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On the Derivations of Lie Algebras

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1. Introduction

Let $\mathfrak{L}$, $\mathfrak{M}$ be Lie algebras over a field K of characteristic 0. A linear mapping D of $\mathfrak{L}$ into $\mathfrak{M}$ is called a derivation of $\mathfrak{L}$ into $\mathfrak{M}$ if $D(x \circ y) = D(x) \circ y + x \circ D(y)$ for all x, y in $\mathfrak{L}$. A derivation of $\mathfrak{L}$ into itself is simply called a derivation of $\mathfrak{L}$. The set $\mathfrak{D}(\mathfrak{L})$ of all derivations of $\mathfrak{L}$ forms a Lie algebra with the commutator product $D_1 \circ D_2 = D_2 D_1 - D_1 D_2$, which is called the derivation algebra of $\mathfrak{L}$. For any element x of $\mathfrak{L}$, the adjoint mapping $D_x : y \rightarrow y \circ x$ is a derivation of $\mathfrak{L}$. Such a derivation is called inner. It is easy to see that the inner derivations of $\mathfrak{L}$ form an ideal in $\mathfrak{D}(\mathfrak{L})$ which we denote by $\mathfrak{J}(\mathfrak{L})$. Let $\mathfrak{L}_1$ be a subalgebra of $\mathfrak{L}$. We shall denote by $D|\mathfrak{L}_1$ the restriction to $\mathfrak{L}_1$ of a derivation D of $\mathfrak{L}$ and, for any subset $\mathfrak{E}$ of $\mathfrak{D}(\mathfrak{L})$, denote by $\mathfrak{E}|\mathfrak{L}_1$ the set of $D|\mathfrak{L}_1$ for all D in $\mathfrak{E}$. A subset of $\mathfrak{L}$ is called characteristic if it is mapped into itself by every derivation of $\mathfrak{L}$. The radical $\mathfrak{R}$ of $\mathfrak{L}$ is a characteristic ideal [2] so that $\mathfrak{D}(\mathfrak{L})|\mathfrak{R}$ is a subalgebra of $\mathfrak{D}(\mathfrak{R})$. If there exists a subalgebra $\mathfrak{L}_2$ such that $\mathfrak{L} = \mathfrak{L}_1 + \mathfrak{L}_2$ and $\mathfrak{L}_1 \cap \mathfrak{L}_2 = 0$, then we say that $\mathfrak{L}$ splits over $\mathfrak{L}_1$ and that $\mathfrak{L}_2$ is a complement of $\mathfrak{L}_1$ in $\mathfrak{L}$.

The purpose of this paper is to study the relations between the derivation algebras of Lie algebras and their radicals. By a well-known theorem of E. Cartan, every derivation of a semi-simple Lie algebra is an inner derivation. We give a necessary and sufficient condition for a derivation of $\mathfrak{L}$ to be inner (Theorem 1) and show that every derivation of $\mathfrak{L}$ is inner if and only if $\mathfrak{D}(\mathfrak{L})|\mathfrak{R} = \mathfrak{J}(\mathfrak{L})|\mathfrak{R}$ (Theorem 2). Recently G. F. Leger [5] has proved that, if $\mathfrak{D}(\mathfrak{R})$ splits over $\mathfrak{J}(\mathfrak{R})$, $\mathfrak{D}(\mathfrak{L})$ splits over $\mathfrak{J}(\mathfrak{L})$. We show that, in order that $\mathfrak{D}(\mathfrak{L})$ may split over $\mathfrak{J}(\mathfrak{L})$, each of the following conditions is necessary and sufficient: (1) $\mathfrak{D}(\mathfrak{L})|\mathfrak{R}$ splits over $\mathfrak{J}(\mathfrak{L})|\mathfrak{R}$; (2) $\mathfrak{D}(\mathfrak{L})|\mathfrak{R}$ splits over $\mathfrak{J}(\mathfrak{R})$ (Theorem 3). We also generalize a result of G. Hochschild [2] and study the derivation algebras of reductive Lie algebras.