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On a function space related to the Hardy-Littlewood inequality for Riemannian symmetric spaces

Dedicated to Professor Kiyosato Okamoto on his 60th birthday

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ABSTRACT. On Riemannian symmetric spaces G/K we define an L^q type Schwartz space $\mathscr{J}^q(G)$ which corresponds to the Schwartz space with weight $|x|^{n(q-2)}$ on $\boldsymbol{R}^n$. We study some properties of $\mathscr{J}^q(G)$ and we prove if $2 \leq q < 4$ and p and q are conjugate, then $J^q(G)$ equals to the L^p-type Schwartz space $\mathscr{S}^p(G)$ defined by Harish–Chandra.

1. Introduction

For a real number q $(2 \leq q < \infty)$ and a Borel function f on $\boldsymbol{R}^n$ we put

$$\|f\|_{(q)} = \left(\int_{\boldsymbol{R}^n} |f(x)|^q |x|^{n(q-2)} dx\right)^{1/q}$$

and denote by $J^q(\boldsymbol{R}^n)$ the Banach space of all Borel functions f on $\boldsymbol{R}^n$ satisfying $\|f\|_{(q)} < \infty$. The Hardy-Littlewood theorem ([3]) says that if $f \in J^q(\boldsymbol{R}^n)$, then the Fourier transform $\tilde{f}$ of f is well-defined and there exists a constant $C_q > 0$ such that

$$\|\tilde{f}\|_q \leq C_q \|f\|_{(q)} .$$

On the other hand, if $1 \leq p \leq 2$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then the Fourier transform $\tilde{f}$ of $f \in L^p(\boldsymbol{R}^n)$ is well-defined and there exists a constant $B_p > 0$ such that

$$\|\tilde{f}\|_q \leq B_p \|f\|_p .$$

This is the Hausdorff-Young theorem. These two theorems suggest the resemblance between $L^p(\boldsymbol{R}^n)$ and $J^q(\boldsymbol{R}^n)$. In fact, if we put $f_\alpha(x) = (1 + |x|^2)^\alpha$ and $g_\beta(x) = |x|^\beta (|x| \leq 1), \; = 0 (|x| > 1)$, then

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