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Module Spectra over the Moore Spectrum

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Introduction

Let q be an odd integer and $M_q(=M)$ be the Moore spectrum of type Z_q. It is a ring spectrum with multiplication $m_M: M_q \wedge M_q \to M_q$ and unit $i: S \to M_q$ [1], where S is the sphere spectrum. A finite CW-spectrum X is called an M_q-*module spectrum* if X is equipped with a left inverse $m_X: M_q \wedge X \to X$, which we call an M_q-*action* on X, of $i \wedge 1_X$. It is clear that X is an M_q-module spectrum if and only if $q1_X=0$ in $[X, X]$. When q is a prime, our M_q-module spectrum is just the Z_q-spectrum introduced by H. Toda [10].

The main purpose of this note is to investigate conditions under which an M_q-module spectrum is (non-)associative. Here an M_q-module spectrum (X, m_X) is called *associative* if $m_X(1_M \wedge m_X)=m_X(m_M \wedge 1_X)$. For an M_q-module spectrum X, the order r of 1_X is a divisor of q and the homology group $H_i(X)$ is a finite Z_r-module, and we shall obtain in §6 the following theorems on the associativity and on the non-associativity according as the case $q \not\equiv \pm 3 \bmod 9$ or $(r, 3)=1$ and the case $q \equiv \pm 3 \bmod 9$ and $3|r$.

THEOREM 6.6. *Let X be an M_q-module spectrum and in the case of $q \equiv \pm 3$* mod 9 *assume that the order of 1_X is relatively prime to* 3. *If X satisfies the following two conditions, then X admits an associative M_q-action.*

(i) *$\sharp H_i(X)$ is relatively prime to $\sharp H_{i-1}(X)$ and to $\sharp H_{i-2}(X)$.*

(ii) *For any prime p, the p-component of $H_i(X)$ is free over the p-component of Z_q.*

Here $\sharp G$ denotes the order of a finite group G. Furthermore we shall see that in the dual Postnikov system $\{X_i\}$ of X (X_i is a subspectrum of X realizing $\sum_{j \leqq i} H_j(X)$ as its homology group) each X_i is also an associative M_q-module spectrum, (cf. Remark 6.7). We shall also construct, for every prime $q>3$, an example which does not satisfy the condition (i) and has a unique M_q-action, which is not associative (Example 6.8).

THEOREM 6.3. *Assume that $q \equiv \pm 3$* mod 9. *Let X be an M_q-module spectrum such that the order of 1_X is a multiple of* 3. *Then every M_q-action on X is not associative.*

In §1, we shall study elementary properties of M_q-module spectra. In §2,