HIROSHIMA MATH. J.
8 (1978), 91–100

Axiomatic Characterizations of Grade for Commutative Rings

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(Received September 13, 1977)

Introduction

Let R be a commutative ring and I an ideal of R. Suppose that R is noetherian. Then, for every R-module M, we can define the grade of I on M in two different ways using M-regular sequences in I and $\mathrm{Ext}_R(R/I, M)$ (or $\mathrm{H}_I(M)$). In the case that R is not necessarily noetherian, there are two notions of grade which may be regarded as good generalizations of the one in the noetherian case. One is the homological grade (or 'Rees' grade) which is defined by using $\mathrm{Ext}_R(R/I, \ )$ or $\mathrm{H}_I(\ )$ in the same way as in the noetherian case. Another one is the polynomial grade which is a successful generalization of the notion of the longest M-regular sequence in I. As for the last one, a characterization was given by M. Hochster by making use of the Koszul complex in [4].

In this paper, we shall give axiomatic characterizations of the above two different notions of grade, which show a relationship between these two notions from another point of view.

In § 1 we shall see that the concept of localizing subcategories plays an essential role in the theory of homological grade in an abelian category. In § 2 we are mainly concerned with the homological grade in the category of R-modules. In this case we use Gabriel topologies on R instead of localizing subcategories. We shall also give a proof of the Auslander-Buchsbaum theorem on finite free resolutions in terms of homological grade. In § 3 we shall study the polynomial grade originally introduced by M. Hochster.

Throughout this paper all rings and algebras are commutative with identity and modules are unitary.

1. Homological grade in an abelian category

In this section, we shall discuss a homological theory of grade in an abelian category. Throughout this section $\mathscr{A}$ is a locally small abelian category with injective envelopes and products.

Let $\mathscr{C}$ be a localizing subcategory of $\mathscr{A}$ (cf. [8]). Then for each object M in $\mathscr{A}$ we can assign a non-negative integer or ∞ as follows:

$hgr(\mathscr{C}, M)=$ the least integer n such that $\mathrm{R}^n\mathrm{L}(M)\neq 0$ if there is such an in-