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On products of the β-elements in the stable homotopy of spheres

Shichirô OKA[1] and Katsumi SHIMOMURA[2]

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§1. Introduction

In his paper [20], H. Toda introduced the elements β_s, $1 \leqq s \leqq p-1$, in the p-primary component of the stable homotopy of spheres for an odd prime p, and L. Smith [18] extended them to an infinite family $\{\beta_s\}_{s \geqq 1}$, in case $p \geqq 5$. Later, with the development and plentiful knowledge of the Adams-Novikov spectral sequence based on the Brown-Peterson homology BP such as [5], it is clarified that these β-elements are detected in $\mathrm{Ext}^2_{BP_*BP}(BP_*, BP_*)$, the second line of the E_2-term of the spectral sequence, which consists of an extensive family of elements $\beta_{s/r,i}$ with suitable triple indices including $\beta_s = \beta_{s/1,1}$ (cf. (4.1)). The construction of the homotopy elements β_s is immediate from the one of the 4-cell complex called $V(1)$ and appropriate stable self-maps of $V(1)$ [18], and in this way, L. Smith [19], R. Zahler [23] and the first author [9], [11], [12] constructed homotopy elements which correspond with the generalized β's in Ext^2 including

$$\beta_{sp/r}\ (s \geqq 1,\ 1 \leqq r < p), \quad \beta_{sp/p}\ (s \geqq 2), \quad \beta_{sp^2/p,2}\ (s \geqq 2),$$

where $\beta_{sp/r,1} = \beta_{sp/r}$ and some of these were called ε's and ρ's in earlier literatures (see (2.4), (2.5)).

The purpose of this paper is to study the products $\beta_s \beta_{tp/r}$ with $r \leqq p$ and $\beta_s \beta_{tp^2/p,2}$ in π_*^S, the stable homotopy ring of spheres, in case $p \geqq 5$. In particular, we shall study whether they are trivial or not. In this direction, H. Toda [21] obtained a formula of $\beta_s \beta_t$ extending the earlier work of N. Yamamoto [22] and including the relation $\beta_s \beta_{tp} = 0$ which is the case $r = 1$ of ours.

THEOREM A. *Let p be a prime $\geqq 5$, and r, s, t be positive integers with $r \leqq p$ and $r \leqq p-1$ if $t = 1$. Then the element $\beta_s \beta_{tp/r}$ in π_*^S is trivial, if one of the following holds*:

(i) $r \leqq p-2$.
(ii) $r = p-1$ *and* $s \not\equiv -1 \mod p$.
(iii) $r = p-1,\ p$ *and* $t \equiv 0 \mod p$.

The next cases we have to investigate are (iv) $r = p-1$, $s \equiv -1 \mod p$ and $t \not\equiv 0 \mod p$; and (v) $r = p$ and $t \not\equiv 0 \mod p$. For the case (iv), we obtain a weak