REGULARITY OF FUNCTION-TO-FUNCTION TRANSFORMATIONS*

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1. **Introduction.** In a recent note Hill† considered the transformation

$$(1) \qquad U_x(f) = \int_0^\eta K(x, y)f(y)dy$$

operating on the class $\mathfrak{B}_1$ of measurable, essentially bounded, complex-valued functions f of one real variable y defined for $0<y<\eta$, and satisfying the condition that $\lim_{y\to\eta} f(y)$ exists. The kernel $K(x, y)$ is defined for $0<x<\xi$, $0<y<\eta$, and the integral is interpreted in the Lebesgue sense. Hill derived a set of conditions on $K(x, y)$ necessary and sufficient that the transformation (1) be *regular* on $\mathfrak{B}_1$, that is, that for every f in $\mathfrak{B}_1$, $U_x(f)$ be defined for all x, and $\lim_{x\to\xi} U_x(f) = \lim_{y\to\eta} f(y)$.

In §2 of the present paper we generalize Hill's results for a transformation on a class $\mathfrak{B}_m$ of bounded measurable functions of m real variables to a class of functions of n real variables. This transformation can be expressed in the form (1) with x standing for $x^1, x^2, \cdots, x^n$ and y for $y^1, y^2, \cdots, y^m$. In §3 we define for each kernel $K(x, y)$ its domain of regularity $\mathfrak{K}$ as the largest class of functions on which (1) is regular, and we determine some conditions necessary and sufficient that a function f be in $\mathfrak{K}$. We employ these results in §4 to derive conditions on $K(x, y)$ necessary and sufficient for the transformation (1) to be regular on certain classes of functions more inclusive than $\mathfrak{B}_m$. Finally, in §5, we consider several particular cases of the problem of determining a kernel with a specified class of functions as its domain of regularity.

2. **Hill's theorem in many variables.** We use the following notation throughout this paper: x stands for the ordered set of n real variables $x^1, x^2, \cdots, x^n$, and y for $y^1, y^2, \cdots, y^m$; for $0, 0, \cdots, 0$ we write 0. The equality $a=b$ means that for each j, $a^j=b^j$; $a<b$ that for each j, $a^j<b^j$; $a \not< b$ that for at least one j, $a^j \geqq b^j$; and $a>b$ that $b<a$. We define the *interval* (a, b) as the set of points c such that $a<c<b$.

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† J. D. Hill, *A theorem in the theory of summability*, this Bulletin, vol. 42 (1936), pp. 225–228.