $h: T \rightarrow N$ be a homeomorphism. Then S^3 is homeomorphic to $\overline{S^3 - N} \cup_{h|\mathrm{Bd}\, T} T$. Now let $h_1: \mathrm{Bd}\, T \rightarrow \mathrm{Bd}\, N$ be a homeomorphism with the property that $h^{-1} \cdot h_1: \mathrm{Bd}\, T \rightarrow \mathrm{Bd}\, T$ does not extend to a homeomorphism of T onto T. Let $M^3 = \overline{S^3 - N} \cup_{h_1} T$, then we say that M^3 is obtained from S^3 by performing an elementary surgery along K.

Consider now the fundamental group of the complement of the knot $\pi_1(\overline{S^3 - N})$ with base point $m \cap l$, where m and l are considered as elements of $\pi_1(\overline{S^3 - N}) = G$. Then the coset $\bar{m} = mG'$ generates the commutator quotient group $G/G' = H_1(\overline{S^3 - N})$, and the longitude l is in the second commutator subgroup G''. The fundamental group of M^3 is obtained by adjoining the relation $l^p = m^q$ to $\pi_1(\overline{S^3 - N})$ where $pl - qm$ is the image under h_1 of the boundary of a meridianal disk of T, p and q are relatively prime, and $p > 0$. The first homology group of M^3 is generated by $\bar{m}$ with the relation $\bar{m}^q = 1$.