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LINEAR OPERATORS FOR WHICH T^*T AND TT^* COMMUTE (II)

STEPHEN L. CAMPBELL

Let (BN) denote the class of all bounded linear operators on a Hilbert space such that T^*T and TT^* commute. Let $(BN)^+$ be those $T \in (BN)$ which are hyponormal. Embry has observed that if $T \in (BN)$, then $0 \in W(T)$ or T is normal. This is used to show that if $T \in (BN)$, then $(T + \lambda I) \notin (BN)$ unless T is normal. It is also shown that if $T \in (BN)^+$, then T^n is hyponormal for $n \geqq 1$. An example of a $T \in (BN)^+$ such that $T^2 \notin (BN)$ is given. Paranormality of operators in (BN) is shown to be equivalent to hyponormality. The relationship between T being in (BN) and T being centered is discussed. Finally, all 3×3 matrices in (BN) are characterized.

This paper is a continuation of [3]. In that paper we studied bounded linear operators T acting on a separable Hilbert space $\mathscr{H}$ such that T^*T and TT^* commute. Such operators are called bi-normal and the class of all such operators is denoted (BN). This paper will explore some of the properties of hyponormal bi-normal operators. In addition, we will show that no translate of a non-normal bi-normal operator is bi-normal and characterize all 2×2 and 3×3 bi-normal matrices.

It has been pointed out to the author that the term bi-normal has been used earlier by Brown [2]. However, his usage does not appear to be in the current literature so we will continue to use bi-normal for operators in (BN).

1. All shifts, weighted and unweighted, bilateral and unilateral, are in (BN). Further, operators in (BN), if completely nonnormal, have a tendency to be "shift-like". Our first result, due to Embry, is an example of this.

THEOREM 1. *If $T \in (BN)$, then either T is normal or zero is in the interior of the numerical range of T, $W(T)$.*

Proof. Embry has shown that if $T \in (BN)$ and T is not normal, then $0 \in W(T)$ [7, Theorem 1]. She has also shown that if $T \in (BN)$ and $T + T^* \geqq 0$, then T is normal [5, Theorem 2]. Thus if 0 were on the boundary of $W(T)$, by a suitable choice of α, $|\alpha| = 1$, we could consider $T_1 = \alpha T$ where $T_1 \in (BN)$ and $T_1 + T_1^* \geqq 0$. Then T would be normal.