CONTINOUSLY SPLITTABLE DISTRIBUTIONS IN HILBERT SPACE

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1. Introduction

1.1. This paper is concerned with a class of weak distributions on Hilbert space. Let H be a real Hilbert space. A weak distribution is a linear mapping on H which takes each linear function $(x, \cdot)$ on H into a random variable $m(x)$ on a probability measure space. It is supposed that σ-algebra of measurable sets is the smallest such that all the $m(x)$ are measurable. See [2], [4] and [5].

The normal distribution n is characterized, up to a variance parameter c, by the property that orthogonal vectors x and y correspond to stochastically independent random variables $n(x)$ and $n(y)$. Then each $n(x)$ is normally distributed with variance $c\| x \|^2$ and mean zero. See [5, Theorem 3].

1.2. By a spectral measure $\mathcal{E}$ we mean a completely additive Boolean algebra of commuting projections. We say that $\mathcal{E}$ splits a weak distribution m if, for each x in H and each P in $\mathcal{E}$, $m(Px)$ and $m((I - P)x)$ are stochastically independent. Every spectral measure splits the normal distribution.

One way splittable distributions arise is from suitably smooth stochastic processes with independent increments. For example let X_t, $0 \leq t \leq 1$, be such a process. Let $H = L_2(0, 1)$. Let $m(f) = \int f(t)\, dX_t$. Then m is split by the natural spectral measure on $L_2(0, 1)$.

1.3. A non-atomic spectral measure is one without any non-zero minimal projections. Our main result says if $\mathcal{E}$ is a non-atomic spectral measure which splits a weak distribution m, and if m is absolutely continuous with regard to the normal distribution n, then m is equivalent to n and is actually a translate of n by an element of H. Our proof makes use of two properties of the normal distribution both due to I. E. Segal. They are the duality transform [4, Theorem 3], and the ergodicity theorem [3, Theorem 1].

1.4. Let $x_1, \cdots, x_n$ be orthogonal vectors in H. Let

$$\varphi(t_1, \cdots, t_n)$$

be a bounded Baire function. Then $f(x) = \varphi(t_1, \cdots, t_n)$ with $t_1 = (x_1, x), \cdots t_n = (x_n, x)$ is called a tame function on H. It clearly corresponds to a random variable with regard to the normal distribution. Given a trans-

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