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On the differentiability of Riesz potentials of functions

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In the n-dimensional euclidean space R^n, we define the Riesz potential of order α of a nonnegative measurable function f on R^n by

$$R_\alpha f(x) = \int R_\alpha(x-y)f(y)dy,$$

where $R_\alpha(x)=|x|^{\alpha-n}$ if $\alpha<n$ and $R_n(x)=\log(1/|x|)$. It is known (cf. [2]) that if $f\in L^p(R^n)$, $p\geqq 1$, and $|R_\alpha f|\not\equiv\infty$, then $R_\alpha f$ is (m, p)-semi finely differentiable almost everywhere, where m is a positive integer such that $m\leqq\alpha$. In the case $\alpha p>n$, this fact implies that $R_\alpha f$ is totally m times differentiable almost everywhere. A function u is said to be totally m times differentiable at x_0 if there exists a polynomial P for which $\lim_{x\to x_0}|x-x_0|^{-m}[u(x)-P(x)]=0$.

In this note, we are concerned with the case where $\alpha p=n$ and α is a positive integer m, and aim to give a condition on f which assures the total m times differentiability of $R_\alpha f$.

THEOREM. *Let m be a positive integer, $p=n/m>1$ and f be a nonnegative measurable function on R^n such that $R_m f\not\equiv\infty$ and*

$$\int f(y)^p(\log(2+f(y)))^\delta dy < \infty \qquad \textit{for some} \quad \delta > p-1.$$

Then $R_m f$ is totally m times differentiable almost everywhere.

The proof of the theorem will be carried out along the same lines as in that of Theorem 3 in [2].

We first prepare the following lemmas.

LEMMA 1. *If m, p and f are as in the Theorem, then*

$$\int_{E(f)} R_m(x-y)f(y)dy \leqq M\left(\int f(y)^p[\log(2+f(y))]^\delta dy\right)^{1/p}$$

for all $x\in R^n$, where $E(f)=\{y; f(y)\geqq 1\}$ and M is a positive constant independent of f and x.

PROOF. We may assume that $x=0$. We set

$$E_j = \{y;\ 2^{j-1}\leqq f(y) < 2^j\}$$