ON A CONJECTURE OF BERBERIAN

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1. In [1], S. K. Berberian conjectured that the closure of the numerical range of a hyponormal operator coincides with the convex hull of its spectrum. The purpose of this note is to give an affirmative answer to his conjecture.

Throughout this paper, operator means a bounded linear operator on a Hilbert space. The spectrum of an operator T is denoted by $\sigma(T)$, and its convex hull is denoted by $\sum(T)$. The numerical range of an operator T, denoted by $W(T)$, is the set $W(T) = \{(Tx, x) : \|x\| = 1\}$. We write $\overline{W}(T)$ for the closure of $W(T)$. An operator T is called normaloid if $\|T\| = \sup\{|\lambda| : \lambda \in W(T)\}$. For a compact convex subset X of the plane, a point $\lambda \in X$ is bare if there is a circle through λ such that no points of X lie outside this circle. A closed subset X of the plane is a spectral set for an operator T if $\|u(T)\| \leqq \sup\{|u(z)| : z \in X\}$ for every rational function $u(z)$ having no poles in X.

2. In this section, we shall prove the following theorem.

THEOREM. *Let T be an operator such that $T - \lambda I$ is normaloid for every complex number λ, then we have $\overline{W(T)} = \sum(T)$.*

A key of our proof is the following lemma.

LEMMA 1. *Let T be an operator and $\lambda \in \overline{W(T)}$ a bare point of $\overline{W(T)}$, then there exists a complex number λ_0 satisfying $|\lambda - \lambda_0| = \sup\{|\mu - \lambda_0| : \mu \in \overline{W(T)}\}$.*

PROOF. By the definition of bare point, there is a circle through λ such that no points of $\overline{W(T)}$ lie outside this circle. The center λ_0 of this circle satisfies our requirement.

For convenience we state the following known result as a lemma ([4: Corollary to Theorem 4]).

LEMMA 2. *For an operator T, $\lambda \in \overline{W(T)}$ and $|\lambda| = \|T\|$ imply $\lambda \in \sigma(T)$.*