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Lie structures on differential algebras

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1. Introduction

Let L be a finite-dimensional Lie algebra over a field $\mathfrak{k}$ of characteristic zero, A a commutative associative algebra over $\mathfrak{k}$ with an identity, and $L \subseteq A$. In this paper, we first extend the Lie structure on L to A by means of some derivations of A. After presenting examples of such Lie algebras and showing a way to give a Lie structure on a localization of A, we study the Lie structures on the formal power series ring and some factor algebras of polynomial algebras.

2. Notations and preliminaries

Poisson Lie structure (Berezin [1]): Let L be a finite-dimensional Lie algebra over a field $\mathfrak{k}$ of characteristic zero and c_k^{ij} the structure constants with respect to a basis $\{x_1,\dots,x_n\}$ of L. Let $C^\infty(\boldsymbol{R}^n)$ be the set of all C^∞ function on $\boldsymbol{R}^n$. Then the Poisson Lie structures on $C^\infty(\boldsymbol{R}^n)$ is given by

$$[f, g] = \sum_{i,j,k} c_k^{ij} x_k (\partial f/\partial x_i)(\partial g/\partial x_j) \qquad \text{for} \quad f, g \in C^\infty(\boldsymbol{R}^n).$$

Let $U(L)$ be the universal enveloping algebra of L and $U_n(L)$ the vector space spanned by the products $y_1 \dots y_p$, where $y_1,\dots,y_p \in L$ and $p \leqslant n$. Let $S(L)$ be the symmetric algebra of the vector space L and $S^n(L)$ the set of elements of $S(L)$ which are homogeneous of degree n. By making use of the canonical mapping π_n of $U_n(L)$ onto $S^n(L)$, we can obtain a Lie structure on $S(L)$ as follows: Let $p \in S^m(L)$ and $q \in S^n(L)$, and take elements $\tilde{p} \in U_m(L)$ and $\tilde{q} \in U_n(L)$ such that $\pi_m(\tilde{p}) = p$ and $\pi_n(\tilde{q}) = q$. The Poisson bracket $[p, q]$ of p and q is defined to be $\pi_{m+n-1}([\tilde{p}, \tilde{q}])$ ([3, p. 97]). By a simple computation we can see that this Lie structure on $S(L)$ is the same as the Poisson Lie structure on the polynomial algebra $\mathfrak{k}[x_1,\dots, x_n]$.

Profinite Lie algebra (Christdoulou [2]): Let A_m ($m \in \boldsymbol{N}$) be a finite-dimensional Lie algebra and f_{mn} a homomorphism of A_m into A_n for $m \geqslant n$. Let A be the inverse limit $\varprojlim \{A_m; f_{mn}\}$. Then A is a profinite Lie algebra in the following sense: Let f_m be a canonical homomorphism of A onto A_m and $K_m = \mathrm{Ker} f_m$. Then the set $\{K_m : m \in \boldsymbol{N}\}$

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