ON APPROXIMATELY CONTINUOUS FUNCTIONS

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In a very interesting paper *Sur l'équation fonctionnelle* $g(x) = f\phi(x)$, S. Braun* established a series of theorems on the functional equation

$$g(x) = f[\phi(x)],$$

where $g(x)$ and $f(y)$ are given functions, and $\phi(x)$ is a function sought for. In this note, we consider the case for which $\phi(x)$ is an approximately continuous function.† A function $f(x)$ is said to be approximately continuous at x_0 if the density at x_0 of the set $E[f(x_0), \epsilon]$ of all points x such that $|f(x) - f(x_0)| < \epsilon$ is equal to 1, no matter what ϵ is.

Let $f(x)$ be a finite function of class 1 in $[0, 1] = [0 \leqq x \leqq 1]$, and let $\{y_n\}$ be the sequence of all rational numbers y_n such that there are two points x_n' and x_n'' belonging to $[0, 1]$ and satisfying the condition $f(x_n') < y_n < f(x_n'')$. Let E_{y_n} (E^{y_n}), $(n = 1, 2, 3, \cdots)$, denote the set of all points x such that $f(x) < y_n$ $(f(x) > y_n)$. If z is an irrational number, let E_z (E^z) denote the sum of all the sets E_{y_n} (E^{y_n}) such that $y_n < z$ $(y_n > z)$. We now prove the following theorem:

THEOREM 1. *A necessary and sufficient condition that a finite function* $\phi(x)$ *be approximately continuous in* $[0, 1]$ *is that there exist a system of perfect sets*

$$(\mathfrak{P}): \quad \mathfrak{P}^n_{y_r}, \quad \mathfrak{P}^{y_r}_n, \qquad r = 1, 2, 3, \cdots, n;\ n = 1, 2, 3, \cdots,$$

such that

(i) $E_{y_r} = \lim_{n=\infty} \mathfrak{P}^n_{y_r}$, $E^{y_r} = \lim_{n=\infty} \mathfrak{P}^{y_r}_n$, $\mathfrak{P}^n_{y_r} \subset \mathfrak{P}^{n+1}_{y_r} \subset E_{y_r}$, $\mathfrak{P}^{y_r}_n \subset \mathfrak{P}^{y_r}_{n+1} \subset E^{y_r}$;

(ii) *if* $y_r < y_s$ *and* M *is the greater of the integers* r, s, *every point of the set* $\mathfrak{P}^n_{y_r}$ $(\mathfrak{P}^{y_s}_n)$ *is a density point of the set* $\mathfrak{P}^n_{y_s}$ $(\mathfrak{P}^{y_r}_n)$ *for all* $n \geqq M$.

A point x of a set E will be called a density point in E if

$$\lim_{h=0} \left[\frac{1}{2h} \text{meas } [(x - h, x + h)E] \right] = 1.$$

PROOF. Let x_0 be an arbitrary point in $[0, 1]$, and let $f(x_0) = y_0$.

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* Fundamenta Mathematicae, vol. 28 (1937), pp. 294–302.

† A. Denjoy, *Sur les fonctions dérivées sommables*, Bulletin de la Société Mathématique de France, vol. 43 (1915), pp. 161–247, especially p. 165.