PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 2, 1974

ARITHMETIC PROPERTIES OF CERTAIN RECURSIVELY DEFINED SETS

D. A. KLARNER AND R. RADO

Let R denote a set of linear operations defined on the set N of nonnegative integers; for example, a typical element of R has the form $\rho(x_1, \cdots, x_r) = m_0 + m_1x_1 + \cdots + m_rx_r$ where $m_0, \cdots, m_r$ denote certain integers. Given a set A of positive integers, there is a smallest set of positive integers denoted $\langle R: A\rangle$ which contains A as a subset and is closed under every operation in R. The set $\langle R: A\rangle$ can be constructed recursively as follows: Let $A_0 = A$, and define

$$A_{k+1} = A_k \cup \{\rho(\bar{a}): \rho \in R, \bar{a} \in A_k \times \cdots \times A_k\} \quad (k = 0, 1, \cdots),$$

then it can be shown that $\langle R: A\rangle = A_0 \cup A_1 \cup \cdots$. The sets $\langle R: A\rangle$ sometimes have an elegant form, for example, the set $\langle 2x + 3y: 1\rangle$ consists of all positive numbers congruent to 1 or 5 modulo 12. The objective is to give an arithmetic characterization of elements of a set $\langle R: A\rangle$. This paper is a report on progress made on this problem when the authors collaborated at Reading University in the academic year 1970-71.

Many of the questions left open here have since been resolved; see [2]. We start with a review of certain notions from universal algebra which are going to be used in the precise formulation of our problems. We would like to point out at the outset that only the language and very little of the theory of universal algebra seem to enter our work.

Consider a set R of finitary operations defined on a set X, and suppose A is a subset of X. It can be shown that there is a "smallest" set $\langle R: A\rangle$ with $A \subseteqq \langle R: A\rangle \subseteqq X$ such that $\langle R: A\rangle$ is closed under all operations in R. This is a rough version of the "definition from above" of the set $\langle R: A\rangle$. However, there is an alternative "definition from below" which involves iteration of the operations in R. We define a sequence of sets $A_0, A_1, \cdots$ recursively so that $A = A_0 \subseteqq A_1 \subseteqq \cdots$ and $A_0 \cup A_1 \cup \cdots = \langle R: A\rangle$.

Even though we have a constructive definition of $\langle R: A\rangle$ it is often very difficult to decide whether a given element x of X is an element of $\langle R: A\rangle$. Such a situation may lead to a search for a simple characterization of the elements of $\langle R: A\rangle$ which avoids the recursive construction. In general, we seek an arithmetic characterization of sets $\langle R: A\rangle$ of natural numbers where R is a finite set of finitary linear operations defined on the set of natural numbers, and A is a finite set of natural numbers.