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# Multiplicity and Hilbert-Kunz Multiplicity of Monoid Rings

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In this paper, we will give a method to compute the multiplicity and the Hilbert-Kunz multiplicity of monoid rings. The multiplicity and the Hilbert-Kunz multiplicity are fundamental invariants of rings. For example, the multiplicity (resp. the Hilbert-Kunz multiplicity) of a regular local ring equals to one. Monoid rings are defined by lattice ideals, which are binomial ideals $I$ in a polynomial ring $R$ over a field such that any monomial is a non zero divisor on $R/I$. Affine semigroup rings are monoid rings. Hence we want to extend the thoery of affine semigroup rings to that of monoid rings.

## 1. Main Result.

Let $N > 0$ be an integer and $\mathbf{Z}$ the ring of integers. For $\alpha \in \mathbf{Z}^N$, we denote the $i$-th entry of $\alpha$ by $\alpha_i$. We say $\alpha > 0$ if $\alpha \neq 0$ and $\alpha_i \geq 0$ for each $i$. And $\alpha > \alpha'$ if $\alpha - \alpha' > 0$. Let $R = k[X_1, \cdots, X_N]$ be a polynomial ring over a field $k$. For $\alpha > 0$, we simply write $X^\alpha$ in place of $\prod_{i=1}^N X_i^{\alpha_i}$.

For a positive submodule $V$ of $\mathbf{Z}^N$ of rank $r$, we define an ideal $I(V)$ of $R$, which is generated by all binomials $X^\alpha - X^\beta$ with $\alpha - \beta \in V$ (we say that $V$ is positive if it is contained in the kernel of a map $\mathbf{Z}^N \to \mathbf{Z}$ which is defined by positive integers). Put $d = N - r$. Then $R/I(V)$ is naturally a $\mathbf{Z}^d$-graded ring, which is called a monoid ring. Further, there is a positive submodule $V'$ of $\mathbf{Z}^N$ of rank $r$ containing $V$ such that $\mathbf{Z}^N/V'$ is torsion free. That is, $\mathbf{Z}^N/V \cong \mathbf{Z}^N/V' \oplus T$, where $\mathbf{Z}^N/V' \cong \mathbf{Z}^d$ and $T$ is a torsion module. Hence we can see an element of $\mathbf{Z}^N/V$ as a pair $(\alpha, \beta)$ where $\alpha \in \mathbf{Z}^d$ is a degree element and $\beta \in T$ is a torsion element. Put $t = |T|$(if $T = \{0\}$, put $t = 1$). Let $A = R/I(V)$ and $A' = R/I(V')$. For each $\alpha \in \mathbf{Z}^d$, we denote the degree $\alpha$ component of the $\mathbf{Z}^d$-graded ring $A$ (resp. $A'$) by $A_\alpha$ (resp. $A'_\alpha$). It is clear $\dim_k A_\alpha \leq t$ and $\dim_k A'_\alpha \leq 1$ for $\alpha \in \mathbf{Z}^d$ and $\dim_k A_\alpha \geq \dim_k A_{\alpha'}$ if $\alpha > \alpha'$ and if there is a monomial of $A$ of the degree $\alpha - \alpha'$.

EXAMPLE. Let $V$ be a submodule of $\mathbf{Z}^3$ generated by $-e_1 + 2e_2 - e_3$, $-2e_1 - e_2 + 3e_3$ and $-3e_1 + e_2 + 2e_3$. Then $\mathbf{Z}^3/V \cong \mathbf{Z} \oplus \mathbf{Z}/5\mathbf{Z}$. And there is an isomorphism which corresponds $e_1$, $e_2$ and $e_3$ to $(1, 0)$, $(1, 1)$ and $(1, 2)$, respectively.

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