ON THE DIMENSION OF MODULES AND ALGEBRAS (III)

GLOBAL DIMENSION[1]

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Let Λ be a ring with unit. If A is a left Λ-module, the ***dimension*** of A (notation: $\mathrm{l.dim}_\Lambda A$) is defined to be the least integer n for which there exists an exact sequence

$$0 \longrightarrow X_n \longrightarrow \dots \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

where the left Λ-modules $X_0, \dots, X_n$ are projective. If no such sequence exists for any n, then $\mathrm{l.dim}_\Lambda A = \infty$. The *left global dimension* of Λ is

$$\mathrm{l.gl.dim}\, \Lambda = \sup\ \mathrm{l.dim}_\Lambda A$$

where A ranges over all left Λ-modules. The condition $\mathrm{l.dim}_\Lambda A < n$ is equivalent with $\mathrm{Ext}^n_\Lambda(A, C) = 0$ for all left Λ-modules C. The condition $\mathrm{l.gl.dim}\, \Lambda < n$ is equivalent with $\mathrm{Ext}^n_\Lambda = 0$. Similar definitions and theorems hold for right Λ-modules.

In the first section of this paper it is shown that the global dimension of Λ is completely determined by the dimensions of the cyclic modules over Λ, i.e., the modules generated by a single element. In the next section the notion of *weak global dimension* of Λ (notation: $\mathrm{w.gl.dim}\, \Lambda$) is introduced, and using the previous result it is proven that if Λ is both left and right Noetherian, then $\mathrm{l.gl.dim}\, \Lambda = \mathrm{w.gl.dim}\, \Lambda = \mathrm{r.gl.dim}\, \Lambda$.

The rest of the paper, which is independent of the first two sections, is devoted to a study of the global dimension of semi-primary rings. The principal result here is that $\mathrm{l.dim}_\Lambda \Gamma = \mathrm{l.gl.dim}\, \Lambda = \mathrm{w.gl.dim}\, \Lambda = \mathrm{r.gl.dim}\, \Lambda = \mathrm{r.dim}_\Lambda \Gamma$, where $\Gamma = \Lambda/N$, N being the radical of Λ.

The definitions and notations employed in this paper are based on those

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