125.   Dependent Elements of an Automorphism of a C*-algebra

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(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1972)

1. **Introduction.** Let A be a unital C^*-algebra. For an (*-preserving) automorphism α of A, an element a of A is called a *dependent element* of α if
(1) $ax = x^\alpha a$ for any $x \in A$.
If α is an inner automorphism of A induced by a, then clearly a is a dependent element.

In [5], Nakamura and Takeda recognized the importance of the following implication:
(*)   *If a is a dependent element of α then $a = 0$.*
They proved, among many others, in a finite factor A α satisfies (*) if α is outer, using a sophisticated argument. Recently, Kallman [3] called, when A is a von Neumann algebra, α *freely acting* if (*) is satisfied. His definition of free action agrees with the usual one due to von Neumann if A is an abelian von Neumann algebra. He proved, among others, every automorphism of a von Neumann algebra is directly decomposed into the freely acting and inner parts.

In the present note, we shall study some properties of dependent elements of automorphisms on C^*-algebras. We shall show, by elementary calculations, dependent elements are normal and invariant under the automorphism, in § 2. We shall discuss some applications in § 3, which include a completely elementary proof of a theorem of Nakamura, Takeda and Kallman. In § 4, we shall give a few remarks, one of which is a slight improvement of a proof of a theorem of Kallman.

2. **Dependent elements.** We shall prove some elementary lemmas some of which are already known. In this section, we shall assume that A is a C^*-algebra with the center Z.

Lemma 1 (Kallman). *If a is a dependent element of an automorphism α of A, then a^*a and aa^* belong to Z.*

Proof. The following proof is a slight improvement of Kallman's. From (1), we have

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