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A Generalization of the Classical Moment Problem on *-Algebras with Applications to Relativistic Quantum Theory. I.

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Abstract. A (non-commutative) generalization of the classical moment problem is formulated on arbitrary *-algebras with units. This is used to produce a C^*-algebra associated with the space of test functions for quantum fields. This C^*-algebra plays a role in theories of bounded localized observables in Hilbert space which is similar to that of the space of test functions in quantum field theories (namely it is represented in Hilbert space). The case of local quantum fields which satisfy a slight generalization of the growth condition is investigated.

1. Introduction and Notations

This paper deals with a sort of non-commutative generalization of measure theory and of the classical moment problem on arbitrary *-algebras. The connexion between the classical moment problem and the hermitian representations (in the sense of Powers [1]) of the algebras of polynomials is well known. The generalization given here has some similar connexions with hermitian representations of *-algebras.

In the usual one dimensional classical moment problem [2, 3], one starts with a sequence of numbers S_n $(n \geqq 0)$ and a closed subset S of $\mathbb{R}$ and one asks the following question: Is there a positive measure μ supported by S such that

$$S_n = \int t^n d\mu(t), \quad \text{for any integer} \quad n \geqq 0?$$

Remembering that there is a bijection $(S_n) \mapsto \phi_{(S_n)}$ from $\mathbb{C}^{\mathbb{N}}$ on the set of all the linear forms on the *-algebra $\mathbb{C}[X]$ of complex polynomials with respect to an indeterminate X,

$$\phi_{(S_n)}(\Sigma a_n X^n) = \Sigma a_n S_n, \quad \forall \Sigma a_n X^n \in \mathbb{C}[X].$$

The classical moment problem may be put in the following form. Let ϕ be a linear form on $\mathbb{C}[X]$ and S be a closed subset of $\mathbb{R}$, is there a positive measure μ on S such that,

$$\phi(P(X)) = \int P(t)\, d\mu(t), \quad \forall P(X) \in \mathbb{C}[X]?$$

In the generalization given in this paper: $\mathbb{C}[X]$ is replaced by an arbitrary *-algebra with unit $\mathfrak{A}$, ϕ is a linear form on $\mathfrak{A}$, μ is replaced by a positive linear

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