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# CHARACTERIZATION OF A FUNCTION BY CERTAIN INFINITE SERIES IT GENERATES

CHARLES K. CHUI AND PHILIP W. SMITH

**Let $A$ be a set of real numbers and $F$ be a class of complex-valued functions defined on the real line such that for each $f \in F$ the infinite series $S(x, f) = \sum_{k=1}^{\infty} f(kx)$ converges for every nonzero $x$ in $A$. If $0 \in A$, we set $S(0, f) = f(0)$. It seems to be an interesting problem to study the different sets $A$ and function classes $F$ such that each $f \in F$ is uniquely determined by the sums $S(x, f)$ where $x \in A$. Clearly, the larger the class $F$ is studied, the larger set $A$ is needed to guarantee uniqueness. We have positive results for a class of entire functions of exponential type and for fairly large classes of continuous functions. Some examples are also given to show that in general $A$ cannot be too small.**

**1. Introduction.** For a function $f$ holomorphic in the open unit disc $U$ of the complex plane and continuous on the closure of $U$, let

$$s_n(f, \delta) = \frac{1}{n} \sum_{k=1}^{n} f(e^{i2\pi\delta k/n}) , \qquad n = 1, 2, \cdots, 0 < \delta \leqq 1 .$$

Sufficient conditions on the function $f$ were given in [2, 5] to guarantee that $f$ is uniquely determined by the means $s_n(f, 1)$, and in [3] both positive and negative results were given for the case $0 < \delta < 1$. The annulus case was also studied in [4]. In this paper we study a related problem for an unbounded interval.

Let $A$ be a set of real numbers and $F$ be a class of complex-valued functions defined on the real line $R$ such that for each $f \in F$, the infinite series $S(x, f) = \sum_{k=1}^{\infty} f(kx)$ converges for every nonzero $x \in A$. If $0 \in A$, we denote $S(0, f) = f(0)$. We study various sets $A$ and various function classes $F$ such that each $f \in F$ is uniquely determined by the sums $S(x, f)$ where $x \in A$. Some examples will be given to show that in general $A$ cannot be too small.

**2. Results for entire functions.** We start with a fairly small function class. For $r > 0$, let $PW(r\pi, 1)$ be the class of all entire functions $f(z)$ of exponential type at most $r\pi$ such that $f(x) \in L^2(R)$ and that for some $p > 1$, $f(n/r) = 0(|n|^{-p})$ for $n = \pm 1, \pm 2, \cdots$. Let $Z$ be the set of all integers. We have the following theorem of the Carlson type.

THEOREM 1. *Every function $f$ in $PW(\pi, 1)$ is uniquely determined by the sequence $S(n, f)$ where $n \in Z$.*